Algebra 2025/2026: Exercise sheet 4

Exercise 1.

Let G be a group acting (on the left) on a set X. Let x and y be elements of X belonging to the same orbit. Show that the isotropie groups $I_x \subset G$ and $I_y \subset X$ are conjugate subgroups.

Exercise 2.

Give all the possible orders of elements of the symmetric groups S_3, S_4, S_5, S_6 and S_7.

Exercise 3.

Let p and q be two distinct prime numbers, with $p < q$.

1) Describe up to isomorphism all the finite groups of order $p.q$. [Hint: use Sylow's theorems, or Cauchy's theorem and Exercise 3 of sheet 3]. In particular show that if q is not congruent to 1 modulo p, G is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/q\mathbb{Z}$.

2) Let $r \geq 1$ be an integer and G be a finite group of order $p.q^r$. Show that G is a semi-direct product $H \rtimes \mathbb{Z}/p\mathbb{Z}$ of a group H of order q^r and $\mathbb{Z}/p\mathbb{Z}$ acting on H by group automorphisms. [Hint: use Sylow's theorems]

Exercise 4.

Prove that a nilpotent group is solvable. Is the converse true ?