Algebra 2025/2026: Exercise sheet 10

Exercise 1.

Let R and S be integral domains. Let $\phi : R \to S$ be a morphism of rings. Let $P \in R[X]$ be a monic polynomial of degree ≥ 1. Show that if $\phi(P)$ is irreducible in $S[X]$, P is irreducible in $R[X]$.

Exercise 2.

Let p be a prime number.

1) Let $n \geq 2$ be an integer. Show that p is never an n-power in $\mathbb{Q}$ (that is, there is no $x \in \mathbb{Q}$ such that $x^n = p$).

2) Show that $X^2 - p$ and $X^3 - p$ are irreducible in $\mathbb{Q}[X]$, as well as $\mathbb{Z}[X]$.

Exercise 3.

Let K be a field, let $P \in K[X]$ be a polynomial of degree $n \geq 1$. Show that there is a field extension $K \subset L$ of degree $\leq n$, and an element $x \in L$ with $P(x) = 0$.

Exercise 4.

Let K be a field, let p be a prime number. Let $a \in K$ be an element which is not a p-th power in K. Prove that $X^p - a$ is irreducible in $K[X]$. [Hint: assume $P = Q \times R$ in $K[X]$, with $deg(Q) \geq 1$ and $deg(R) \geq 1$ with Q irreducible. Consider the multiplication by X in the field $K[X]/(Q)$ as a K-endomorphism $\phi : K[X]/(Q) \to K[X]/(Q)$. Observe that $\phi^p = a. : K[X]/(Q) \to K[X]/(Q)$ and use the determinant to get a contradiction...]