Algebra 2025/2026: Exercise sheet 7

Exercise 1.

Show that $\mathbb{Z}[X]$ is not a P.I.D. In the same way, if K is a field, show that $K[X,Y]$ is not a P.I.D.

Exercise 2.

Let R be a noetherian ring. Let $R \twoheadrightarrow S$ be an epimorphism of (commutative) rings. Show that S is also noetherian.

Exercise 3.

Let $n \geq 2$ be an integer. Let $A := \mathbb{Z}[\mathbb{Z}/n.\mathbb{Z}]$ be the group ring of the cyclic group $\mathbb{Z}/n.\mathbb{Z}$. Show that it is isomorphic to the quotient ring $\mathbb{Z}[X]/(X^n - 1)$.

Exercise 4.

1) Let $R \twoheadrightarrow S$ be an epimorphism between commutative rings. Show that the image of an ideal $I \subset R$ is an ideal in S.

2) Let R be a commutative ring and I and J ideals of R. We assume $I \subset J$; show that R/J is canonically isomorphic to the quotient of R/I by the ideal $\overline{J} \subset R/I$ image of J.

3) Let R be a commutative ring and I and J ideals of R. Show that $R/(I+J)$ is isomorphic to $(R/I)/\overline{J}$, where $\overline{J}$ is the image of J in (R/I).

4) Let R be a commutative ring and I and J ideals of R. Show that the kernel of the canonical ring homomorphism $R \to R/I \times R/J$ is $I \cap J$. If moreover $I + J = R$ (one says I and J are coprime) prove that the induced homomorphism

$$R/(I \cap J) \to R/I \times R/J$$

is an isomorphism, and moreover that $I \cap J = I.J$.