

Exercise sheet 1

Exercise 1.

Let X be a topological space.

1) Let $\mathcal{F}$ be a sheaf of sets on X . Let U and V be open subsets of X , and let $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ be sections of $\mathcal{F}$ respectively on U and V . Show that the set $\{x \in U \cap V \mid s_x = t_x \in \mathcal{F}_x\}$ is open (in X).

2) Conclude that if $\mathcal{A}$ is a sheaf of abelian groups on X and $s \in \mathcal{A}(U)$ a section of $\mathcal{A}$ over an open subset U , the set

$$\text{supp}(s) := \{x \in U \mid s_x \neq 0 \in \mathcal{A}_x\}$$

is closed in U .

Exercise 2.

1) Describe the category $\text{PreShv}(\emptyset)$ of presheaves of sets and $\text{Shv}(\emptyset)$ of sheaves of sets on the empty topological space $\emptyset$. Then describe the sheafification functor $a : \text{PreShv}(\emptyset) \rightarrow \text{Shv}(\emptyset)$.

2) Describe the category $\text{PreShv}(*)$ of presheaves of sets and $\text{Shv}(*)$ of sheaves of sets over the topological space $*$ with one element. Then describe the sheafification functor $a : \text{PreShv}(*) \rightarrow \text{Shv}(*)$.

3) Let $T = \{\eta, *\}$ be a set with two elements. We consider the topology on T whose open subsets are $\emptyset, T, \{\eta\}$. η is called the generic point of T . Observe that $\overline{\{\eta\}} = T$ (closure). Describe the category $\text{PreShv}(T)$ of presheaves of sets and $\text{Shv}(T)$ of sheaves of sets over the topological space $*$ with one element. Then describe the sheafification functor $a : \text{PreShv}(T) \rightarrow \text{Shv}(T)$.

Exercise 3.

Let X be a topological space and let $x \in X$ be a closed point of X .

We consider the direct image functor $x_* : \text{Shv}(\{x\}) \rightarrow \text{Shv}(X)$, that is to say which maps a sheaf of sets $\mathcal{F}$ on the space reduced to a point $\{x\}$ to the sheaf of sets $x_*(\mathcal{F})$ on X : $U \mapsto *$ is $x \notin U$ and $U \mapsto \mathcal{F}(\{x\})$ if $x \in U$.

Let S be a set. Compute the stalks of $x_*(\underline{S})$ at any point $y \in X$ (where $\underline{S}$ denotes the constant sheaf associated to S on the topological space $\{x\}$). [Hint: consider the two cases: $y \notin \overline{\{x\}}$ and $y \in \overline{\{x\}}$]

Exercise 4.

Let X be a topological space.

1) If $(\mathcal{F}_i)_{i \in I}$ is an arbitrary family of sheaves of sets on X , show that the presheaf

$$U \mapsto \prod_{i \in I} \mathcal{F}_i(U)$$

is a sheaf. It will be denoted by $\prod_{i \in I} \mathcal{F}_i$ and called the product of the family $(\mathcal{F}_i)_{i \in I}$.

2) If $\mathcal{F}$ and $\mathcal{G}$ are presheaves of sets on X show that for any point $x \in X$ the canonical map

$$(\mathcal{F} \times \mathcal{G})_x \rightarrow \mathcal{F}_x \times \mathcal{G}_x$$

is a bijection.