Algebra 2025/2026: Exercise sheet 12

Exercise 1.

Let $K \subset \mathbb{C}$ be the splitting field of $X^3 - 2$ over $\mathbb{Q}$.

1) Let $j = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Observe $j^3 = 1$. Compute $[\mathbb{Q}[j] : \mathbb{Q}]$ and the minimal polynomial of j (over $\mathbb{Q}$).

2) Show that $K = \mathbb{Q}[\sqrt[3]{2}, j]$.

3) Compute $[K : \mathbb{Q}]$.

Exercise 2.

Let $K \subset L$ be an algebraic extension. Assume that L is algebraically closed. Show that a K-morphism $\sigma : L \to L$ is automatically an isomorphism (or K-automorphism).

Exercise 3.

Let p be a prime number. Let $\zeta := exp^{\frac{2i\pi}{p}} \in \mathbb{C}$. Observe that $\zeta^p = 1$.

1) Let $\Phi_p := \Sigma_{i=0}^{p-1} X^i \in \mathbb{Q}[X]$. Show that Φ_p is irreducible in $\mathbb{Q}[X]$ by using the Eisenstein criterium to the polynomial $\Phi_p(X+1)$ with the prime p.

2) Show that Φ_p is the minimal polynomial of ζ. Conclude that $[\mathbb{Q}[\zeta] : \mathbb{Q}] = p - 1$.

3) Show that $\mathbb{Q}[\zeta]$ is the splitting field (over $\mathbb{Q}$) of $X^p - 1$.

4) (*) Show that $Aut_{\mathbb{Q}}(\mathbb{Q}[\zeta])$ is isomorphic to the multiplicative group $(\mathbb{F}_p)^\times$ of $\mathbb{F}_p$.

Exercise 4.

Let K be a field of characteristic $\neq 2$.

1) Let $K \subset L$ be a quadratic extension, that is to say a field extension of degree 2. Show that there exists $\alpha \in L \setminus K$ with $\alpha^2 \in K$.

2) Show that if $K \subset L$ and $K \subset M$ are quadratic extensions, α is an element of $L \setminus K$ with $\alpha^2 \in K$ and β an element in $M \setminus K$ with $\beta^2 \in K$, then L and M are K-isomorphic if an only if $\frac{\alpha}{\beta} \in K$.

3) Conclude that the set of K-isomorphism classes of quadratic extensions of K is canonically in bijection with the set $K^\times/(K^\times)^2$. Describes this set for $K = \mathbb{Q}$.