Exercise sheet 1

Exercise 1.

Let $\mathcal{M}$ be the category of differentiable manifolds M for which[1] there is some open subset U of some $\mathbb{R}^n$ and a closed proper differentiable embedding $M \subset U$.

Let $\mathcal{T}_1$ be the collection of all (usual) open coverings of all $M \in \mathcal{M}$.

Let $\mathcal{T}_2$ be the collection of families $\{U_i \to M\}_i$ for all $M \in \mathcal{M}$, where the maps $f_i : U_i \to M$ are differentiable maps with differential $d(f_i)$ invertible at any point of U_i and such that the image of the f_i cover M.

Finally let $\mathcal{T}_3$ be the collection of families $\{U_i \to M\}_i$ for all $M \in \mathcal{M}$, where the maps $f_i : U_i \to M$ are submersions (= differentiable maps with differential $d(f_i)$ surjective at any point of U_i) and such that the image of the f_i cover M.

1) Show that $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are topologies on $\mathcal{M}$, and that $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_3$.

2) Show that $Shv_{T_3}(\mathcal{M}) = Shv_{T_2}(\mathcal{M}) = Shv_{T_1}(\mathcal{M})$. [Hint: you might use the "inverse function theorem"]

3) Assume G is a finite group, and $G - \mathcal{M}$ is the category of differentiable manifolds (in $\mathcal{M}$) endowed with an action of G by differentiable automorphisms. Does the analogue of the result of question 2) holds for G non trivial ?

Exercise 2.

Let $(\mathcal{C}, \mathcal{T})$ be a site endowed with a Grothendieck topology.

1) Prove that the category of coverings for $\mathcal{T}$, defined in the lecture, of a given object $X \in \mathcal{C}$ is left filtering.

2) Prove that the correspondance $X \mapsto L(P)(X)$, also defined in the lecture, is a presheaf.

3) Show that the natural map $P(X) \to L(P)(X)$ defined $\forall X \in \mathcal{C}$ defines a morphism of presheaves.

Exercise 3.

Let $\mathcal{C}$ be a category admitting finite limits. Show that the category $pro-\mathcal{C}$ of pro-objects in $\mathcal{C}$ admits all limits. [Hint: reduce to the case of arbitrary product and equalisers]

[1]This technical condition is only there to be sure that $\mathcal{M}$ is essentially small! You can ignore it...