Exercise sheet 8

Exercise 1.

Let $f : S' \to S$ be a finite morphism of noetherian schemes. Show that the higher direct image functors $R^*f_* : \mathcal{A}b_{Nis}(Sch^{ft}_{S'}) \to \mathcal{A}b_{Nis}(Sch^{ft}_S)$ are zero for $* > 0$. [Hint: for $M \in \mathcal{A}b_{Nis}(Sch^{ft}_{S'})$ choose an injective resolution $M \to I^*$ and compute the stalks of $f_*(I^*)$ at each "usual" point of the site $(Sch^{ft}_S, \mathcal{T}_{Nis})$]

Exercise 2.

Let $G_\bullet : \mathcal{I} \to \mathcal{FG}r$ be a pro-object of the category of finite groups.

1) Show that $G_\bullet$ is (canonically) isomorphic in $pro - \mathcal{FG}r$ to a pro-object whose transistion morphisms are all epimorphisms.

From now on we assume that $G_\bullet$ satisfies the previous property (thus for any morphism $i \to j$ in $\mathcal{I}$, $G_i \to G_j$ is an epimorphism.

2) We let G be the limit of $G_\bullet$ in the category of groups. As G is canonically a subset of the product $\Pi_{i \in \mathcal{I}} G_i$ we may consider the induced topology on G from that of the product (product topology), which is a compact topological space (Tychonoff theorem). Show that G is a closed subset and thus compact.

3) For each $i \in \mathcal{I}$ show that the (obvious) morphism $G \to G_i$ is an epimorphism [Hint: you may use without proof, the fact that the limit of a left filtering system of non empty finite sets is non empty...].

4) For each $i \in \mathcal{I}$, let $U_i := Ker(G \to G_i)$. Show that the $(U_i)_i$ form a basis of open neighborhood of the neutral element of G.

5) Prove that a continuous map $G \to S$ to a discrete set factorises through one of the morphisms $G \to G_i$ and conclude that the natural map $colim_i Map(G_i, S) \to C^0(G, S)$ is a bijection. In the same way, if $S : \mathcal{J} \to Set$ is an "ind set" (a functor from a right filtering small category $\mathcal{J}$) show that the natural map $colim_j C^0(G, S_j) \to C^0(G, colim_j S_j)$ is a bijection.

Exercise 3.

(*) Let k be a field, let Sm_k be the category of smooth, finite type, separated k-schemes. Let $X \in Sm_k$ and let $x \in X$ be a rational k-point (a point such that $k \subset \kappa(x)$ is an isomorphism).

1) Show that the Hensel ring $\mathcal{O}^h_{X,x}$ is isomorphic to $\mathcal{O}^h_{\mathbb{A}^n_k,0}$ (the Henselisation of the local ring of $\mathbb{A}^n_k$ at 0).

2) Show that in the category $Shv_{Nis}(Sm_k)$ the quotient sheaf $X/(X - \{x\})$ (collapsing the open subscheme $X - \{x\}$ to the point) is isomorphic to $\mathbb{A}^n_k/(\mathbb{A}^n_k - \{0\})$