

Exercise sheet 2

Exercise 1.

Denote by $\mathcal{F} \subset \text{Set}$ the full subcategory of finite sets. Show that a functor $F : \mathcal{F} \rightarrow \text{et}$ which takes finite limits to finite limits is pro-representable. Let $S \in \text{et}$ be a set; how would you define its profinite completion \hat{S} ?

Denote by $\mathcal{FG}r \subset \mathcal{G}r$ the full subcategory of the category of groups consisting of finite groups. Show that a functor $F : \mathcal{FG}r \rightarrow \text{Set}$ which takes finite limits to finite limits is pro-representable. Let $G \in \mathcal{G}r$ be a group ; how would you define its profinite completion \hat{G} ?

Give an example!

Exercise 2.

Let $(\mathcal{C}, \mathcal{T})$ be a site endowed with a Grothendieck topology. Let P be a presheaf of sets on \mathcal{C} . Let \mathcal{C}/P be the category whose objects are pairs (C, x) with C an object of \mathcal{C} and $x \in P(C)$ and whose morphisms $(C, x) \rightarrow (D, y)$ are morphisms $\phi : C \rightarrow D$ in \mathcal{C} such that $P(\phi)(y) = x$. We have the obvious (forgetful) functor $(\mathcal{C}/P)^{op} \rightarrow \text{Pshv}(\mathcal{C}), (C, x) \mapsto C$ (by which we mean $C \in \text{Pshv}(\mathcal{C})$ is the presheaf $\text{Hom}_{\mathcal{C}}(-, C)$ represented by C). Show that the obvious morphism:

$$\text{colim}_{(C,x) \in \mathcal{C}/P} C \rightarrow P$$

is an isomorphism.

Exercise 3.

Let $(\mathcal{C}, \mathcal{T})$ be a site endowed with a Grothendieck topology. Let $f : F \rightarrow G$ be a morphism in $\text{Shv}_{\mathcal{T}}(\mathcal{C})$.

Show that :

$(f \text{ is an epimorphism}) \iff (f \text{ is an effective epimorphism}) \iff (f \text{ is a universal effective epimorphism})$

[Hint: consider the diagram $F \times_G F \rightrightarrows F \rightarrow G$ also in the category of presheaves and use a remark of the lecture concerning monomorphisms of presheaves...]