Algebra 2025/2026: Exercise sheet 11

Exercise 1.

Let $K \subset L$ and $L \subset M$ be field extensions. Show that:

$$(K \subset M \text{ is a finite extension}) \Leftrightarrow (K \subset L \text{ and } L \subset M \text{ are finite extensions})$$

Moreover in that case

$$[M : K] = [M : L] \times [L : K]$$

(In particular $[L : K]$ and $[M : L]$ always divide $[M : K]$.) [Hint: choose a basis $\{e_1, \dots, e_n\}$ of L over K and $\{f_1, \dots, f_m\}$ of M over L and show that $\{e_i.f_j\}$ is a basis of M over K.]

Exercise 2.

Let $K \subset L$ and $K \subset M$ be field extensions.Let $\sigma : L \to M$ be a morphism of K-extensions. Show that if $x \in L$ is algebraic over K, then so is $\sigma(x) \in M$. In particular the group $Aut_K(L)$ of K-automorphisms of L operates on the algebraic closure of K in L.

Exercise 3.

Let K be a finite field.

1) Show that its characteristic is non zero. Let $p = char(K)$ be the characteristic of K, a prime number. Let q be the cardinality of K.

2) Show that $q = p^{[K:\mathbb{F}_p]}$.

3) Show that any non zero element x of K satisfies $x^{q-1} = 1$ [Hint: use the multiplicative group of K, a finite group...]. Conclude that in $K[X]$:

$$X^q - X = \Pi_{x \in K}(X - x)$$

Exercise 4.

Let α and β be the real numbers defined by $\alpha = \sqrt{2}$ and $\beta = \sqrt[3]{2}$. Let $K = \mathbb{Q}[\alpha, \beta] \subset \mathbb{R}$ be the sub-$\mathbb{Q}$-algebra generated by α and β.

1) Show that α and β are algebraic over $\mathbb{Q}$, and that $K \subset \mathbb{R}$ is a subfield, and a finite extension of $\mathbb{Q}$.

2) Prove that $[K : \mathbb{Q}] = 6$.

3) Prove that $\mathbb{Q}[\alpha + \beta] = K$, in other words $\alpha + \beta$ generates K over $\mathbb{Q}$. [Hint: if not, then $\alpha + \beta$ has degree 2 or 3 over $\mathbb{Q}$. Use the basis $\{1, \alpha, \beta, \alpha.\beta, \beta^2, \alpha.\beta^2\}$ of K over $\mathbb{Q}$ to show it is not possible...]

4) Give the minimal polynomial of $\alpha + \beta$ over $\mathbb{Q}$.