Exercise sheet 6

Exercise 1.

Recall that $\mathbb{A}^1_{\mathbb{Z}} = Spec(\mathbb{Z}[T])$ and that $(\mathbb{G}_m)_{\mathbb{Z}} = \mathbb{A}^1_{\mathbb{Z}} - \{0\}$ is the open subscheme $D(T)$ of $\mathbb{A}^1_{\mathbb{Z}}$. Let X a scheme; recall that for any $x \in X$, there is a canonical morphism of schemes

$$Spec(\mathcal{O}_{X,x}) \to X$$

and also that we identify $\mathcal{O}_X(X)$ and $Hom_{Sch}(X, \mathbb{A}^1_{\mathbb{Z}})$.

1) Let $f \in \mathcal{O}_X(X)$. Show that using the previous identifications, for any $x \in X$ the stalk $f_x \in \mathcal{O}_{X,x}$ of f at x (that is the image of f through the canonical morphism $\mathcal{O}_X(X) \to \mathcal{O}_{X,x}$) corresponds to the composition

$$Spec(\mathcal{O}_{X,x}) \to X \xrightarrow{f} \mathbb{A}^1_{\mathbb{Z}}$$

2) Let X_f be the inverse image of the open subset $\mathbb{A}^1_{\mathbb{Z}} - \{0\}$ of $\mathbb{A}^1_{\mathbb{Z}}$ through $f : X \to \mathbb{A}^1_{\mathbb{Z}}$. Show that

$$X_f = \{x \in X | f_x \in \mathcal{O}^\times_{X,x}\}$$

Observe that in case $X = Spec(A)$ is affine $X_f = D(f)$.

3) For A a commutative ring, show that: ($f \in A$ is a unit in A) $\Leftrightarrow$ ($\forall \mathcal{P} \in Spec(A)$, $f_{\mathcal{P}} \in A^\times_{\mathcal{P}}$).

4) Show that $f|_{X_f}$ is invertible in $\mathcal{O}_X(X_f)$ and that if U is an open subset of X such that $f|U$ is invertible in $\mathcal{O}_X(U)$ then $U \subset X_f$.

Exercise 2.

(sequel to Ex 1))

Let X be a scheme and let A be the ring $\mathcal{O}_X(X)$. Let $f \in A$.

1) If X is quasi-compact, let $a \in A$ be such that $a|_{X_f} = 0$. Show that there exists $n > 0$ such that $f^n.a = 0 \in A$. [Hint: use an open affine covering of X and the previous exercise]

2) Assume that X can be covered by finitely many open affines suchemes U_i such that $U_i \cap U_j$ is quasi-compact. Let $b \in \mathcal{O}_X(X_f)$. Show that there exists $n > 0$such that $f^n.b \in \mathcal{O}_X(X_f)$ is the restriction of an element of A.

3) with the assumptions of 2) conclude that $\mathcal{O}_X(X_f) = A_f$

4) (*) Conclude that if X is a scheme such that there are elements f_1,... , f_r in $A = \mathcal{O}_X(X)$ such that the open subsets X_{f_i} are affine and the f_1,... , f_r generate the unit ideal of A, then X is affine. [Hint: consider the canonical morphism $X \to Spec(A)$ ...]