Algebra 2025/2026: Exercise sheet 6

Exercise 1.

1) Let R be a finite commutative ring. Show that (R is an integral domain) $\Leftrightarrow$ (R is a field)

2) Let $\mathbb{F}$ be a finite field, and $R \subset \mathbb{F}$ be a subring. Show that R is a field.

Exercise 2.

Let R be an integral domain. Show that the canonical ring monomorphism $R \subset R[X]$ taking an element λ of R to the corresponding contant polynomial, induces an isomorphism on the multiplicative groups:

$$R^{\times} \cong (R[X])^{\times}$$

If R is not assumed to be an integral domain, is it still true ? [Hint: Assume there is an element x in $R \setminus \{0\}$ such that $x^2 = 0$. Construct from that a polynomial in $R[X]$ which is not constant and is invertible]

Exercise 3.

Let $\mathcal{P} \subset \mathbb{N}$ be the subset consisting of prime numbers.

1) Let $\mathbb{Z}[\mathcal{P}]$ be the free abelian group on $\mathcal{P}$. Show that the multiplicative group $\mathbb{Q}^{\times}$ of $\mathbb{Q}$ is canonically isomorphic to the product $\{\pm 1\} \times \mathbb{Z}[\mathcal{P}]$.

2) Let $S \subset \mathcal{P}$ be a subset. Denote by $\mathbb{Z}_S \subset \mathbb{Q}$ the subset of rational numbers which can be written as $\frac{a}{b}$ with a and b integers, $b \neq 0$, and such that the prime divisors of b are all in S. Show the $\mathbb{Z}_S$ is a subring of $\mathbb{Q}$ and that its multiplicative group $(\mathbb{Z}_S)^{\times}$ is isomorphic to $\{\pm 1\} \times \mathbb{Z}[S]$, where $\mathbb{Z}[S]$ denotes the free abelian group on the set S.

Exercise 4.

Let R be a (non necessarily commutative) ring, let G be a group. Show that the map:

$$Hom_{Ring}(\mathbb{Z}[G], R) \to Hom_{Grp}(G, R^{\times})$$

from the set of ring homomorphisms from the group ring $\mathbb{Z}[G]$ to the ring R to the set of group homomorphisms $Hom_{Grp}(G, R^{\times})$ from G to the multiplicative group $R^{\times}$ of R, taking a ring homomorphism $\phi : \mathbb{Z}[G] \to R$ to the composition

$$G \subset (\mathbb{Z}[G])^{\times} \xrightarrow{\phi^{\times}} R^{\times}$$

is a bijection.

In case $G = \mathbb{Z}/2\mathbb{Z}$ and $R = \mathbb{Z}$, show that there are exactly two (commutative) ring homomorphisms $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \to \mathbb{Z}$, ϕ and ψ, and show that the product homomorphism $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \to \mathbb{Z} \times \mathbb{Z}$, $x \mapsto (\phi(x), \psi(x))$ is a monomorphism of rings but is not surjective.