Exercise sheet 5

Exercise 1.

Let X be a scheme and let A be the ring $\mathcal{O}_X(X)$. Show that X is affine if and only if the canonical morphism of schemes:

$$X \to Spec(A)$$

is an isomorphism.

Exercise 2.

Let X be a scheme and K be a commutative field. Describe a natural bijection:

$$Hom_{Sch}(Spec(K), X) \cong \{(x, \phi) | x \in X,\ \phi : \kappa(x) \subset K\}$$

(where $\phi : \kappa(x) \subset K$ is a field extension)

Exercise 3.

Let $f : X \to Y$ be a morphism of schemes. Suppose $V \subset Y$ is an open subset of Y containing the (set theoretical) image of X. Let $j : (V, \mathcal{O}_Y|_V) \to (Y, \mathcal{O}_Y)$ be the induced morphism of schemes. Show that there exists a unique morphism of schemes $g : X \to V$ such that

$$f = j \circ g$$

Exercise 4.

Let k be a field and let $A := k[X, Y]/(Y^3 - X^2)$ and let $B := k[T]$.

1) Show that the ring homomorphism $\phi : A \to B$ with $\phi(X) = T^3$ and $\phi(Y) = T^2$ is injective. [Hint: use that $k[X, Y]$ is a UFD of which $(Y^3 - X^2) \in k[X, Y]$ is an irreducible element]

2) Show that the ideal $\overline{(X, Y)}$ in A generated the class of X and Y is a maximal ideal denoted by $\mathcal{M}_0$. Show it is not a principal ideal. Conclude that $\phi : A \to B$ is not an isomorphism.

3) Show that there is an element f in A such that $A_f \to B_f$ is an isomorphism. Let $k(T)$ be the fraction field of B. Conclude that $k(T)$ is also the fraction field of A.

4) Show that the inverse image of $\mathcal{M}_0$ is the maximal ideal (T). Conclude that the restriction of $Spec(\phi)$ to $D(T)$ is an isomorhism onto $Spec(A) \setminus \{\mathcal{M}_0\}$, that $Spec(\phi)$ is an homeomorphism but is not an isomorphism of schemes.