

CELLULAR \mathbb{A}^1 -HOMOLOGY AND THE MOTIVIC VERSION OF MATSUMOTO'S THEOREM

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1. INTRODUCTION

The aim of this work is to determine the \mathbb{A}^1 -fundamental sheaf of groups $\pi_1^{\mathbb{A}^1}(G)$ of a split reductive group G over a field k . Our computation of $\pi_1^{\mathbb{A}^1}(G)$ relies on three facts: the combinatorics of the root datum of a split reductive group, the \mathbb{A}^1 -homotopy purity theorem of [24] and the structure of \mathbb{A}^1 -homotopy sheaves of connected spaces over a perfect field k [23]; the latter two being foundational results from \mathbb{A}^1 -algebraic topology. The main novelty of our

Anand Sawant acknowledges support of the Department of Atomic Energy, Government of India, under project no. 12-R&D-TFR-5.01-0500. He was supported by DFG Individual Research Grant SA3350/1-1 while he was affiliated to Ludwig-Maximilians-Universität, München, where a part of this work was carried out.

approach is defining a new version of \mathbb{A}^1 -homology, called *cellular \mathbb{A}^1 -homology*, for smooth schemes admitting a *nice* stratification (more precisely, a *cellular structure*). The cellular \mathbb{A}^1 -homology sheaves are obtained as the homology sheaves of the *cellular \mathbb{A}^1 -chain complex*, which is associated to the cellular structure. The cellular \mathbb{A}^1 -homology theory, inspired by cellular homology in classical topology, is independent of the cellular structure, and is very often entirely computable. Some desired but elusive properties of \mathbb{A}^1 -homology can be seen to hold for cellular \mathbb{A}^1 -homology; for instance, it follows immediately from definitions that cellular \mathbb{A}^1 -homology of a scheme vanishes in degrees beyond the dimension of the scheme.

In this work, we will freely use the terminology introduced and used in [24] and [23] (we refer the reader to the section on notation and conventions at the end of the introduction). Throughout this paper, k will denote a fixed base field (which will soon be assumed to be perfect) and Sm_k will denote the category of smooth k -schemes, endowed with the Nisnevich topology. Unless otherwise stated, all sheaves are meant to be sheaves for the Nisnevich topology on Sm_k . Given a split reductive group G over k , let G_{der} denote the derived subgroup of G with universal simply connected cover $G_{\text{sc}} \rightarrow G_{\text{der}}$ (in the sense of the theory of algebraic groups) with kernel μ , which is known to be a finite group scheme of multiplicative type (it is thus a product of group schemes of the form μ_n). Let \mathbf{K}_n^{M} (respectively, \mathbf{K}_n^{MW}) denote the n th unramified Milnor K -theory (respectively, Milnor-Witt K -theory) sheaf.

Theorem 1. *For a split reductive group G over a field k , we have an exact sequence*

$$1 \rightarrow \pi_1^{\mathbb{A}^1}(G_{\text{sc}}) \rightarrow \pi_1^{\mathbb{A}^1}(G) \rightarrow \mu \rightarrow 1$$

of Nisnevich sheaves of abelian groups. If G is a split, semisimple, almost simple, simply connected algebraic group over a field k , then there exists an isomorphism of sheaves

$$\pi_1^{\mathbb{A}^1}(G) \simeq \begin{cases} \mathbf{K}_2^{\text{M}}, & \text{if } G \text{ is not of symplectic type;} \\ \mathbf{K}_2^{\text{MW}}, & \text{if } G \text{ is of symplectic type.} \end{cases}$$

The first assertion of Theorem 1 follows from standard arguments. Indeed, one has a short exact sequence

$$1 \rightarrow G_{\text{der}} \rightarrow G \rightarrow \text{corad}(G) \rightarrow 1,$$

where $\text{corad}(G)$ denotes the coradical torus of G . Since $\text{corad}(G)$ is \mathbb{A}^1 -rigid, it follows that $\pi_1^{\mathbb{A}^1}(G) \simeq \pi_1^{\mathbb{A}^1}(G_{\text{der}})$, reducing the computation of the \mathbb{A}^1 -fundamental group of a split reductive group to its semisimple part. The \mathbb{A}^1 -fiber sequence

$$G_{\text{sc}} \rightarrow G_{\text{der}} \rightarrow B_{gm}\mu,$$

where $B_{gm}\mu$ is the geometric classifying space considered by Totaro [29] and also studied in [24], gives rise to a short exact sequence of sheaves of groups of the form

$$1 \rightarrow \pi_1^{\mathbb{A}^1}(G_{\text{sc}}) \rightarrow \pi_1^{\mathbb{A}^1}(G_{\text{der}}) \rightarrow \mu \rightarrow 1.$$

Furthermore, observe that G_{sc} is a product of its almost simple factors. As a consequence, the computation of the \mathbb{A}^1 -fundamental group of a split reductive group reduces to the computation of the \mathbb{A}^1 -fundamental group of a split, semisimple, almost simple, simply connected algebraic group.

Remark 2. A classical theorem of Matsumoto [19] determines a presentation of the center of the universal central extension of the group of rational points of a split, semisimple, simply connected algebraic group over an infinite field k . Let G be a such a group over k .

Matsumoto's theorem can be rephrased as saying that the center of the universal central extension

$$(1.1) \quad 1 \rightarrow A \rightarrow E \rightarrow G(k) \rightarrow 1$$

of $G(k)$ is given by

$$A = \begin{cases} \mathbf{K}_2^{\mathbf{M}}(k), & \text{if } G \text{ is not of symplectic type;} \\ \mathbf{K}_2^{\mathbf{MW}}(k), & \text{if } G \text{ is of symplectic type.} \end{cases}$$

Note that Matsumoto only considers infinite base fields; indeed, $\mathbf{K}_2^{\mathbf{M}}$ and $\mathbf{K}_2^{\mathbf{MW}}$ of a finite field are trivial. Moreover, note that for some small finite fields F , the group $G(F)$ is at times not perfect and hence, does not admit a universal central extension.

It is natural to look for central extensions (1.1) that are functorial in k and in G . Brylinski and Deligne [8] have shown the existence of a central extension of any reductive algebraic group G by $\mathbf{K}_2^{\mathbf{M}}$ as Zariski sheaves and determined the category of all such extensions in terms of Weyl-invariant integer-valued quadratic forms on the coroot lattice associated with the semisimple part of G . A standard fact from topology asserts that the fundamental group of a path-connected topological group is abelian, and its universal covering is in fact a topological central extension of it by its fundamental group. If the topological group is moreover perfect, this implies the existence of a canonical morphism from the universal central extension to the universal covering.

In view of this heuristic, it is natural to search for analogues of these facts in \mathbb{A}^1 -homotopy theory of smooth algebraic varieties. More specifically, one should expect that the central extensions of a reductive group determined by Matsumoto and Brylinski-Deligne are closely related to its universal cover in the sense of \mathbb{A}^1 -homotopy theory and its \mathbb{A}^1 -fundamental group in the sense of [24]. This is precisely what Theorem 1 achieves. We stress that we do not use the theorem of Matsumoto in any form; our result is entirely independent of it.

Remark 3. We use some of the results of [23] in our arguments, which only hold when the base field is perfect. However, by Chevalley's existence theorem [11, XXV, 1.2], any split reductive k -group is obtained (up to isomorphism) by extension of a split reductive group defined over \mathbb{Z} and consequently, a split reductive group defined over the prime field k_0 of k , which is perfect. Using standard facts on essentially smooth base change (see, for example, [18, Appendix A]), we are reduced to proving Theorem 1 in case the base field is perfect. Therefore, we will always work over a perfect field throughout the article.

Remark 4. If G is a split, semisimple, simply connected group such that every component of the relative root system of G has isotropic rank at least 2, then the group $\pi_1^{\mathbb{A}^1}(G)(k)$ can be directly computed (see [30]). The proof in [30] uses the \mathbb{A}^1 -locality of the singular construction $\text{Sing}_*^{\mathbb{A}^1} G$ [2], Matsumoto's results [19] and some work by Petrov and Stavrova [25] (which requires the hypothesis on the isotropic rank of the root system). However, these computations are unsatisfactory because they do not determine $\pi_1^{\mathbb{A}^1}(G)$ as a sheaf and do not work if a component of the relative root system of G has rank 1. Moreover, the proof of \mathbb{A}^1 -locality of $\text{Sing}_*^{\mathbb{A}^1} G$ depends on several nontrivial results in algebraic geometry. Our proof of Theorem 1 does not make any use of the results of Matsumoto and Brylinski-Deligne, as mentioned above. It is worthwhile to mention that we never use any model of G in \mathbb{A}^1 -homotopy theory (such as $\text{Sing}_*^{\mathbb{A}^1} G$) in our proofs. Our proof of Theorem 1 is much more elementary and direct; see Remark 8 below.

Remark 5. Observe that the \mathbb{A}^1 -fundamental group of an algebraic group G only depends on the \mathbb{A}^1 -connected component of the neutral element of G . We recall the description of the sheaf $\pi_0^{\mathbb{A}^1}(G)$ obtained in [22, Appendix A]. Let G be a split, semisimple algebraic group over k with universal simply connected cover $G_{\text{sc}} \rightarrow G$. Let T denote a maximal k -split torus of G and let T_{sc} be its inverse image in G_{sc} . It has been shown in [22, Theorem A.2] that there is a canonical isomorphism of the form

$$T/\text{Nis}T_{\text{sc}} \cong \pi_0^{\mathbb{A}^1}(G),$$

where the left hand-side is the quotient of T by the image of the induced morphism $T_{\text{sc}} \rightarrow T$ as a Nisnevich sheaf of abelian groups. In particular, G is \mathbb{A}^1 -connected if and only if it is simply connected in the sense of algebraic groups theory. However, Theorem 1 shows that G is never \mathbb{A}^1 -simply connected!

We now briefly outline the main ideas in our proof of the main theorem. It is an elementary observation (analogous to the one in classical topology) that for any simplicial sheaf of groups \mathcal{G} on Sm_k , the sheaf of groups $\pi_1^{\mathbb{A}^1}(\mathcal{G})$ is abelian. If \mathcal{G} is \mathbb{A}^1 -connected, then the Hurewicz morphism $\pi_1^{\mathbb{A}^1}(\mathcal{G}) \rightarrow \mathbf{H}_1^{\mathbb{A}^1}(\mathcal{G})$ is an isomorphism [23, Theorem 6.35]. Consequently, determination of the \mathbb{A}^1 -homology sheaf $\mathbf{H}_1^{\mathbb{A}^1}(G)$ is tantamount to the proof of Theorem 1. However, it turns out that the \mathbb{A}^1 -homology sheaves are very hard to compute and even the basic computations of them are quite involved.

We define *cellular \mathbb{A}^1 -homology* (denoted by $\mathbf{H}_*^{\text{cell}}$) in Section 2.3 for smooth schemes admitting a *cellular structure*; that is, an increasing filtration by open subschemes with a condition on the closed complements that their cohomology with values in any strictly \mathbb{A}^1 -invariant sheaf is trivial (see Sections 2.2 and 2.3 for precise definitions and more details). The normal bundles of these closed strata are then automatically trivial. The first nontrivial cellular \mathbb{A}^1 -homology sheaf of a *cellular scheme* clearly agrees with its corresponding \mathbb{A}^1 -homology sheaf. Cellular \mathbb{A}^1 -homology is then the homology of an explicit chain complex of *strictly \mathbb{A}^1 -invariant sheaves* on Sm_k directly defined using the cellular structure. However, these computations are delicate, since one has to keep track of the chosen orientations (or trivializations) of the normal bundles of the closed strata appearing in the cellular structure. In the setting of our main theorem, the cellular structures are naturally provided by the Bruhat decomposition of a split semisimple group, which yields an explicit chain complex $C_*^{\text{cell}}(G)$ whose homology sheaves are $\mathbf{H}_*^{\text{cell}}(G)$ (see Section 3). For any split, semisimple, almost simple, simply connected algebraic group G over k is \mathbb{A}^1 -connected and hence, we have isomorphisms

$$\pi_1^{\mathbb{A}^1}(G) \cong \mathbf{H}_1^{\mathbb{A}^1}(G) \cong \mathbf{H}_1^{\text{cell}}(G).$$

The differentials in $C_*^{\text{cell}}(G)$ are determined by the orientations of the appropriate Bruhat cells and certain entries of the associated Cartan matrix. We describe in Section 3.2 how a slightly weaker version of a *pinning* of an algebraic group determines these orientations. After orienting $C_*^{\text{cell}}(G)$ in low degrees using a *weak-pinning* of G and a choice of the reduced expression of the longest word in the Weyl group of G , the differentials in degrees ≤ 2 can be explicitly computed. This is carried out in Section 4, which is the technical heart of our proof of Theorem 1. These delicate computations culminate in the proof of Theorem 1 in Section 5 (see Theorem 5.1) in the following precise form.

Theorem 6. *Let G be a split, semisimple, almost simple, simply connected k -group and fix a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ containing T . Let α be a long root in*

the corresponding system of simple roots of G and let $S_\alpha \subset G$ be the split, semisimple, almost simple, simply connected k -subgroup of rank 1 generated by U_α and $U_{-\alpha}$. The inclusion $S_\alpha \subset G$ induces an epimorphism $\pi_1^{\mathbb{A}^1}(S_\alpha) \rightarrow \pi_1^{\mathbb{A}^1}(G)$. Moreover:

(a) This morphism induces a canonical isomorphism

$$\mathbf{K}_2^{\mathbb{M}} \xrightarrow{\cong} \pi_1^{\mathbb{A}^1}(G),$$

if G is not of symplectic type.

(b) This morphism induces a non-canonical isomorphism

$$\mathbf{K}_2^{\text{MW}} \xrightarrow{\cong} \pi_1^{\mathbb{A}^1}(G),$$

if G is of symplectic type.

Remark 7. Let G be as in Theorem 6 and of symplectic type. The Dynkin diagram of G contains a unique long root, so the α in the statement of Theorem 6 is canonical. However, the isomorphism $\pi_1^{\mathbb{A}^1}(G) \cong \mathbf{K}_2^{\text{MW}}$ is not canonical. For instance, in the case $G = \mathbf{SL}_2$, it can be verified that for any $a \in k^\times$, the automorphism (as schemes and not as groups) of \mathbf{SL}_2 corresponding to

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} x & ay \\ a^{-1}z & w \end{pmatrix}$$

induces the morphism $\langle a \rangle \in \mathbf{GW}(k) = \text{Hom}_{\text{Ab}_k}(\mathbf{K}_2^{\text{MW}}, \mathbf{K}_2^{\text{MW}})$ on $\pi_1^{\mathbb{A}^1}(\mathbf{SL}_2)$. This morphism is the identity map precisely when a is a square in k .

Remark 8. In order to prove Theorem 6, we need very few properties of the sheaves \mathbf{K}_2^{MW} and $\mathbf{K}_2^{\mathbb{M}}$ (described for instance in [23]) in the computations. The reader who is not familiar with these descriptions could consider \mathbf{K}_2^{MW} to be defined as

$$\mathbf{K}_2^{\text{MW}} := \pi_1^{\mathbb{A}^1}(\mathbf{SL}_2).$$

Since $\pi_1^{\mathbb{A}^1}(\mathbf{SL}_2) = \pi_1^{\mathbb{A}^1}(\Sigma(\mathbb{G}_m^{\wedge 2}))$, for any pair of units (u, v) in a smooth k -algebra A (for instance, a field extension), we have a corresponding symbol $(u)(v) \in \mathbf{K}_2^{\text{MW}}$ (see conventions and Notation at the end of the introduction). We then need the action of the units modulo squares on $\mathbf{K}_2^{\text{MW}} = \pi_1^{\mathbb{A}^1}(\mathbf{SL}_2)$ (described in Remark 7), that is, the morphism $\mathbf{GW}(k) \rightarrow \text{Hom}_{\text{Ab}_k}(\mathbf{K}_2^{\text{MW}}, \mathbf{K}_2^{\text{MW}})$ (we do not need the fact that it is an isomorphism). In other words, if α, u and v are units defined in A , we may define the multiplication of α modulo squares, denoted by $\langle \alpha \rangle$, with the symbol $(u)(v)$, which will be denoted by $\langle \alpha \rangle \cdot (u)(v)$. We finally need to use the morphism $\eta : \mathbf{K}_1^{\text{MW}} \otimes_{\mathbb{A}^1} \mathbf{K}_2^{\text{MW}} \rightarrow \mathbf{K}_2^{\text{MW}}$, which appears in the differentials of the cellular \mathbb{A}^1 -chain complexes. However, this morphism precisely appears as one of the differentials in the cellular \mathbb{A}^1 -chain complex of \mathbf{SL}_3 , as we will see later. The only additional property that we use in the computations, which also follows from a careful but simple analysis of the cellular chain complex of \mathbf{SL}_3 , is that for any units α, u and v as above, the action of $\langle \alpha \rangle$ on the symbol $(u)(v)$ is given by

$$\langle \alpha \rangle \cdot (u)(v) = (u)(v) + \eta(\alpha)(u)(v).$$

Once this is observed, one may define $\mathbf{K}_2^{\mathbb{M}}$ to be the quotient

$$\mathbf{K}_2^{\mathbb{M}} = \mathbf{K}_2^{\text{MW}} / \eta = \pi_1^{\mathbb{A}^1}(\mathbf{SL}_2) / \eta.$$

These formal properties are sufficient for our proof of Theorem 6. The reader should notice that we do not explicitly use the Steinberg relation, though it is encoded in the sheaf \mathbf{K}_2^{MW} .

Remark 9. Let G be of rank 2 and type A_2 , that is, $G \cong \mathbf{SL}_3$. Let α be one of the two simple roots in the root system of G . The inclusion $S_\alpha \subset G$ induces an epimorphism $\pi_1^{\mathbb{A}^1}(S_\alpha) \rightarrow \pi_1^{\mathbb{A}^1}(G)$, which in turn induces a canonical isomorphism

$$\mathbf{K}_2^{\mathbb{M}} = \pi_1^{\mathbb{A}^1}(S_\alpha)/\eta \xrightarrow{\simeq} \pi_1^{\mathbb{A}^1}(G).$$

The properties noted in Remark 8 imply that the above isomorphism $\mathbf{K}_2^{\mathbb{M}} \xrightarrow{\simeq} \pi_1^{\mathbb{A}^1}(G)$ is canonical and does not depend on any choices. In the case G is of type G_2 , Theorem 6 follows from a direct computation (see the proof of Theorem 5.1). If G is of rank > 2 and is not of symplectic type, one may explicitly describe part (a) of Theorem 6 as follows. Note that we can always find a pair of long roots α and β that are connected by an edge in the Dynkin diagram of G . Then the semisimple subgroup-scheme $S_{\alpha,\beta} \subset G$ is simply connected and of type A_2 and the inclusion $S_{\alpha,\beta} \subset G$ induces a canonical isomorphism $\mathbf{K}_2^{\mathbb{M}} = \pi_1^{\mathbb{A}^1}(S_{\alpha,\beta}) \xrightarrow{\simeq} \pi_1^{\mathbb{A}^1}(G)$. Surjectivity of this morphism follows from by observing that the composite $\pi_1^{\mathbb{A}^1}(S_\alpha) \rightarrow \pi_1^{\mathbb{A}^1}(S_{\alpha,\beta}) \rightarrow \pi_1^{\mathbb{A}^1}(G)$ induced by the natural inclusions is an epimorphism. Injectivity of this epimorphism follows from the observation that adding an edge connecting α or β to another root (long or short) does not change the morphism (see the proof of Theorem 5.1; see also Remark 5.2).

We now briefly discuss the case of non-split, semisimple algebraic groups. As mentioned in Remark 5, the sheaf $\pi_1^{\mathbb{A}^1}(G)$ depends upon the \mathbb{A}^1 -connected component of the neutral element of G . Note that at least over a field of characteristic 0, a reductive algebraic group G is \mathbb{A}^1 -connected if and only if G is semisimple, simply connected and every almost k -simple factor of G is R -trivial [4, Theorem 5.2]. Using this criterion, one can see that over special classes of fields (such as local fields and global fields) isotropic, semisimple, simply connected groups are \mathbb{A}^1 -connected (see [14] for a survey of known positive results and a counterexample in general due to Platonov). It seems reasonable to conjecture the following regarding the \mathbb{A}^1 -fundamental group of an \mathbb{A}^1 -connected isotropic group.

Conjecture 10. Let G be an \mathbb{A}^1 -connected, isotropic, semisimple, almost simple, simply connected algebraic group over k , split by a finite separable field extension F/k .

- (1) If G is not of symplectic type, then the universal $\mathbf{K}_2^{\mathbb{M}}$ -torsor on G constructed by Brylinski-Deligne [8] gives the universal covering of G in the sense of \mathbb{A}^1 -homotopy theory.
- (2) If G is of symplectic type, then there exists a $\mathbf{K}_2^{\mathbb{M}^{\text{W}}}$ -torsor on G giving the \mathbb{A}^1 -universal covering of G .
- (3) More precisely, suppose that the relative root system of G admits a coroot α , which becomes a coroot of small length in the root system of G_F . Then the inclusion $S_\alpha \subset G$ induces an isomorphism of $\mathbf{K}_2^{\mathbb{M}^{\text{W}}}$ (respectively, $\mathbf{K}_2^{\mathbb{M}}$) with $\pi_1^{\mathbb{A}^1}(G)$, if G is of symplectic type (respectively, not of symplectic type).

Remark 11. The study of central extensions of a (locally compact) topological group and its relationship with some version of fundamental group has a long history, the works of Steinberg [28], Moore [20] and Matsumoto [19] being some of the major milestones. One says that the *topological fundamental group* $\pi_1(G)$ of a locally compact topological group G exists if the endofunctor $\text{CExt}(G, -)$ on the category of abelian groups associating with every abelian group A the abelian group of topological central extensions of G by A under the operation of Baer sum is representable. The group representing $\text{CExt}(G, -)$ is then defined to be $\pi_1(G)$;

see [20] for a systematic study and more results. Let G be the group of k -rational points of a split, semisimple, almost simple, simply connected algebraic group over a locally compact nondiscrete field k (for example, a local field). Then $\pi_1(G)$ exists and the explicit description of the universal central extension of G by generators and relations was given by Steinberg in [28]. In [20, p. 194], Moore constructed the 2-cocycle representing Steinberg's universal central extension. Furthermore, with the notation of Theorem 6, it was shown in [20, 8.1-8.4] that for a long root α in the Dynkin diagram of G , the inclusion $S_\alpha \subset G$ induces a surjective homomorphism

$$\pi_1(S_\alpha) \twoheadrightarrow \pi_1(G).$$

Moore also showed that $\pi_1(G)$ is a quotient of $\mathbf{K}_2^{\text{MW}}(k)$ or $\mathbf{K}_2^{\text{M}}(k)$, depending upon whether G is of symplectic type or otherwise. These results were shortly improved by Matsumoto (see Remark 2) by constructing certain explicit central extensions over an infinite base field. Notice the striking analogy with Theorem 6, which shows that the \mathbb{A}^1 -fundamental group $\pi_1^{\mathbb{A}^1}(G)$ is indeed the correct analogue of $\pi_1(G)$ in the algebro-geometric/sheaf-theoretic realm.

We end the introduction by mentioning some direct applications of our main results. An immediate consequence of Theorem 1 is the computation of the first cohomology group of G with coefficients in a strictly \mathbb{A}^1 -invariant sheaf (see Section 2.1 and conventions for precise definitions).

Corollary 12. *For any split, semisimple, almost simple, simply connected algebraic group G over k and a strictly \mathbb{A}^1 -invariant sheaf of abelian groups \mathbf{M} on Sm_k , we have:*

$$H_{\text{Zar}}^1(G, \mathbf{M}) \simeq \begin{cases} \mathbf{M}_{-2}(k), & \text{if } G \text{ is of symplectic type;} \\ {}_\eta \mathbf{M}_{-2}(k), & \text{if } G \text{ is not of symplectic type.} \end{cases}$$

Indeed, since any G as in Theorem 1 is \mathbb{A}^1 -connected, for any strictly \mathbb{A}^1 -invariant sheaf \mathbf{M} on Sm_k , the universal coefficient formula gives an identification

$$H_{\text{Nis}}^1(G, \mathbf{M}) = \text{Hom}_{\text{Ab}_{\mathbb{A}^1}(k)}(\pi_1^{\mathbb{A}^1}(G), \mathbf{M})$$

and Corollary 12 follows from the fact that $\text{Hom}_{\text{Ab}_{\mathbb{A}^1}(k)}(\mathbf{K}_2^{\text{MW}}, \mathbf{M}) = \mathbf{M}_{-2}(k)$ and that the change of topology morphism induces an isomorphism $H_{\text{Zar}}^1(G, \mathbf{M}) \simeq H_{\text{Nis}}^1(G, \mathbf{M})$ [23, Corollary 5.43].

Remark 13. It has been shown in the work of Esnault-Kahn-Levine-Viehweg [13, Proposition 3.20] and Brylinski-Deligne [8, Proposition 4.6] that for a cycle module \mathbf{M}_* in the sense of Rost [26], one has

$$H_{\text{Zar}}^1(G, \mathbf{M}_*) = \mathbf{M}_{*-2}(k).$$

On the other hand, S. Gille has shown in [15] if k has characteristic different from 2 and if \mathbf{W} determines the unramified Witt sheaf of quadratic forms on Sm_k , then there is a dichotomy:

$$H_{\text{Zar}}^1(G, \mathbf{W}) = \begin{cases} \mathbf{W}(k), & \text{if } G \text{ is of symplectic type;} \\ 0, & \text{if } G \text{ is not of symplectic type.} \end{cases}$$

Corollary 12 uniformly explains and generalizes these computations of Esnault-Kahn-Levine-Viehweg, Brylinski-Deligne and S. Gille.

Remark 14. Let G be split, semisimple, almost simple, simply connected over k and let G_{ad} denote the adjoint group of G , that is, the quotient of G by its center. The group G_{ad} acts on G by conjugation, thereby giving an action of G_{ad} on $\pi_1^{\mathbb{A}^1}(G)$. If G is not of symplectic type, this action is trivial. In case G is of symplectic type, this action is nontrivial and it can be shown that one has a canonical isomorphism

$$\pi_1^{\mathbb{A}^1}(G)/G_{\text{ad}} \cong \mathbf{K}_2^{\text{M}}$$

of Nisnevich sheaves of abelian groups. This will be established in Section 5.4. In fact, analyzing the method of our proof applied through appropriate realization functors, we obtain a simple proof of the following result by S. Gille [16].

Corollary 15. *For any split, semisimple, almost simple, simply connected algebraic group G over k , there is a canonical isomorphism*

$$\mathbf{K}_2^{\text{M}} \cong \mathbf{H}_1^{\text{S}}(G)$$

of Nisnevich sheaves of abelian groups with transfers in the sense of Voevodsky, where \mathbf{H}_1^{S} denotes the first Suslin homology sheaf of quasi-projective schemes over k .

Remark 16. Using our method relying on the cellular structure given by the Bruhat decomposition, it is also possible to deduce (or reprove) the following two classical results from topology (see Section 5.4):

- (1) For any split, semisimple, simply connected algebraic group G over \mathbb{R} , the topological space $G(\mathbb{R})$ is path-connected. Moreover, if G is almost simple and not of symplectic type, there is a canonical isomorphism

$$\pi_1(G) = \mathbb{Z}/2\mathbb{Z}.$$

If G is of symplectic type, there is a non-canonical isomorphism

$$\pi_1(G) = \mathbb{Z}.$$

The dichotomy in our main theorem is somehow explained by this fact. Over the reals, \mathbf{K}_2^{MW} “becomes” a free abelian group of rank one (think about $\pi_1(\mathbf{SL}_2(\mathbb{R}))$), and η becomes the multiplication by 2. Now, the quotient of a free abelian group of rank one by 2 is canonically isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

- (2) (A theorem of E. Cartan [9]) For any split, semisimple, almost simple, algebraic group G over \mathbb{C} , the topological space $G(\mathbb{C})$ is 1-connected if and only if G is simply connected in the sense of algebraic groups. Moreover, if G is simply connected in the sense of algebraic groups, then $\pi_2(G(\mathbb{C})) = 0$ and $\pi_3(G(\mathbb{C}))$ is a free abelian group of rank one.¹

Another proof of Theorem 1 (and Theorem 6) can be obtained by using the five-term exact sequence of low-degree terms associated with the Serre spectral sequence in cellular \mathbb{A}^1 -homology for the \mathbb{A}^1 -fibration $G \rightarrow G/T \rightarrow BT$, where T is a maximal k -split torus of G and by explicitly computing the cellular \mathbb{A}^1 -homology of G/T (which is \mathbb{A}^1 -weak equivalent to G/B , where B is a Borel subgroup of G). The details regarding this approach and the determination of cellular \mathbb{A}^1 -homology of the generalized flag variety G/B will be taken up in the sequel. See Remarks 5.7–5.10 for more information.

¹The statement on $\pi_3(G(\mathbb{C}))$ was only observed without proof by Cartan, see [12, page 496]; see [6] for a proof using Morse theory.

Conventions and notation.

We will freely use the standard notation in \mathbb{A}^1 -homotopy theory developed in [24] and [23]. For the sake of convenience, we fix a perfect base field k . The results about \mathbb{A}^1 -homotopy and \mathbb{A}^1 -homology sheaves we use all extend to the case of arbitrary base fields by standard results on essentially smooth base change (see [18, Appendix A]).

We will denote by Sm_k the big Nisnevich site of smooth, finite-type schemes over k . We will denote the category of simplicial Nisnevich sheaves of sets over Sm_k (called *spaces*) by $\Delta^{\text{op}}Shv_{\text{Nis}}(Sm_k)$. A morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of simplicial sheaves of sets on Sm/k is a *Nisnevich local weak equivalence* if it induces an isomorphism on every stalk for the Nisnevich topology. The category $\Delta^{\text{op}}Shv_{\text{Nis}}(Sm_k)$ admits a proper closed model structure, called the *Nisnevich local injective model structure*, in which cofibrations are monomorphisms and weak equivalences are local weak equivalences. The associated homotopy category is called the *simplicial homotopy category* and is denoted by $\mathcal{H}_s(k)$. The left Bousfield localization of the Nisnevich local injective model structure with respect to the collection of all projection morphisms $\mathcal{X} \times \mathbb{A}^1 \rightarrow \mathcal{X}$, as \mathcal{X} runs over all simplicial sheaves, is called the \mathbb{A}^1 -*model structure*. The associated homotopy category is called the \mathbb{A}^1 -*homotopy category* and is denoted by $\mathcal{H}(k)$. There is an obvious pointed analogue of this construction starting with the category whose objects are (\mathcal{X}, x) where $x : \text{Spec} k \rightarrow \mathcal{X}$ is a base-point, which gives rise to the *pointed \mathbb{A}^1 -homotopy category* $\mathcal{H}_\bullet(k)$.

For a field F , we will denote its Milnor-Witt K -theory by $\mathbf{K}_*^{\text{MW}}(F) = \bigoplus_{n \in \mathbb{Z}} \mathbf{K}_n^{\text{MW}}(F)$. It is the associative graded ring generated by a symbol η of degree -1 and symbols (u) of degree 1 for each $u \in F^\times$ with the relations $(u)(1-u) = 0$; $(uv) = (u) + (v) + \eta(u)(v)$; $\eta(u) = (u)\eta$ and $\eta h = 0$, where $h = 2 + \eta(-1)$, for all $u, v \in F^\times$. The symbol $\langle u \rangle$ will denote the element $1 + \eta(u) \in \mathbf{K}_0^{\text{MW}}(F) = \text{GW}(F)$. We will denote by \mathbf{K}_n^{M} the n th unramified Milnor K -theory sheaf and by \mathbf{K}_n^{MW} the n th unramified Milnor-Witt K -theory sheaf; see [23, Chapter 3] for a detailed exposition. Beware that we use a slightly different notation for Milnor-Witt K -theory than [23], the reasons for which will become clear later (see Convention 3.22 for the choice of this notation for mostly computational reasons). The sheaves \mathbf{K}_n^{M} and \mathbf{K}_n^{MW} are strictly \mathbb{A}^1 -invariant sheaves in sense of [23, Chapter 2] over any field. Recall that for $n \geq 1$, \mathbf{K}_n^{MW} is the free strictly \mathbb{A}^1 -invariant sheaf on $\mathbb{G}_m^{\wedge n}$ in the sense of [23, Theorem 3.37], and \mathbf{K}_n^{M} the quotient of \mathbf{K}_n^{MW} by η . This can in fact be considered here as the definition of these sheaves. We will freely use the results from [23] about the category $Ab_{\mathbb{A}^1}(k)$ of strictly \mathbb{A}^1 -invariant sheaves; particularly, the fact that it is an abelian category. We will also repeatedly use the fact that to verify that a morphism is an isomorphism, or that two morphisms are equal in $Ab_{\mathbb{A}^1}(k)$, it suffices to verify the same on sections over finitely generated separable field extensions of the base field k .

2. \mathbb{A}^1 -HOMOLOGY THEORIES

We begin with a brief recollection of the construction and basic properties of the \mathbb{A}^1 -derived category and \mathbb{A}^1 -homology sheaves developed in [21] and [23]. These are essential for the formulation of our main results and the techniques used in the proofs. We will then describe the construction of the cellular complexes and cellular \mathbb{A}^1 -homology sheaves. The results of this section will be applied to the case of cellular complexes associated with the Bruhat decomposition of a split, semisimple, simply connected group and its flag variety in Sections 3, 4 and 5.

2.1. Preliminaries on \mathbb{A}^1 -homotopy and \mathbb{A}^1 -homology sheaves.

Definition 2.1. [24] For any pointed space (\mathcal{X}, x) , the n th \mathbb{A}^1 -homotopy sheaf of (\mathcal{X}, x) , denoted by $\pi_n^{\mathbb{A}^1}(\mathcal{X}, x)$, is defined to be the Nisnevich sheaf associated with the presheaf $U \mapsto \text{Hom}_{\mathcal{H}_\bullet(k)}(S^n \wedge U_+, (\mathcal{X}, x))$.

We will suppress the base-point from the notation whenever it is clear from the context. For any pointed space \mathcal{X} , the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is a sheaf of (pointed) sets, and $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is a sheaf of groups for $n \geq 1$, and $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is a sheaf of abelian groups for $n \geq 2$.

Definition 2.2. We say that a Nisnevich sheaf \mathcal{F} on Sm_k of abelian groups is \mathbb{A}^1 -invariant if the projection map $U \times \mathbb{A}^1 \rightarrow U$ induces a bijection

$$\mathcal{F}(U) \rightarrow \mathcal{F}(U \times \mathbb{A}^1),$$

for every $U \in Sm_k$. We say that \mathcal{F} is *strictly* \mathbb{A}^1 -invariant if for every integer $i \geq 0$, the projection map $U \times \mathbb{A}^1 \rightarrow U$ induces a bijection

$$H_{\text{Nis}}^i(U, \mathcal{F}) \rightarrow H_{\text{Nis}}^i(U \times \mathbb{A}^1, \mathcal{F}),$$

for every $U \in Sm_k$.

One of the main results of [23] is that for any pointed space \mathcal{X} (over a perfect base field), the sheaf $\pi_n^{\mathbb{A}^1}(\mathcal{X})$ is strictly \mathbb{A}^1 -invariant for $n \geq 2$. The same is true for $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ if it happens to be a sheaf of abelian groups. Observe that this is the case if \mathcal{X} is a group in the pointed \mathbb{A}^1 -homotopy category. We refer the reader to [23] for more details on the \mathbb{A}^1 -homotopy sheaves.

Notation 2.3. We will denote by $Ab(k)$ the abelian category of Nisnevich sheaves of abelian groups on Sm_k . We will denote the category of strictly \mathbb{A}^1 -invariant sheaves on Sm_k by $Ab_{\mathbb{A}^1}(k)$.

The category of strictly \mathbb{A}^1 -invariant sheaves on Sm_k happens to be an abelian category. This assertion is a consequence of the fact that the category of strictly \mathbb{A}^1 -invariant sheaves on Sm_k can be identified as the heart of the homological t -structure on the \mathbb{A}^1 -derived category [21, Lemma 6.2.11].

We will use homological conventions while working with complexes of Nisnevich sheaves of abelian groups on Sm_k . Let $Ch_{\geq 0}(Ab(k))$ denote the category of chain complexes C_* of objects in $Ab(k)$ (with differentials of degree -1) such that $C_n = 0$, for all $n < 0$. Recall that the normalized chain complex functor

$$C_* : \Delta^{\text{op}} Shv_{\text{Nis}}(Sm_k) \rightarrow Ch_{\geq 0}(Ab(k))$$

admits a right adjoint called the Eilenberg-MacLane functor

$$K : Ch_{\geq 0}(Ab(k)) \rightarrow \Delta^{\text{op}} Shv_{\text{Nis}}(Sm_k).$$

For every $\mathbf{M} \in Ab(k)$, we set $K(\mathbf{M}, n) := K(\mathbf{M}[n])$. It is a fact that the normalized chain complex functor induces a functor

$$C_* : \mathcal{H}_s(k) \rightarrow D(Ab(k))$$

which on \mathbb{A}^1 -localization yields a functor

$$C_*^{\mathbb{A}^1} : \mathcal{H}(k) \rightarrow D_{\mathbb{A}^1}(k),$$

where $D_{\mathbb{A}^1}(k)$ is the full subcategory of $D(\text{Ab}(k))$ consisting of \mathbb{A}^1 -local complexes. The inclusion of $D_{\mathbb{A}^1}(k)$ into $D(\text{Ab}(k))$ admits a left adjoint, called the \mathbb{A}^1 -localization functor

$$L_{\mathbb{A}^1} : D(\text{Ab}(k)) \rightarrow D_{\mathbb{A}^1}(k).$$

Thus for a space $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\text{Sm}_k)$, one has $C_*^{\mathbb{A}^1}(\mathcal{X}) = L_{\mathbb{A}^1}(C_*(\mathcal{X}))$. If \mathcal{X} is a pointed space, the reduced chain complex $\tilde{C}_*(\mathcal{X})$ is defined to be the kernel of the canonical morphism $C_*(\mathcal{X}) \rightarrow \mathbb{Z}$, which is in fact a direct summand of $C_*(\mathcal{X})$. Indeed, the inclusion of the point induces a canonical isomorphism $C_*(\mathcal{X}) = \mathbb{Z} \oplus \tilde{C}_*(\mathcal{X})$. In the same way, the reduced \mathbb{A}^1 -chain complex of \mathcal{X} is $\tilde{C}_*^{\mathbb{A}^1}(\mathcal{X}) = L_{\mathbb{A}^1}(\tilde{C}_*(\mathcal{X}))$ and we continue to have a canonical isomorphism $C_*^{\mathbb{A}^1}(\mathcal{X}) = \mathbb{Z} \oplus \tilde{C}_*^{\mathbb{A}^1}(\mathcal{X})$.

Definition 2.4. For any $\mathcal{X} \in \Delta^{\text{op}}\text{Shv}_{\text{Nis}}(\text{Sm}_k)$ and $n \in \mathbb{Z}$, the n th \mathbb{A}^1 -homology sheaf of \mathcal{X} is defined to be the n th homology sheaf of the \mathbb{A}^1 -chain complex $C_*^{\mathbb{A}^1}(\mathcal{X})$. If \mathcal{X} is pointed, then we define the n th reduced \mathbb{A}^1 -homology sheaf of \mathcal{X} by $\tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\mathcal{X}) := \mathbf{H}_n^{\mathbb{A}^1}(\tilde{C}_*^{\mathbb{A}^1}(\mathcal{X}))$.

Since $\mathbf{H}_0^{\mathbb{A}^1}(\text{Spec } k) \simeq \mathbb{Z}$ and $\mathbf{H}_n^{\mathbb{A}^1}(\text{Spec } k) = 0$, for $n \neq 0$, we have an isomorphism

$$\mathbf{H}_*^{\mathbb{A}^1}(\mathcal{X}) \simeq \mathbb{Z} \oplus \tilde{\mathbf{H}}_*^{\mathbb{A}^1}(\mathcal{X})$$

of graded abelian sheaves. Since the \mathbb{A}^1 -localization functor commutes with the simplicial suspension functor in $D(\text{Ab}(k))$, for any pointed space \mathcal{X} and any $n \in \mathbb{Z}$, we have a canonical isomorphism

$$\tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\mathcal{X}) \simeq \tilde{\mathbf{H}}_{n+1}^{\mathbb{A}^1}(S^1 \wedge \mathcal{X}).$$

As a consequence of the \mathbb{A}^1 -connectivity theorem [23, Theorem 6.22], one knows that for every space \mathcal{X} and every integer n , the \mathbb{A}^1 -homology sheaves $\mathbf{H}_n^{\mathbb{A}^1}(\mathcal{X})$ are strictly \mathbb{A}^1 -invariant sheaves that vanish if $n < 0$.

Notation 2.5. For any Nisnevich sheaf of sets \mathcal{F} on Sm_k , we write

$$\mathbf{Z}_{\mathbb{A}^1}[\mathcal{F}] := \mathbf{H}_0^{\mathbb{A}^1}(\mathcal{F}).$$

For a pointed Nisnevich sheaf of sets \mathcal{F} on Sm_k , we write

$$\mathbf{Z}_{\mathbb{A}^1}(\mathcal{F}) := \tilde{\mathbf{H}}_0^{\mathbb{A}^1}(\mathcal{F}).$$

The sheaf $\mathbf{Z}_{\mathbb{A}^1}[\mathcal{F}]$ is the *free strictly \mathbb{A}^1 -invariant sheaf on \mathcal{F}* in the sense that we have a canonical bijection

$$\text{Hom}_{\text{Ab}(k)}(\mathbf{Z}_{\mathbb{A}^1}[\mathcal{F}], \mathbf{M}) \xrightarrow{\sim} \text{Hom}_{\text{Shv}_{\text{Nis}}(\text{Sm}_k)}(\mathcal{F}, \mathbf{M})$$

for every $\mathbf{M} \in \text{Ab}(k)$. The obvious variant holds for the *free strictly \mathbb{A}^1 -invariant sheaf on the pointed sheaf of sets \mathcal{F}* .

Remark 2.6. With the above notations, one has the following result (see [23]): the canonical morphism $\mathbb{Z}((\mathbb{G}_m)^{\wedge n}) \rightarrow \mathbf{K}_n^{\text{MW}}$ induces an isomorphism

$$\mathbf{Z}_{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge n}) \cong \mathbf{K}_n^{\text{MW}}.$$

As already noted above, the reader who is not acquainted with the sheaves \mathbf{K}_n^{MW} may consider this as a definition of the sheaf \mathbf{K}_n^{MW} .

If \mathbf{M} is strictly \mathbb{A}^1 -invariant, then $K(\mathbf{M}, n)$ is \mathbb{A}^1 -local for every integer $n \geq 0$ and the Dold-Kan correspondence yields a bijection

$$H_{\text{Nis}}^n(\mathcal{X}, \mathbf{M}) = [\mathcal{X}, K(\mathbf{M}, n)]_s \xrightarrow{\sim} \text{Hom}_{D(\text{Ab}(k))}(C_*^{\mathbb{A}^1}(\mathcal{X}), \mathbf{M}[n]).$$

Moreover, if \mathcal{X} is \mathbb{A}^1 - n -connected, then by the strong \mathbb{A}^1 -Hurewicz theorem [23, Theorem 6.57] we have $\mathbf{H}_i^{\mathbb{A}^1}(\mathcal{X}) = 0$ for $i \leq n$ and

$$H_{\text{Nis}}^{n+1}(\mathcal{X}, \mathbf{M}) \simeq \text{Hom}_{\text{Ab}_{\mathbb{A}^1}(k)}(\mathbf{H}_{n+1}^{\mathbb{A}^1}(\mathcal{X}), \mathbf{M}) \simeq \text{Hom}_{\text{Ab}_{\mathbb{A}^1}(k)}(\pi_{n+1}^{\mathbb{A}^1}(\mathcal{X}), \mathbf{M}).$$

2.2. Cellular structures and orientations.

Definition 2.7. Let k be a field and let $X \in \text{Sm}_k$ be an irreducible smooth k -scheme of Krull dimension n . A *strict cellular structure* on X consists of an increasing filtration

$$\emptyset = \Omega_{-1} \subset \Omega_0(X) \subset \Omega_1(X) \subset \cdots \subset \Omega_n(X) = X$$

by open subschemes such that for each $i \in \{0, \dots, n\}$ the reduced induced closed subscheme $X_i := \Omega_i(X) - \Omega_{i-1}(X)$ of $\Omega_i(X)$ is k -smooth and each of its irreducible components is isomorphic to \mathbb{A}^{n-i} . We say that X is a *strictly cellular scheme* if X is endowed with a strict cellular structure. We will simply write Ω_i for $\Omega_i(X)$ whenever there is no confusion.

Example 2.8. The most basic example of a strict cellular scheme is the projective space \mathbb{P}^n . Consider the obvious increasing filtration of linear projective subspaces $\emptyset \subset \mathbb{P}^0 \subset \cdots \subset \mathbb{P}^i \subset \cdots \subset \mathbb{P}^n$ corresponding to the canonical flag of subspaces of \mathbb{A}^{n+1} and set $\Omega_i = \mathbb{P}^n - \mathbb{P}^{n-i-1}$. Since $\Omega_i - \Omega_{i-1} = \mathbb{P}^{n-i} - \mathbb{P}^{n-i-1} \simeq \mathbb{A}^{n-i}$, the sequence of open subschemes $\emptyset = \Omega_{-1} \subset \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_n = \mathbb{P}^n$ gives a strict cellular structure on \mathbb{P}^n .

Another classical example of (strict) cellular structure arises from the Bruhat decomposition on the flag variety G/B of a split semisimple algebraic group G . We now generalize this notion of strict cellular schemes by allowing more general schemes as “cells” in order to get a “cellular structure” on G itself.

Definition 2.9. We say that $X \in \text{Sm}_k$ is *cohomologically trivial* if $H_{\text{Nis}}^n(X, \mathbf{M}) = 0$, for every $n \geq 1$ and for every strictly \mathbb{A}^1 -invariant sheaf $\mathbf{M} \in \text{Ab}_{\mathbb{A}^1}(k)$.

Remark 2.10. Clearly, X is cohomologically trivial if and only if the sheaf $\mathbf{Z}_{\mathbb{A}^1}[X] = \mathbf{H}_0^{\mathbb{A}^1}(X)$ is a projective object of the abelian category $\text{Ab}_{\mathbb{A}^1}(k)$. Examples of cohomologically trivial schemes include $\text{Spec } L$, where L is a finite separable field extension of k , \mathbb{A}^1 , \mathbb{G}_m ; and more generally, open subschemes of \mathbb{A}^1 . It is easy to show that the product of two cohomologically trivial smooth k -schemes is again cohomologically trivial.

We can now formulate a slightly more general definition of a cellular structure:

Definition 2.11. Let k be a field and let $X \in \text{Sm}_k$ be a smooth k -scheme. A *cellular structure* on X consists of an increasing filtration

$$\emptyset = \Omega_{-1} \subsetneq \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_s = X$$

by open subschemes such that for each $i \in \{0, \dots, s\}$, the reduced induced closed subscheme $X_i := \Omega_i - \Omega_{i-1}$ of Ω_i is k -smooth, affine, everywhere of codimension i and cohomologically trivial. We call X a *cellular scheme* if X is endowed with a cellular structure.

Remark 2.12. Let the notation be as in Definition 2.11.

- (1) In the decomposition $X_i = \Omega_i - \Omega_{i-1} = \coprod_{j \in J_i} X_{ij}$ of X_i in irreducible components, each k -scheme X_{ij} is clearly affine, smooth, cohomologically trivial and of codimension i in Ω_i . Observe, however, that here s need not be the Krull dimension of X .
- (2) Clearly, every strict cellular structure is a cellular structure. These definitions are rather classical; here are some comments. One might be tempted, given a cellular structure on X as above, to define the i -th skeleton of that structure as the closed subset $sk_i(X) = \coprod_{r \in \{i, \dots, s\}, j \in J_r} Y_{rj}$ (disjoint) union of the strata with higher codimension. The problem is that in general, these skeletons are not k -smooth (for example, consider Schubert cells in a flag variety), and there is no obvious way to express $sk_i(X)$ as obtained from $sk_{i-1}(X)$ by “attaching” i -cells.

Contrary to the first impression, the above definition relying on an increasing filtration $\emptyset = \Omega_{-1} \subset \Omega_0 \subset \Omega_1 \subset \dots \subset \Omega_d = X$ of open subsets is closer to the notion of CW -structure on a topological space X . If a topological space X is endowed with a CW -structure, with i -skeleton $sk_i(X) \subset X$, and where $sk_i(X)$ is obtained from $sk_{i-1}(X)$ by attaching i -cells, one may choose a (small) open neighborhood U_i of $sk_i(X)$ such that $sk_i(X) \subset U_i$ is an homotopy equivalence and so that $U_i - U_{i-1}$ is a disjoint sum of contractible spaces indexed by the number of i -cells. The quotient (or rather the “pair”) U_i/U_{i-1} is then homotopy equivalent to a wedge of pointed spaces homotopy equivalent to i -spheres. In the approach taken in Definition 2.11, the quotient Ω_i/Ω_{i-1} is \mathbb{A}^1 -weakly equivalent to the Thom space of the normal bundle ν_i of the inclusion of $Y_i \subset \Omega_i$, by the \mathbb{A}^1 -homotopy purity theorem [24, Theorem 2.23, page 115]. We will see below that the normal bundles ν_i are always trivial as the Y_i are cohomologically trivial and affine. Thus, the quotient Ω_i/Ω_{i-1} is non canonically \mathbb{A}^1 -weakly equivalent to the space $(\mathbb{A}^{d-i}/(\mathbb{A}^{d-i} - \{0\})) \wedge (Y_{ij} +)$.

These \mathbb{A}^1 -equivalences can be made canonical using the notion of an oriented cellular structure introduced below (see Definition 2.17), making this approach quite close to the topological one!

- (3) The main example of a cellular structure that is not strict cellular is given by the Bruhat decomposition of a split reductive group G , in which the strata Y_{ij} are the Bruhat cells BwB with $\text{length}(w) = \ell - i$, where ℓ is the length of the longest word in the Weyl group of G . See Section 3 below for the details.

Lemma 2.13. *Let X be a cohomologically trivial smooth affine variety over a perfect field k . Then every vector bundle of rank $n \geq 1$ on X is trivial.*

Proof. Since X is smooth affine, by a result proved in [23] for $r \geq 3$ and in [1] in general, the set of isomorphism classes of rank n vector bundles on X can be identified with the set $\text{Hom}_{\mathcal{H}(k)}(X, BGL_n)$. Let ξ be a rank n vector bundle on X and let γ denote the corresponding element in $\text{Hom}_{\mathcal{H}(k)}(X, BGL_n)$. By a Postnikov tower argument, obstructions to lifting through the successive Postnikov fibers of BGL_n lie in Nisnevich cohomology groups of X with coefficients in the higher \mathbb{A}^1 -homotopy sheaves of BGL_n . These cohomology groups vanish by cohomological triviality of X since the higher \mathbb{A}^1 -homotopy sheaves are strictly \mathbb{A}^1 -invariant by [23] and the lemma follows. \square

We now briefly recall some facts about the notion of an orientation of a vector bundle on a smooth k -scheme.

Definition 2.14. Let $r \geq 1$, and let ξ be a rank r vector bundle over a smooth k -scheme X . We say that ξ is *strictly orientable* if the line bundle $\Lambda^r(\xi)$ is trivial. A *strict orientation* of

ξ is then a choice of an isomorphism $\theta : \mathbb{A}_X^1 \cong \Lambda^r(\xi)$, where \mathbb{A}_X^1 is the trivial vector bundle of rank 1 over X . We will denote by $\mathcal{O}r^{st}(\xi)$ the set of strict orientations of ξ .

For instance, a trivialization of the vector bundle ξ on X induces a strict orientation of ξ . Given $\theta \in \mathcal{O}r^{st}(\xi)$ and a unit $\alpha \in \mathcal{O}(X)^\times$, the product $\alpha \cdot \theta$ is clearly a strict orientation of ξ as well. Thus, the group of units $\mathcal{O}(X)^\times$ acts naturally on the set $\mathcal{O}r^{st}(\xi)$.

Definition 2.15. Let ξ be a strictly orientable rank r vector bundle over the smooth k -scheme X . We denote by $\mathcal{O}r(\xi)$ the quotient of the set of strict orientations $\mathcal{O}r^{st}(\xi)$ of ξ by the action of the subgroup $\mathcal{O}(X)^{\times 2}$ of squares in $\mathcal{O}(X)^\times$.

Remark 2.16. Let the notation be as above.

- (1) If a strict orientation θ of a vector bundle ξ exists, then the set $\mathcal{O}r^{st}(\xi)$ is canonically isomorphic to $\mathcal{O}(X)^\times$ through the action. Hence, the set $\mathcal{O}r(\xi)$ is isomorphic to the $\mathbb{Z}/2$ -vector space $\mathcal{O}(X)^\times / (\mathcal{O}(X)^{\times 2})$ of units in X modulo the squares.
- (2) In general, if ξ is not strictly orientable, we may still define the sheaf of orientations of ξ as the quotient sheaf of sets in the category of sheaves of sets in the Nisnevich topology on k -smooth schemes of the k -smooth scheme $\Lambda^r(\xi)^\times$ by the action of the subsheaf $(\mathbb{G}_m)^{(2)}$ of squares in \mathbb{G}_m . This is a sheaf of $\mathbb{Z}/2$ -vector spaces over X and an orientation of ξ is a section of this sheaf over X . There are also other notions of orientability of vector bundles, see for instance [23, Def. 4.3] where an orientation is defined to be an isomorphism of $\Lambda^r(\xi)$ with the square of a line bundle. The former definition is clearly more general.
- (3) Observe that a vector bundle ξ of rank r over a smooth cohomologically trivial k -scheme X is automatically strictly orientable. Indeed, the rank 1 vector bundle $\Lambda^r(\xi)$ corresponds up to isomorphism to an element of the group $H^1(X; \mathbb{G}_m)$, which is trivial. If X is moreover affine, then Lemma 2.13 implies that such a vector bundle is trivial.

In this article, we use only one notion of orientation, namely, the one of *strict orientation* given by Definition 2.14. We will often drop the adjective *strict* for the sake of brevity.

Definition 2.17. Let $X \in Sm_k$ be a smooth k -scheme. An *oriented cellular structure* on X consists of a cellular structure on X ,

$$\emptyset = \Omega_{-1} \subsetneq \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_s = X$$

together with, for each i , an orientation o_i of the normal bundle ν_i of the closed immersion $X_i := \Omega_i - \Omega_{i-1} \subset \Omega_i$. One also says that the choices of the o_i 's is an orientation of the given cellular structure, and that X is an *oriented cellular (smooth) k -scheme*.

Definition 2.18. Let X and Y be smooth k -schemes, both endowed with a cellular structure. A morphism of k -schemes $f : X \rightarrow Y$ is said to be cellular if it preserves the open filtrations, that is, $f(\Omega_i(X)) \subset \Omega_i(Y)$ for every i .

For each i , we denote by X_i the reduced induced closed subscheme of $\Omega_i(X)$ on the closed complement $\Omega_i(X) - \Omega_{i-1}(X)$, and respectively by Y_i , the corresponding closed subschemes of $\Omega_i(Y)$. Since a cellular morphism $f : X \rightarrow Y$ preserves the open filtrations, f induces a morphism of smooth k -schemes $f : X_i \rightarrow Y_i$ for each i , and also a morphism on the induced normal bundles $\nu_f : \nu_{X_i} \rightarrow (f|_{X_i})^*(\nu_{Y_i})$. If the cellular structures of X and Y are oriented, then a cellular morphism f is said to be *oriented* if for each n the induced morphism on normal bundles $\nu_f : \nu_{X_n} \rightarrow (f|_{X_n})^*(\nu_{Y_n})$ preserves the orientations (which implies in particular that $\nu_f : \nu_{X_n} \rightarrow (f|_{X_n})^*(\nu_{Y_n})$ is an isomorphism).

Example 2.19. We consider the n -dimensional projective space \mathbb{P}^n over k . Consider the obvious increasing filtration of linear projective subspaces

$$\emptyset \subset \mathbb{P}^0 \subset \dots \subset \mathbb{P}^i \subset \dots \subset \mathbb{P}^n$$

corresponding to the canonical flag of subspaces of \mathbb{A}^{n+1} and set $\Omega_i = \mathbb{P}^n - \mathbb{P}^{n-i-1}$. If X_0, \dots, X_n denote the usual homogeneous coordinates on \mathbb{P}^n ; or, in other words, the canonical generating sections of $\mathcal{O}(1)$, then clearly $\Omega_i = \bigcup_{j=0}^i D_+(X_j)$. Thus $\Omega_i - \Omega_{i-1} = \mathbb{P}^{n-i} - \mathbb{P}^{n-i-1}$ is the zero locus on Ω_i of the rational functions $\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}$. Since this sequence is a regular sequence, the normal bundle ν_i is trivialized by this sequence, thereby inducing an orientation. This data forms the canonical oriented cellular structure on \mathbb{P}^n .

The following lemma allows us to generate more examples of oriented cellular schemes.

Lemma 2.20. *Let T be a split k -torus, $X \in \text{Sm}_k$ be an oriented cellular scheme, and let $\pi : Y \rightarrow X$ be a T -torsor over X . Let*

$$\emptyset = \Omega_{-1} \subsetneq \Omega_0 \subsetneq \Omega_1 \subsetneq \dots \subsetneq \Omega_s = X$$

be the open filtration of X defining the cellular structure. Define an increasing filtration by open subsets on Y by pulling back the Ω_i : $\tilde{\Omega}_i := \pi^{-1}(\Omega_i)$. Then the sequence

$$\emptyset = \tilde{\Omega}_{-1} \subsetneq \tilde{\Omega}_0 \subsetneq \tilde{\Omega}_1 \subsetneq \dots \subsetneq \tilde{\Omega}_s = Y$$

is a cellular structure on Y . Moreover, with the obvious notation, the normal bundle of $Y_i \subset \tilde{\Omega}_i$ is the pull-back of the normal bundle of $X_i \subset \Omega_i$ so that the oriented cellular structure on X induces a canonical oriented cellular structure on Y .

Proof. The smooth k -scheme $\tilde{\Omega}_i - \tilde{\Omega}_{i-1}$ is clearly affine and cohomologically trivial, as it is a (trivial) \mathbb{G}_m -torsor on Ω_i . The rest easily follows from this. \square

Example 2.21. In this way we get a canonical oriented cell structure on $\mathbb{A}^{n+1} - \{0\}$ using the usual \mathbb{G}_m -torsor $\mathbb{A}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$.

Remark 2.22. By Lemma 2.13, the normal bundles ν_i in the definition above of a oriented cellular structure on a smooth k -scheme are in fact always trivial. The main point is not the orientability, but the choice of an orientation. In all our cases, for instance the oriented cell structure of \mathbb{P}^n described above, the triviality is always proved directly without invoking Lemma 2.13.

We next record the following particular case of the Thom isomorphism theorem for oriented bundles.

Lemma 2.23. *Let ξ be a trivial vector bundle of rank r over a smooth k -scheme X , endowed with an orientation $o \in \mathcal{O}r(\xi)$. Then there exists an isomorphism $\phi : \xi \xrightarrow{\cong} \mathbb{A}_X^r$ of vector bundles over X , which preserves the orientations. Moreover, if X is cohomologically trivial, then the induced pointed \mathbb{A}^1 -homotopy class (which is an \mathbb{A}^1 -homotopy equivalence)*

$$\text{Th}(\xi) \rightarrow \mathbb{A}^r / (\mathbb{A}^r - \{0\}) \wedge (X_+)$$

only depends on $o \in \mathcal{O}r(\xi)$.

Proof. Let $\psi : \xi \xrightarrow{\sim} \mathbb{A}_X^r$ be an isomorphism of vector bundles over X . Taking the top exterior power Λ^r and using the strict orientations on both sides yields an isomorphism $\mathbb{A}_X^1 \xrightarrow{\sim} \mathbb{A}_X^1$ of line bundles over X , which corresponds to an invertible regular function α on X . Multiplying ψ on the first coordinate by α^{-1} yields now an isomorphism $\phi : \xi \xrightarrow{\sim} \mathbb{A}_X^r$ which preserves the two strict orientations.

Now given another isomorphism $\phi' : \xi \xrightarrow{\sim} \mathbb{A}_X^r$ taking θ to the strict orientation of \mathbb{A}_X^r multiplied by a square α^2 of a unit $\alpha \in \mathcal{O}(X)^\times$. The automorphism $\Theta = \phi' \circ \phi^{-1} : \mathbb{A}_X^r \xrightarrow{\sim} \mathbb{A}_X^r$ preserves the strict orientations up to multiplication by α^2 ; that is to say an element of $\text{Hom}_k(X, \mathbf{GL}_r)$ whose determinant is α^2 . Let $\Theta' \in \text{Hom}_k(X, \mathbf{GL}_r)$ be the diagonal matrix with u^2 on the first entry and 1 on the others. Then $\Theta = \Theta' \circ \Theta''$ with $\Theta'' \in \text{Hom}_k(X, \mathbf{SL}_r)$. As X is cohomologically trivial and as \mathbf{SL}_r is \mathbb{A}^1 -connected, any morphism of k -schemes $X \rightarrow \mathbf{SL}_r$ is \mathbb{A}^1 -homotopically trivial (use again the \mathbb{A}^1 -Postnikov tower). This implies that the induced pointed morphism on Thom spaces

$$Th(\Theta) : \mathbb{A}^r/(\mathbb{A}^r - \{0\}) \wedge (X_+) \rightarrow \mathbb{A}^r/(\mathbb{A}^r - \{0\}) \wedge (X_+)$$

induced by Θ is \mathbb{A}^1 -homotopic to the one induced by Θ' . As $\mathbb{A}^r/(\mathbb{A}^r - \{0\})$ is isomorphic to $(\mathbb{A}^1/(\mathbb{A}^1 - \{0\})) \wedge (\mathbb{A}^{r-1}/(\mathbb{A}^{r-1} - \{0\}))$, the induced \mathbb{A}^1 -homotopy class

$$Th(\Theta') : \mathbb{A}^r/(\mathbb{A}^r - \{0\}) \wedge (X_+) \rightarrow \mathbb{A}^r/(\mathbb{A}^r - \{0\}) \wedge (X_+)$$

is clearly the smash product of the identity of $\mathbb{A}^{r-1}/(\mathbb{A}^{r-1} - \{0\})$ with the morphism $\mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \wedge (X_+) \xrightarrow{\sim} \mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \wedge (X_+)$ induced by α^2 . The latter morphism is \mathbb{A}^1 -homotopic to the identity: this follows from the standard argument that $\mathbb{P}^1 \wedge (X_+) \rightarrow \mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \wedge (X_+)$ is an \mathbb{A}^1 -weak equivalence, and that the automorphism $\mathbb{P}^1 \wedge (X_+) \xrightarrow{\sim} \mathbb{P}^1 \wedge (X_+)$ given by $[x, y] \mapsto [\alpha^2 x, y]$ is equal to $[x, y] \mapsto [\alpha x, \alpha^{-1} y]$, which is induced now by an element of $\mathbf{SL}_2(X)$, which is \mathbb{A}^1 -nullhomotopic. \square

Remark 2.24.

- (1) For an oriented cellular scheme $X \in Sm_k$ and a T -torsor $Y \rightarrow X$ over X , with T a split k -torus, the morphism $Y \rightarrow X$ is clearly an oriented cellular morphism for the oriented cellular structure on Y described in Lemma 2.20.
- (2) Using the notion of an oriented cellular morphism described in Definition 2.18, we may also extend the notion of an oriented cellular structure to ind-smooth k -schemes in an obvious way, by requiring that each object has a given (oriented) cellular structure and each morphism are (oriented) cellular. For instance, the closed immersions $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ (defined by $X_n = 0$) is an oriented cellular morphism; observe that Ω_{n-1} already contains \mathbb{P}^{n-1} , and the intersection of the Ω_i 's, for $i \leq (n-1)$, with \mathbb{P}^{n-1} are exactly the Ω_i 's of \mathbb{P}^{n-1} . In this way, we get an oriented cell structure on \mathbb{P}^∞ .
- (3) We may also adapt the notion of an oriented cellular structure to simplicial smooth k -schemes, by requiring that each term of the simplicial smooth k -scheme are given a (oriented) cellular structure and each face morphism is (oriented) cellular. For example, the classifying space BT of any split torus T has a canonical oriented cellular structure.

2.3. Cellular \mathbb{A}^1 -chain complexes.

We are now set to introduce the notions of *cellular \mathbb{A}^1 -chain complex*, *oriented cellular \mathbb{A}^1 -chain complex* and the corresponding *cellular \mathbb{A}^1 -homology*. Let X be a smooth k -scheme

with a cellular structure. Let

$$\emptyset = \Omega_{-1} \subset \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_s = X$$

be the corresponding filtration by open subschemes as in Definition 2.11. Let $\Omega_i/\Omega_{i-1} = \coprod_{j \in J_i} Y_{ij}$ be the decomposition of $\Omega_i - \Omega_{i-1}$ in irreducible components. We can now apply the motivic homotopy purity theorem [24, Theorem 2.23, page 115] to get a canonical pointed \mathbb{A}^1 -weak equivalence

$$(2.1) \quad \Omega_i/\Omega_{i-1} \simeq_{\mathbb{A}^1} Th(\nu_i)$$

with ν_i the normal bundle of the closed immersion $X_i \hookrightarrow \Omega_i$ of smooth k -schemes. Consider the cofibration sequence

$$\Omega_{i-1} \rightarrow \Omega_i \rightarrow \Omega_i/\Omega_{i-1}.$$

In the corresponding long exact sequence of reduced \mathbb{A}^1 -homology sheaves

$$\cdots \rightarrow \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\Omega_{i-1}) \rightarrow \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\Omega_i) \rightarrow \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\Omega_i/\Omega_{i-1}) \xrightarrow{\delta} \tilde{\mathbf{H}}_{n-1}^{\mathbb{A}^1}(\Omega_{i-1}) \rightarrow \cdots$$

we can use the previous identification

$$\tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\Omega_n/\Omega_{n-1}) \simeq \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(Th(\nu_n))$$

Define ∂_n to be the composite

$$\tilde{\mathbf{H}}_n^{\mathbb{A}^1}(Th(\nu_n)) \simeq \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\Omega_n/\Omega_{n-1}) \xrightarrow{\delta} \tilde{\mathbf{H}}_{n-1}^{\mathbb{A}^1}(\Omega_{n-1}) \rightarrow \tilde{\mathbf{H}}_{n-1}^{\mathbb{A}^1}(\Omega_{n-1}/\Omega_{n-2}) \simeq \tilde{\mathbf{H}}_{n-1}^{\mathbb{A}^1}(Th(\nu_{n-1}))$$

where the map $\tilde{\mathbf{H}}_{n-1}^{\mathbb{A}^1}(\Omega_{n-1}) \rightarrow \tilde{\mathbf{H}}_{n-1}^{\mathbb{A}^1}(\Omega_{n-1}/\Omega_{n-2})$ is induced by the previous cofibration sequence

$$\Omega_{n-2} \rightarrow \Omega_{n-1} \rightarrow \Omega_{n-1}/\Omega_{n-2}.$$

It is easy to see that $\partial_{n-1} \circ \partial_n = 0$, for every n , thus we get a chain complex (in the category of strictly \mathbb{A}^1 -invariant sheaves)

$$C_*^{\text{cell}}(X) := (\tilde{\mathbf{H}}_n^{\mathbb{A}^1}(Th(\nu_n)), \partial_n)$$

The chain complex $C_*^{\text{cell}}(X)$ is called the *cellular chain complex* associated to the cellular structure on X , and its homology sheaves $H_n(C_*^{\text{cell}}(X))$ are called the *cellular \mathbb{A}^1 -homology sheaves* of X .

If we choose an orientation of the given cellular structure on X , the chosen orientations (and the canonical \mathbb{A}^1 -weak equivalence $\mathbb{A}^i/(\mathbb{A}^i - \{0\}) = S^i \wedge \mathbb{G}_m^{\wedge i}$) together with the the pointed \mathbb{A}^1 -equivalence of the above Lemma define canonical pointed \mathbb{A}^1 -weak equivalence

$$(2.2) \quad \Omega_i/\Omega_{i-1} \simeq_{\mathbb{A}^1} Th(\nu_i) \simeq_{\mathbb{A}^1} S^i \wedge \mathbb{G}_m^{\wedge i} \wedge (\vee_{j \in J_i} (Y_{ij})_+)$$

Thus we can further identify, for each n :

$$\tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\Omega_n/\Omega_{n-1}) \simeq \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(Th(\nu_n)) \simeq \bigoplus_{j \in J_n} \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(S^n \wedge \mathbb{G}_m^{\wedge n} \wedge (Y_{nj})_+).$$

Using the suspension isomorphisms in \mathbb{A}^1 -homology, and the identification $\mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m^{\wedge n}) = \mathbf{K}_n^{\text{MW}}$, we get isomorphisms of strictly \mathbb{A}^1 -invariant sheaves:

$$\bigoplus_{j \in J_n} \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(S^n \wedge \mathbb{G}_m^{\wedge n} \wedge (Y_{nj})_+) \simeq \bigoplus_{j \in J_n} \tilde{\mathbf{H}}_0^{\mathbb{A}^1}(\mathbb{G}_m^{\wedge n} \wedge (Y_{nj})_+) \simeq \bigoplus_{j \in J_n} \mathbf{K}_n^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_{nj}]$$

(the tensor product being performed in the category of strictly \mathbb{A}^1 -invariant sheaves). Thus using the orientations, the cellular chain complex $C_*^{\text{cell}}(X)$ becomes isomorphic to the chain complex with terms in degree n : $\bigoplus_{j \in J_n} \mathbf{K}_n^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_{nj}]$.

Now we denote by $\tilde{\partial}_n$ the induced composite

$$\bigoplus_{j \in J_n} \mathbf{K}_n^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_{nj}] \simeq \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\text{Th}(\nu_n)) \xrightarrow{\tilde{\partial}_n} \tilde{\mathbf{H}}_{n-1}^{\mathbb{A}^1}(\text{Th}(\nu_{n-1})) \simeq \bigoplus_{j \in J_{n-1}} \mathbf{K}_{n-1}^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_{nj}]$$

Definition 2.25. The chain complex (in the category of strictly \mathbb{A}^1 -invariant sheaves)

$$\tilde{C}_*^{\text{cell}}(X) := \left(\bigoplus_{j \in J_n} \mathbf{K}_n^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_{nj}], \tilde{\partial}_n \right)_n$$

is called the *oriented cellular \mathbb{A}^1 -chain complex* associated to the oriented cellular structure on X . The n th homology sheaf of this complex will be called n -th *oriented cellular \mathbb{A}^1 -homology* of X and will be denoted by $\tilde{\mathbf{H}}_n^{\text{cell}}(X)$; using the orientations there are of course isomorphic to the cellular \mathbb{A}^1 -homology sheaves $\mathbf{H}_n^{\text{cell}}(X)$.

Remark 2.26.

- (1) It is easy to check using the previous description that each term $C_n^{\text{cell}}(X)$ is a projective object of $Ab_{\mathbb{A}^1}(k)$. Indeed, for any cohomologically trivial smooth k -scheme Y , the strictly \mathbb{A}^1 -invariant sheaf $\mathbf{Z}_{\mathbb{A}^1}[Y]$ is clearly a projective object of $Ab_{\mathbb{A}^1}(k)$. More generally, for each $n \geq 1$, the tensor product $\mathbf{K}_n^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y]$ is also a projective object of $Ab_{\mathbb{A}^1}(k)$. This can be seen by using the adjunction formula:

$$\text{Hom}_{Ab_{\mathbb{A}^1}(k)}(\mathbf{K}_n^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y], \mathbf{M}) \cong \text{Hom}_{Ab_{\mathbb{A}^1}(k)}(\mathbf{Z}_{\mathbb{A}^1}[Y], \mathbf{M}_{-n}),$$

for any $\mathbf{M} \in Ab_{\mathbb{A}^1}(k)$ and the fact that the functor $\mathbf{M} \mapsto \mathbf{M}_{-1}$ is an exact functor, which follows from the fact that \mathbb{G}_m is cohomologically trivial.

- (2) Let X be a cellular smooth k -scheme. Without the choice of orientations, the cellular \mathbb{A}^1 -chain complex $C_*^{\text{cell}}(X)$ is practically uncomputable. Its oriented version, $\tilde{C}_*^{\text{cell}}(X)$, depending on the choice of orientations is often more explicitly computable. See also the intrinsic interpretation given below in Section 2.4.
- (3) If one uses the variant of Suslin homology instead of \mathbb{A}^1 -homology, the Thom isomorphisms exist without choice of orientations, so that the cellular Suslin chain complex $C_*^{\text{Sus,cell}}(X)$ is a well defined chain complex of \mathbb{A}^1 -invariant sheaves with transfers in the sense of Voevodsky, and is defined for any cellular structure on X .

Let $f : X \rightarrow Y$ be a cellular morphism of cellular smooth k -schemes (see Definition 2.18). For each i , we denote by X_i the reduced induced closed subscheme of $\Omega_i(X)$ on the closed complement $\Omega_i(X) - \Omega_{i-1}(X)$, and in the same way Y_i , the corresponding closed subschemes of $\Omega_i(Y)$. Then, as f preserves the open filtrations, f induces a morphism of smooth k -schemes $f : X_i \rightarrow Y_i$ for each i , and also a morphism on the induced normal bundles $\nu_f : \nu_{X_i} \rightarrow (f|_{X_i})^*(\nu_{Y_i})$. It follows easily that f induces then a canonical morphism of cellular \mathbb{A}^1 -chain complexes:

$$f_*^{\text{cell}} : C_*^{\text{cell}}(X) \rightarrow C_*^{\text{cell}}(Y)$$

If the cellular structures of X and Y are oriented, the morphism f_*^{cell} induces a morphism $\tilde{f}_*^{\text{cell}} : \tilde{C}_*^{\text{cell}}(X) \rightarrow \tilde{C}_*^{\text{cell}}(Y)$ of oriented cellular \mathbb{A}^1 -chain complexes, so that in degree n , $\tilde{f}_n^{\text{cell}}$ is the morphism

$$\mathbf{K}_n^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[X_n] \cong \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\text{Th}(\nu_{X_n})) \xrightarrow{\tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\text{Th}(\nu_f))} \tilde{\mathbf{H}}_n^{\mathbb{A}^1}(\text{Th}(\nu_{Y_n})) \cong \mathbf{K}_n^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_n]$$

induced by ν_f in the obvious way.

If f is an oriented cellular morphism, that is, if for each n the induced morphism on normal bundles $\nu_f : \nu_{X_n} \rightarrow (f|_{X_n})^*(\nu_{Y_n})$ preserves the orientations, then $\tilde{f}_n^{\text{cell}}$ is just the tensor product of $\tilde{\mathbf{H}}_0^{\mathbb{A}^1}(f|_{X_n}) : \mathbf{Z}_{\mathbb{A}^1}[X_n] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[Y_n]$ with \mathbf{K}_n^{MW} .

2.4. Intrinsic interpretation in the derived category $D(\text{Ab}_{\mathbb{A}^1}(k))$.

Given a sheaf of abelian groups \mathbf{M} on Sm_k , we can construct the cohomological cellular cochain complex $C_{\text{cell}}^*(X; \mathbf{M})$ which is the (cochain) complex of abelian groups with terms $C_{\text{cell}}^n(X; \mathbf{M}) := \text{Hom}_{\text{Ab}_{\mathbb{A}^1}(k)}(C_n^{\text{cell}}(X), \mathbf{M})$ and the obvious induced differentials.

Proposition 2.27. *Let X be a cellular smooth scheme over k . For any strictly \mathbb{A}^1 -invariant sheaf \mathbf{M} on Sm_k , we have canonical isomorphisms*

$$H_{\text{Nis}}^n(X, \mathbf{M}) \xrightarrow{\sim} H^n(C_{\text{cell}}^*(X; \mathbf{M})) \xrightarrow{\sim} \text{Hom}_{D(\text{Ab}_{\mathbb{A}^1}(k))}(C_*^{\text{cell}}(X), \mathbf{M}[n]).$$

Proof. The first isomorphism follows from the cohomological spectral sequence computing $H_{\text{Nis}}^*(X, \mathbf{M})$ associated with the filtration induced by the flag of open subsets

$$\emptyset = \Omega_{-1} \subsetneq \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_s = X$$

and the cohomological triviality of the Y_{ij} , which shows that the spectral sequence collapses from E_2 page onwards, yielding the isomorphism. The second isomorphism follows from the fact that each term $C_n^{\text{cell}}(X)$ is a projective object of $\text{Ab}_{\mathbb{A}^1}(k)$, which follows again from the cohomological triviality of the Y_{ij} . \square

Remark 2.28.

- (1) Note that the proof of the first isomorphism given above exactly parallels the way in which we introduced the cellular \mathbb{A}^1 -chain complex: observe that the E_1 term of the spectral sequence considered in the proof of Proposition 2.4 is the cochain complex $C_{\text{cell}}^*(X; \mathbf{M})$ and the E_2 terms are $H_{\text{Nis}}^*(X, \mathbf{M})$.
- (2) It is known from [23] that for any strictly \mathbb{A}^1 -invariant sheaf \mathbf{M} on Sm_k and any smooth k -scheme X , the natural comparison morphism:

$$H_{\text{Zar}}^n(X, \mathbf{M}) \cong H_{\text{Nis}}^n(X, \mathbf{M})$$

is an isomorphism for any $n \geq 0$.

We first observe the following easy consequence:

Corollary 2.29. *Let X be a cellular smooth k -scheme. Then for any strictly \mathbb{A}^1 -invariant sheaf \mathbf{M} on Sm_k with $M_{-i} = 0$ for some $i \geq 0$ then, for $n \geq i$*

$$H_{\text{Nis}}^n(X, \mathbf{M}) = 0$$

Proof. Indeed, under the assumptions, the cohomological cellular cochain complex $C_{\text{cell}}^*(X; \mathbf{M})$ vanishes in degrees $\geq i$. \square

Corollary 2.30. *Let X be an \mathbb{A}^1 - $(n-1)$ -connected, $n \geq 1$, cellular smooth k -scheme. Then $\mathbf{H}_0^{\text{cell}}(X) = \mathbb{Z}$ and $\mathbf{H}_i^{\text{cell}}(X) = 0$ for $1 \leq i \leq n-1$, and there exists a canonical isomorphism*

$$\mathbf{H}_n^{\text{cell}}(X) \simeq \mathbf{H}_n^{\mathbb{A}^1}(X)$$

and an epimorphism $\mathbf{H}_{n+1}^{\mathbb{A}^1}(X) \twoheadrightarrow \mathbf{H}_{n+1}^{\text{cell}}(X)$.

Proof. If X an \mathbb{A}^1 -($n - 1$)-connected space and \mathbf{M} a strictly \mathbb{A}^1 -invariant sheaf, one has $H^0(X; \mathbf{M}) = \mathbf{M}(k)$ and $H^i(X; \mathbf{M}) = 0$ for $i \in \{1, \dots, n - 1\}$. The first equality shows that the canonical morphism $\mathbf{H}_0^{\text{cell}}(X) \rightarrow \mathbb{Z}$ is an isomorphism. If $\tilde{C}_*^{\text{cell}}(X)$ denotes the reduced cell chain complex of X (that is, the kernel of $C_*^{\text{cell}}(X) \rightarrow \mathbb{Z}$), then Proposition 2.27 easily implies that $H_i(\tilde{C}_*^{\text{cell}}(X)) = \mathbf{H}_i^{\text{cell}}(X) = 0$ for $i \in \{1, \dots, n - 1\}$. Now, since

$$H^n(C_{\text{cell}}^*(X; \mathbf{M})) = H^n(X; \mathbf{M}) = \text{Hom}_{Ab_{\mathbb{A}^1}(k)}(\mathbf{H}_n^{\mathbb{A}^1}(X), \mathbf{M})$$

for every $\mathbf{M} \in Ab_{\mathbb{A}^1}(k)$, the above vanishing gives an isomorphism $\mathbf{H}_n^{\text{cell}}(X) \simeq \mathbf{H}_n^{\mathbb{A}^1}(X)$. Now if C_* is any chain complex in any abelian category \mathcal{A} with $H_i C_* = 0$ for $i \leq n - 1$ and M an object of \mathcal{A} , then one has an exact sequence of the form

$$0 \rightarrow \text{Ext}_{\mathcal{A}}^1(H_n C_*; M) \rightarrow \text{Hom}_{D\mathcal{A}}(C_*; M[n + 1]) \rightarrow \text{Hom}_{\mathcal{A}}(H_{n+1} C_*; M) \rightarrow \text{Ext}_{\mathcal{A}}^2(H_n C_*, M).$$

Using Proposition 2.27 and the above exact sequences for $\mathcal{A} = Ab_{\mathbb{A}^1}(k)$ with $C_* = C_*^{\mathbb{A}^1}(X)$ as well as $C^{\text{cell}}(X)$ we conclude by a diagram chase that the comparison morphism

$$\text{Hom}(\mathbf{H}_{n+1}^{\text{cell}}(X), \mathbf{M}) \rightarrow \text{Hom}(\mathbf{H}_{n+1}^{\mathbb{A}^1}(X), \mathbf{M})$$

is injective for every $\mathbf{M} \in Ab_{\mathbb{A}^1}(k)$. Consequently, the induced morphism $\mathbf{H}_{n+1}^{\mathbb{A}^1}(X) \rightarrow \mathbf{H}_{n+1}^{\text{cell}}(X)$ is an epimorphism. \square

If X and Y are cellular smooth k -schemes, the product $X \times_k Y$ can be naturally endowed with a cellular structure defined by $\Omega_i(X \times_k Y) := \cup_{j=0}^{\infty} (\Omega_j(X) \times_k \Omega_{i-j}(Y))$. In case the cellular structures on X and Y are oriented, the cellular structure on $X \times_k Y$ can be oriented using the obvious tensor products of appropriate orientations for X with those for Y . The proof of the following Lemma is easy and left to the reader.

Lemma 2.31. *Let X and Y be cellular smooth k -schemes. The induced morphism of chain complexes (of strictly \mathbb{A}^1 -invariant sheaves)*

$$C_*^{\text{cell}}(X) \otimes C_*^{\text{cell}}(Y) \rightarrow C_*^{\text{cell}}(X \times_k Y)$$

is an isomorphism. If the cellular structures on X and Y are both oriented, then the induced morphism of chain complexes (of strictly \mathbb{A}^1 -invariant sheaves)

$$\tilde{C}_*^{\text{cell}}(X) \otimes \tilde{C}_*^{\text{cell}}(Y) \rightarrow \tilde{C}_*^{\text{cell}}(X \times_k Y)$$

is an isomorphism.

Remark 2.32. Projective objects in the abelian category $Ab_{\mathbb{A}^1}$ need not be flat. For instance, the unramified Milnor-Witt K -theory sheaves \mathbf{K}_n^{MW} are projective objects, but not flat since the multiplication by an odd integer on \mathbf{K}_1^{MW} is never injective, although it is injective on \mathbb{Z} . However, in the case of cellular smooth k -schemes, the tensor product $C_*^{\text{cell}}(X) \otimes C_*^{\text{cell}}(Y)$ is the derived tensor product of the two complexes $C_*^{\text{cell}}(X)$ and $C_*^{\text{cell}}(Y)$, as the two cellular \mathbb{A}^1 -chain complexes consist of projective objects in each degree.

We will next see that up to chain homotopy, the correspondence $X \mapsto C_*^{\text{cell}}(X)$ is functorial on the category of smooth k -schemes admitting a cellular structure.

Let X be a cellular smooth k -scheme and C_* be a chain complex of strictly \mathbb{A}^1 -invariant sheaves. The cellular \mathbb{A}^1 -chain complex $C_*^{\text{cell}}(X)$ is bounded and consists of terms that are projective objects in $Ab_{\mathbb{A}^1}(k)$. Thus, the group

$$\text{Hom}_{D(Ab_{\mathbb{A}^1}(k))}(C_*^{\text{cell}}(X), C_*)$$

of morphisms in the derived category of $Ab_{\mathbb{A}^1}(k)$ is the group of chain homotopy classes of morphisms of chain complexes from $C_*^{\text{cell}}(X)$ to C_* . On the other hand we may consider the hypercohomology $H_{\text{Nis}}^n(X; C_*)$ of X with coefficients in C_* , which is defined by

$$H_{\text{Nis}}^n(X; C_*) := \text{Hom}_{D(Ab(k))}(\mathbb{Z}(X), C_*[n]).$$

Clearly, C_* is \mathbb{A}^1 -local as it consists of strictly \mathbb{A}^1 -invariant sheaves, so its homology sheaves are strictly \mathbb{A}^1 -invariant sheaves as well. Thus, we get an identification:

$$H_{\text{Nis}}^n(X; C_*) = \text{Hom}_{D(Ab_{\mathbb{A}^1}(k))}(C_*^{\mathbb{A}^1}(X), C_*[n]).$$

Proposition 2.4 above states that if C_* is a sheaf M concentrated in degree 0, then there is a canonical isomorphism

$$H_{\text{Nis}}^n(X; C_*) \cong \text{Hom}_{D(Ab_{\mathbb{A}^1}(k))}(C_*^{\mathbb{A}^1}(X), M[n]).$$

It is thus tempting to extend this isomorphism to any complex C_* of strictly \mathbb{A}^1 -invariant sheaves.

Definition 2.33. We say that a chain complex D_* of abelian sheaves of groups is *strictly \mathbb{A}^1 -solvable* if there is a chain complex C_* of strictly \mathbb{A}^1 -invariant sheaves and a morphism

$$\phi : D_* \rightarrow C_*$$

in $D(Ab_{\mathbb{A}^1}(k))$ such that for any chain complex C'_* of strictly \mathbb{A}^1 -invariant sheaves, the morphism

$$\text{Hom}_{D(Ab_{\mathbb{A}^1}(k))}(C_*, C'_*) \rightarrow \text{Hom}_{D_{\mathbb{A}^1}(k)}(D_*, C'_*)$$

induced by ϕ is an isomorphism. We call the pair (C_*, ϕ) a *strict \mathbb{A}^1 -resolution* of D_* .

Remark 2.34. In other words, D_* is strictly \mathbb{A}^1 -solvable if the functor $D(Ab_{\mathbb{A}^1}(k)) \rightarrow Ab$ defined by $C'_* \mapsto \text{Hom}_{D_{\mathbb{A}^1}(k)}(D_*, C'_*)$ is representable. If this is the case, then the pair (C_*, ϕ) consisting of C_* and the morphism $D_* \rightarrow C_*$ is clearly unique up to a canonical isomorphism. Moreover, the correspondence $D_* \mapsto C_*$ defines a functor on the full subcategory of strictly \mathbb{A}^1 -solvable chain complexes.

Lemma 2.35. *Let D_* be a connective chain complex of sheaves of abelian groups, in the sense that there exists $n_0 \in \mathbb{Z}$ such that $H_n(C_*) = 0$ for $n \leq n_0$. Let $\phi : D_* \rightarrow C_*$ be a morphism in $D_{\mathbb{A}^1}(k)$, where C_* is a chain complex of strictly \mathbb{A}^1 -invariant sheaves. Then (C_*, ϕ) is strict \mathbb{A}^1 -resolution of D_* if and only if for any strictly \mathbb{A}^1 -invariant sheaf M and any $n \in \mathbb{Z}$ the morphism*

$$\text{Hom}_{D(Ab_{\mathbb{A}^1}(k))}(C_*, M[n]) \rightarrow \text{Hom}_{D_{\mathbb{A}^1}(k)}(D_*, M[n])$$

induced by ϕ is an isomorphism.

Proof. Let $\tau_{\geq n_0} C'_* \rightarrow C'_*$ denote the truncation of C'_* with same homology in degrees $\geq n_0$ and with trivial homology in degrees $< n_0$. For any chain complex C'_* of strictly \mathbb{A}^1 -invariant sheaves, the group $\text{Hom}_{D_{\mathbb{A}^1}(k)}(D_*, C'_*)$ is clearly equal to the group $\text{Hom}_{D_{\mathbb{A}^1}(k)}(D_*, \tau_{\geq n_0} C'_*)$. We consider the successive truncations $\tau_{\geq r} C'_*$ for $r \geq n_0$, so that the cone of $\tau_{\geq r+1} C'_* \rightarrow \tau_{\geq r} C'_*$ is a complex quasi-isomorphic to $H_r C'_*$ placed in degree r . Inductively, using the assumptions, we may thus show that for any $r \geq n_0$

$$\text{Hom}_{D(Ab_{\mathbb{A}^1}(k))}(C_*, C'_*/\tau_{\geq r} C'_*) \rightarrow \text{Hom}_{D_{\mathbb{A}^1}(k)}(C_*^{\mathbb{A}^1}(X), C'_*/\tau_{\geq r} C'_*)$$

is an isomorphism, where $C'_*/\tau_{\geq r}C'_*$ denotes the cone of $\tau_{\geq r}C'_* \rightarrow C'_*$. We conclude using the fact that the morphism

$$C'_* \rightarrow \operatorname{holim}_r C'_*/\tau_{\geq r}C'_*$$

is a quasi-isomorphism (see [24, Definition 1.31, page 58 and Theorem 1.37, page 60] for the analogous statement for simplicial sheaves of sets). \square

Example 2.36. Let X be a smooth, cohomologically trivial k -scheme. Then $\mathbf{H}_0^{\mathbb{A}^1}(X)$ is a projective object of $Ab_{\mathbb{A}^1}(k)$ and the pair $(\mathbf{H}_0^{\mathbb{A}^1}(X), \phi_X)$, where $\phi : C_*^{\mathbb{A}^1}(X) \rightarrow \mathbf{H}_0^{\mathbb{A}^1}(X)$ is the obvious projection, is a strict \mathbb{A}^1 -resolution of $C_*^{\mathbb{A}^1}(X)$ by Lemma 2.35.

More generally, for any integer $n \geq 0$, $(\mathbb{A}^n/(\mathbb{A}^n - \{0\})) \wedge (X_+)$ is strictly \mathbb{A}^1 -solvable by $((\mathbf{K}_n^{\text{MW}} \otimes \mathbf{H}_0^{\mathbb{A}^1}(X))[n], \phi_n \otimes \phi_X)$, where $\phi_n : \tilde{C}_*^{\mathbb{A}^1}(\mathbb{A}^n/(\mathbb{A}^n - \{0\})) \rightarrow \mathbf{K}_n^{\text{MW}}[n]$ is the obvious projection. This can be verified by an easy direct computation and the above observations.

Proposition 2.37. *Let X be a cellular smooth k -scheme. Then $C_*^{\mathbb{A}^1}(X)$ is strictly \mathbb{A}^1 -solvable by its cellular \mathbb{A}^1 -chain complex. More precisely, there exists a (unique) morphism*

$$\phi_X : C_*^{\mathbb{A}^1}(X) \rightarrow C_*^{\text{cell}}(X)$$

in $D_{\mathbb{A}^1}(k)$ which is a strict \mathbb{A}^1 -resolution.

Proof. We proceed by induction on the length of the filtration by open subsets of X

$$\emptyset = \Omega_{-1} \subsetneq \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_s = X$$

which defines the cellular structure. If $s = 0$, the statement holds by Example 2.36.

Now assume $s \geq 1$. Let

$$\Omega_{s-1} \rightarrow X \rightarrow Th(\nu_s)$$

be the \mathbb{A}^1 -cofibration sequence deduced from the cellular structure (in degree s). This induces an exact triangle

$$C_*^{\mathbb{A}^1}(\Omega_{s-1}) \rightarrow C_*^{\mathbb{A}^1}(X) \rightarrow \tilde{C}_*^{\mathbb{A}^1}(Th(\nu_s)).$$

Now, the Thom space $Th(\nu_s)$ is (non-canonically) isomorphic to $\mathbb{A}^s/(\mathbb{A}^s - \{0\}) \wedge ((X_s)_+) \simeq_{\mathbb{A}^1} S^s \wedge \mathbb{G}_m^s \wedge ((X_s)_+)$ by Lemma 2.13, since X_s is cohomologically trivial. We consider the connecting morphism

$$(2.3) \quad \tilde{C}_*^{\mathbb{A}^1}(Th(\nu_s)[-1]) \rightarrow C_*^{\mathbb{A}^1}(\Omega_{s-1})$$

in $D_{\mathbb{A}^1}(k)$ obtained from the above triangle. Both the source and the target of this morphism are strictly \mathbb{A}^1 -solvable. The target by the inductive assumption. The source by example 2.36 above, by remembering that $Th(\nu_s)$ is (non canonically) isomorphic in the pointed \mathbb{A}^1 -homotopy to $(\mathbb{A}^s/(\mathbb{A}^s - \{0\})) \wedge ((Y_s)_+)$, so that the strict \mathbb{A}^1 -localization is (non canonically) $\mathbf{K}_s^{\text{MW}} \otimes \mathbf{H}_0^{\mathbb{A}^1}(Y_s)[s]$.

It follows from Remark 2.34 that the morphism (2.3) induces a unique morphism in $D(Ab_{\mathbb{A}^1}(k))$ between the corresponding strict \mathbb{A}^1 -resolutions:

$$(2.4) \quad \mathbf{H}_s^{\mathbb{A}^1}(Th(\nu_s))[s-1] \rightarrow C_*^{\text{cell}}(\Omega_{s-1}),$$

where $C_*^{\text{cell}}(\Omega_{s-1})$ is the cellular \mathbb{A}^1 -chain complex corresponding to the cellular structure on Ω_{s-1} induced by the one on X . Since the chain complex $C_*^{\text{cell}}(\Omega_{s-1})$ is concentrated in degrees $\leq s-1$ and since $\mathbf{H}_s^{\mathbb{A}^1}(Th(\nu_s))$ is a strictly \mathbb{A}^1 -invariant sheaf, the morphism (2.4)

is determined by the morphism it induces on H_{s-1} . Thus, the morphism (2.4) corresponds exactly to a morphism in $Ab_{\mathbb{A}^1}(k)$ given by the composition

$$\mathbf{H}_s^{\mathbb{A}^1}(Th(\nu_s)) = \mathbf{K}_s^{\text{MW}} \otimes \mathbf{H}_0^{\mathbb{A}^1}(Y_s) \rightarrow Z_{s-1}^{\text{cell}}(\Omega_{s-1}) \hookrightarrow C_{s-1}^{\text{cell}}(\Omega_{s-1}) = \mathbf{H}_{s-1}^{\mathbb{A}^1}(Th(\nu_{s-1}))$$

by the construction of the cellular \mathbb{A}^1 -chain complex. This induced morphism $\mathbf{H}_s^{\mathbb{A}^1}(Th(\nu_s)) \rightarrow \mathbf{H}_{s-1}^{\mathbb{A}^1}(Th(\nu_{s-1}))$ is a differential of the complex $C_*^{\text{cell}}(\Omega_{s-1})$. This follows by functoriality of the strict \mathbb{A}^1 -resolution applied to the composition

$$\tilde{C}_*^{\mathbb{A}^1}(Th(\nu_s)[s-1]) \rightarrow C_*^{\mathbb{A}^1}(\Omega_{s-1}) \rightarrow \tilde{C}_*^{\mathbb{A}^1}(\Omega_{s-1}) \cong \tilde{C}_*^{\mathbb{A}^1}(Th(\nu_{s-1}))$$

and the definition of the differential as the morphism $\mathbf{H}_s^{\mathbb{A}^1}(Th(\nu_s)) \rightarrow \mathbf{H}_{s-1}^{\mathbb{A}^1}(Th(\nu_{s-1}))$ induced by the connecting morphism in the \mathbb{A}^1 -cofiber sequence

$$Th(\nu_{s-1}) \cong \Omega_{s-1}/\Omega_{s-2} \subset \Omega_s/\Omega_{s-2} \rightarrow \Omega_s/\Omega_{s-1} \cong Th(\nu_s).$$

Therefore, the cone of the morphism $\mathbf{H}_s^{\mathbb{A}^1}(Th(\nu_s))[s-1] \rightarrow C_*^{\text{cell}}(\Omega_{s-1})$ in $D(Ab_{\mathbb{A}^1}(k))$ is quasi-isomorphic to $C_*^{\text{cell}}(X)$.

This allows us to extend the morphism $\phi_{\Omega_{s-1}}$ to a morphism ϕ_X in $D(Ab_{\mathbb{A}^1}(k))$, which defines a morphism of exact triangles equal to $\phi_{\Omega_{s-1}}$ on one extremity and $\phi_{Th(\nu_s)}$ on the other one. It is then easy to check, using the induction hypothesis for Ω_{s-1} that ϕ_X satisfies the criterion given by Lemma 2.35. This completes the proof. \square

We get the following interpretation of $C_*^{\text{cell}}(X)$ up to quasi-isomorphism in $D(Ab_{\mathbb{A}^1}(k))$ from the above discussion.

Corollary 2.38. *Let X be a cellular smooth k -scheme. Then for any chain complex C_* of strictly \mathbb{A}^1 -invariant sheaves on Sm_k , one has a canonical isomorphism*

$$\text{Hom}_{D_{\mathbb{A}^1}(k)}(C_*^{\mathbb{A}^1}(X), C_*) \cong \text{Hom}_{D(Ab_{\mathbb{A}^1}(k))}(C_*^{\text{cell}}(X), C_*).$$

In other words, if X admits a cellular structure, the functor

$$D(Ab_{\mathbb{A}^1}(k)) \rightarrow Ab; \quad C_* \mapsto \text{Hom}_{D_{\mathbb{A}^1}(k)}(C_*^{\mathbb{A}^1}(X), C_*)$$

is represented by $C_*^{\text{cell}}(X)$.

Remark 2.39. If X is not cellular, then in general the functor in Corollary 2.38 is not representable by a complex, but by a pro-object of $D(Ab_{\mathbb{A}^1}(k))$. This follows directly from standard ‘‘Brown representability type’’ results. One of the main points here is that for smooth k -schemes admitting a cellular structure, this representing pro-object in the category $D(Ab_{\mathbb{A}^1}(k))$ is in fact constant, as well as its homology, and the cellular \mathbb{A}^1 -chain complex is an explicit model for it.

The following lemma, quite analogous to the classical properties of cellular chain complexes associated to a CW-structure on a topological space X , is an easy consequence of the above observations.

Lemma 2.40. *Let $f : X \rightarrow Y$ be a morphism between smooth k -schemes in the \mathbb{A}^1 -homotopy category $\mathcal{H}(k)$ and assume that X and Y are both equipped with cellular structures. Then there exists a canonical homotopy class of morphisms of cellular \mathbb{A}^1 -chain complexes*

$$[f]_*^{\text{cell}} : C_*^{\text{cell}}(X) \rightarrow C_*^{\text{cell}}(Y)$$

which only depends on the class of f in $\mathrm{Hom}_{\mathcal{H}(k)}(X, Y)$. In particular f induces a canonical morphism

$$\mathbf{H}^{\mathrm{cell}}(f)_* : \mathbf{H}_*^{\mathrm{cell}}(X) \rightarrow \mathbf{H}_*^{\mathrm{cell}}(Y)$$

Remark 2.41. In case f is a cellular morphism of smooth k -schemes, in the sense of Definition 2.18, then the canonical morphism of cellular \mathbb{A}^1 -chain complexes $f_*^{\mathrm{cell}} : C_*^{\mathrm{cell}}(X) \rightarrow C_*^{\mathrm{cell}}(Y)$ induced by f has $[f]_*^{\mathrm{cell}}$ for its chain homotopy class.

We also observe two trivial consequences of Lemma 2.40, obtained by applying it to $f = \mathrm{Id}_X : X \rightarrow X$.

Corollary 2.42. *Let X be a smooth k -scheme endowed with two cellular structures. Then there exists a canonical homotopy equivalence between the two corresponding cellular \mathbb{A}^1 -chain complexes, and consequently, a canonical isomorphism between the cellular \mathbb{A}^1 -homologies of X corresponding to the two structures.*

Corollary 2.43. *The correspondence $X \mapsto C_*^{\mathrm{cell}}(X)$ defines a functor from the full subcategory of $\mathcal{H}(k)$ with objects the smooth k -schemes admitting at least one cellular structure to the homotopy category of (non-negative) chain complexes of projective objects in $\mathrm{Ab}_{\mathbb{A}^1}(k)$. Consequently, it induces a functor $X \mapsto \mathbf{H}_*^{\mathrm{cell}}(X)$ from the full subcategory of $\mathcal{H}(k)$ with objects the smooth k -schemes admitting at least one cellular structure to the category of graded strictly \mathbb{A}^1 -invariant sheaves.*

Remark 2.44. As noted in Remark 2.39, the functor $X \mapsto C_*^{\mathrm{cell}}(X)$ in Corollary 2.43 extends to a functor defined on the category $\mathcal{H}(k)$, but with target the category of pro-objects in the homotopy category of (non-negative) chain complexes of strictly \mathbb{A}^1 -invariant sheaves:

$$\mathrm{Sm}_k \rightarrow \mathrm{pro}\text{-}D(\mathrm{Ab}_{\mathbb{A}^1}(k)); \quad X \mapsto C_*^{\mathrm{st}\text{-}\mathbb{A}^1}(X)$$

and the corresponding homology $H_*^{\mathrm{st}\text{-}\mathbb{A}^1}$ takes its values in the category $\mathrm{pro}\text{-}\mathrm{Ab}_{\mathbb{A}^1}(k)$. Observe that $H_n^{\mathrm{st}\text{-}\mathbb{A}^1}(X) = 0$ for $n > \dim(X)$ by the analogue of Proposition 2.27, since the Nisnevich cohomological dimension of a smooth scheme X is $\dim(X)$. These objects seem to be much more computable compared to the corresponding \mathbb{A}^1 -homology sheaves. In fact, there are reasons to believe that for a smooth (projective) k -scheme X , the pro-objects $H_*^{\mathrm{st}\text{-}\mathbb{A}^1}(X)$ are always constant.

2.5. Cellular \mathbb{A}^1 -homology of projective spaces and punctured affine spaces.

In this section, we explicitly compute the cellular \mathbb{A}^1 -chain complex and the cellular \mathbb{A}^1 -homology sheaves of the n -dimensional projective space \mathbb{P}^n with respect to the canonical oriented cellular structure described in Example 2.19. If Ω_\bullet denotes the filtration of \mathbb{P}^n by open subschemes described in Example 2.19, we have $\Omega_{n-1} = \mathbb{P}^n - \{[0, \dots, 0, 1]\}$. Thus, Ω_{n-1} is the total space of the canonical line bundle over \mathbb{P}^{n-1} , $\Omega_{n-1} = \mathbb{A}^n - \{0\} \times_{\mathbb{G}_m} \mathbb{A}^1$. The projection morphism $\Omega_{n-1} \rightarrow \mathbb{P}^{n-1}$ is cellular with respect to the cellular structure on Ω_{n-1} induced by the one on \mathbb{P}^n and the canonical cellular structure on \mathbb{P}^{n-1} . In fact, the morphism $\pi_{n-1} : \Omega_{n-1} \rightarrow \mathbb{P}^{n-1}$ is oriented cellular in the sense of Definition 2.18 by Lemma 2.20. Since π_{n-1} is a vector bundle of rank one, the induced morphism

$$(\tilde{\pi}_{n-1}^{\mathrm{cell}})_* : \tilde{C}_*^{\mathrm{cell}}(\Omega_{n-1}) \rightarrow \tilde{C}_*^{\mathrm{cell}}(\mathbb{P}^{n-1})$$

is an isomorphism of (oriented) cellular \mathbb{A}^1 -chain complexes. Let $\tilde{\Omega}_\bullet$ denote the filtration of $\mathbb{A}^{n+1} - \{0\}$ by open subschemes obtained by pulling back Ω_\bullet by the canonical \mathbb{G}_m -torsor

$\mathbb{A}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$ as described in Lemma 2.20. In exactly the same way as above, one can verify that the canonical projection $\tilde{\Omega}_{n-1} \rightarrow \mathbb{A}^n - \{0\}$, which is the projection of a (trivial) vector bundle of rank one, induces an isomorphism

$$\tilde{C}_*^{\text{cell}}(\tilde{\Omega}_{n-1}) \xrightarrow{\cong} \tilde{C}_*^{\text{cell}}(\mathbb{A}^n - \{0\})$$

of (oriented) cellular \mathbb{A}^1 -chain complexes. Observe that $\tilde{\Omega}_i = (\mathbb{A}^{i+1} - \{0\}) \times \mathbb{A}^{n-i}$, and consequently, it follows that $\tilde{\Omega}_i - \tilde{\Omega}_{i-1} = \mathbb{A}^i \times \mathbb{G}_m \times \mathbb{A}^{n-i}$. We now point out that for the morphism $\mathbb{A}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$ to be oriented cellular, the orientations for $\mathbb{A}^{n+1} - \{0\}$ have to be the ones induced by those on \mathbb{P}^n , which are described in example 2.19. If X_0, \dots, X_n denote the $n+1$ -coordinate functions on $\mathbb{A}^{n+1} - \{0\}$, then the normal bundle ν_i of the closed immersion of $\tilde{\Omega}_i - \tilde{\Omega}_{i-1}$ in $\tilde{\Omega}_i$ is trivialized (or rather, oriented) by the functions $\frac{X_j}{X_n}$ (and not by the functions X_j) for $j \in \{0, \dots, i-1\}$. Thus,

$$(2.5) \quad \tilde{C}_i^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}) \cong \mathbf{K}_i^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m]$$

for each $i \leq n$, where the isomorphism $\mathbf{K}_i^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] \cong \mathbf{H}_i^{\mathbb{A}^1}(\text{Th}(\nu_i))$ is coming from the above trivializations.

Remark 2.45. Let $\mathbf{H} := \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m]$ denote the free strictly \mathbb{A}^1 -invariant sheaf on \mathbb{G}_m . Clearly, \mathbf{H} is a commutative and cocommutative Hopf algebra in $Ab_{\mathbb{A}^1}$ for the \mathbb{A}^1 -tensor product. The structure morphism $\mathbb{G}_m \rightarrow \text{Spec } k$ induces an augmentation morphism, the counit of the structure, which is split by the unit of \mathbb{G}_m . The augmentation ideal of \mathbf{H} , that is, the kernel of this augmentation morphism, is canonically $\mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) = \mathbf{K}_1^{\text{MW}}$ (where we use 1 as the base point of \mathbb{G}_m). The above splitting thus induces a decomposition

$$\mathbf{H} = \mathbf{K}_1^{\text{MW}} \oplus \mathbb{Z}.$$

In this decomposition, the product $\mathbf{H} \otimes \mathbf{H} \rightarrow \mathbf{H}$ is easily computed to be the following: it is the obvious morphism on the factor

$$\mathbb{Z} \oplus (\mathbf{K}_1^{\text{MW}} \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes \mathbf{K}_1^{\text{MW}}) \rightarrow \mathbf{H}$$

and it is given by

$$\eta : \mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} = \mathbf{K}_2^{\text{MW}} \rightarrow \mathbf{K}_1^{\text{MW}} \subset \mathbf{H}$$

on the remaining factor $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} \subset \mathbf{H}$; this is in fact precisely the definition of η (see [23]). The diagonal $\Psi : \mathbf{H} \rightarrow \mathbf{H} \otimes \mathbf{H}$ giving the Hopf algebra structure is easily checked to be the inclusion

$$\mathbb{Z} = \mathbb{Z} \otimes \mathbb{Z} \subset \mathbf{H} \otimes \mathbf{H}$$

on the factor $\mathbb{Z} \subset \mathbf{H}$ and on the factor $\mathbf{K}_1^{\text{MW}} \subset \mathbf{H}$, it is given by the morphism

$$\begin{aligned} \mathbf{K}_1^{\text{MW}} &\rightarrow (\mathbf{K}_1^{\text{MW}} \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes \mathbf{K}_1^{\text{MW}}) \oplus (\mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}}) \subset \mathbf{H} \otimes \mathbf{H} \\ (t) &\mapsto (t) \otimes 1 + 1 \otimes (t) + (-1) \cdot (t). \end{aligned}$$

Observe that for a unit t , we have $(t) \cdot (t) = (-1) \cdot (t) = (t) \cdot (-1)$ in \mathbf{K}_2^{MW} .

Using this structure, if \mathbf{M} and \mathbf{N} are strictly \mathbb{A}^1 -invariant sheaves with an \mathbf{H} -module structure, one defines as usual a \mathbf{H} -module structure on the (\mathbb{A}^1 -) tensor product $\mathbf{M} \otimes \mathbf{N}$ by the composition:

$$\mathbf{M} \otimes \mathbf{N} \otimes \mathbf{H} \xrightarrow{\text{Id}_{\mathbf{M}} \otimes \text{Id}_{\mathbf{N}} \otimes \Psi} \mathbf{M} \otimes \mathbf{N} \otimes \mathbf{H} \otimes \mathbf{H} \cong (\mathbf{M} \otimes \mathbf{H}) \otimes (\mathbf{N} \otimes \mathbf{H}) \rightarrow \mathbf{M} \otimes \mathbf{N}.$$

For instance, if \mathbf{K}_1^{MW} is endowed with the \mathbf{H} -module structure as the kernel of the augmentation $\mathbf{H} \rightarrow \mathbb{Z}$, then the n -th tensor product $(\mathbf{K}_1^{\text{MW}})^{\otimes n} = \mathbf{K}_n^{\text{MW}}$ admits an induced \mathbf{H} -module structure.

The action of \mathbb{G}_m on $\mathbb{A}^{n+1} - \{0\}$ preserves the oriented cellular structure by Lemma 2.20, and induces an action of \mathbf{H} on the entire cellular \mathbb{A}^1 -chain complex. Thus, $\tilde{C}_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$ (and also $C_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$) is a complex of \mathbf{H} -modules in $Ab_{\mathbb{A}^1}(k)$. One may show that the isomorphism (2.5) above is an isomorphism of \mathbf{H} -modules where the action of \mathbf{H} on \mathbf{K}_i^{MW} is the trivial one. This is contained in the following result, which computes the oriented cellular \mathbb{A}^1 -chain complex of $\mathbb{A}^{n+1} - \{0\}$ and its cellular \mathbb{A}^1 -homology.

Theorem 2.46. *The oriented cellular \mathbb{A}^1 -chain complex $\tilde{C}_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$ of $\mathbb{A}^{n+1} - \{0\}$, as a complex of (say, right) \mathbf{H} -modules, is canonically isomorphic to:*

$$\mathbf{K}_n^{\text{MW}} \otimes \mathbf{H} \xrightarrow{\partial_n} \mathbf{K}_{n-1}^{\text{MW}} \otimes \mathbf{H} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_{i+1}} \mathbf{K}_i^{\text{MW}} \otimes \mathbf{H} \xrightarrow{\partial_i} \mathbf{K}_{i-1}^{\text{MW}} \otimes \mathbf{H} \dots \mathbf{K}_1^{\text{MW}} \otimes \mathbf{H} \xrightarrow{\partial_1} \mathbf{H}$$

where for each i the \mathbf{H} -module structure on $\mathbf{K}_i^{\text{MW}} \otimes \mathbf{H}$ is the tensor product of the trivial structure on \mathbf{K}_i^{MW} and the canonical one on \mathbf{H} and where the differential

$$\partial_i : (\mathbf{K}_{i+1}^{\text{MW}} \oplus \mathbf{K}_i^{\text{MW}}) \cong \mathbf{K}_i^{\text{MW}} \otimes \mathbf{H} \rightarrow \mathbf{K}_{i-1}^{\text{MW}} \otimes \mathbf{H} \cong (\mathbf{K}_i^{\text{MW}} \oplus \mathbf{K}_{i-1}^{\text{MW}})$$

is given for i odd by the matrix (with obvious notations)

$$\begin{pmatrix} \eta & 0 \\ \text{Id}_{\mathbf{K}_i^{\text{MW}}} & 0 \end{pmatrix}$$

and for i even by the matrix

$$\begin{pmatrix} 0 & 0 \\ \langle -1 \rangle \cdot \text{Id}_{\mathbf{K}_i^{\text{MW}}} & \eta \end{pmatrix}.$$

Remark 2.47. It immediately follows from Theorem 2.46 that the complex $\tilde{C}_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$ is a complex of “free” \mathbf{H} -modules. Moreover, it is acyclic in degree $\neq 0$ and n , because the space $\mathbb{A}^{n+1} - \{0\}$ is $(n-1)$ -connected. In other words $\tilde{C}_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$ is an $(n+1)$ -extension of \mathbb{Z} by $\mathbf{K}_{n+1}^{\text{MW}} = \mathbf{H}_n^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$. One has a chain homotopy equivalence

$$\tilde{C}_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}) \otimes_{\mathbf{H}} \mathbb{Z} \cong \tilde{C}_*^{\text{cell}}(\mathbb{P}^n).$$

We will need some preliminaries before embarking upon the proof of Theorem 2.46. Consider the automorphism of the rank n trivial vector bundle over \mathbb{G}_m defined by

$$\begin{aligned} \Theta_n : \mathbb{A}^n \times \mathbb{G}_m &\xrightarrow{\cong} \mathbb{A}^n \times \mathbb{G}_m \\ ((\lambda_0, \dots, \lambda_{n-1}), t) &\mapsto ((\lambda_0 t, \dots, \lambda_{n-1} t), t). \end{aligned}$$

and let

$$Th(\Theta_n) : (\mathbb{A}^n / (\mathbb{A}^n - \{0\})) \wedge (\mathbb{G}_m)_+ \cong (\mathbb{A}^n / (\mathbb{A}^n - \{0\})) \wedge (\mathbb{G}_m)_+$$

denote the associated isomorphism of Thom spaces. The space $(\mathbb{A}^n / (\mathbb{A}^n - \{0\})) \wedge (\mathbb{G}_m)_+$ is clearly $(n-1)$ -connected and the Künneth morphism

$$(2.6) \quad \mathbf{K}_n^{\text{MW}} \otimes \mathbf{H} \xrightarrow{\cong} \mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^n / (\mathbb{A}^n - \{0\})) \wedge (\mathbb{G}_m)_+)$$

is an isomorphism (of strictly \mathbb{A}^1 -invariant sheaves) because of the Hurewicz Theorem [23, §6.3]. Now using the cofibration sequence

$$(\mathbb{A}^n - \{0\}) \times \mathbb{G}_m \rightarrow \mathbb{A}^n \times \mathbb{G}_m \rightarrow (\mathbb{A}^n / (\mathbb{A}^n - \{0\})) \wedge (\mathbb{G}_m)_+$$

and its associated long exact sequence in \mathbb{A}^1 -homology, we deduce an exact sequence²

$$(2.7) \quad \mathbf{H}_n^{\mathbb{A}^1}(\mathbb{G}_m) \rightarrow \mathbf{K}_n^{\text{MW}} \otimes \mathbf{H} \rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^n - \{0\}) \times \mathbb{G}_m) \rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{G}_m) \rightarrow 0.$$

The automorphism

$$(\mathbb{A}^n - \{0\}) \times \mathbb{G}_m \xrightarrow{\cong} (\mathbb{A}^n - \{0\}) \times \mathbb{G}_m; \quad ((\lambda_0, \dots, \lambda_{n-1}), t) \mapsto ((\lambda_0 t, \dots, \lambda_{n-1} t), t)$$

given by restricting Θ_n induces an automorphism of the exact sequence (2.7) (the action on the factors $\mathbf{H}_n^{\mathbb{A}^1}(\mathbb{G}_m)$ and $\mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{G}_m)$ being trivial). Thus, the induced automorphism

$$(2.8) \quad \theta_n : \mathbf{K}_n^{\text{MW}} \otimes \mathbf{H} \cong \mathbf{K}_n^{\text{MW}} \otimes \mathbf{H}$$

is the same as the one induced by $Th(\Theta_n)$ and the identification (2.6). The projection of θ_n onto the factor $\mathbf{K}_n^{\text{MW}} \otimes \mathbb{Z} = \mathbf{K}_n^{\text{MW}}$ is clearly the morphism induced by the projection of Θ_n on the first factor

$$(2.9) \quad (\mathbb{A}^n - \{0\}) \times \mathbb{G}_m \rightarrow \mathbb{A}^n - \{0\}; \quad ((\lambda_0, \dots, \lambda_{n-1}), t) \mapsto (\lambda_0 t, \dots, \lambda_{n-1} t)$$

on $\mathbf{H}_{n-1}^{\mathbb{A}^1}$:

$$\mathbf{K}_n^{\text{MW}} \otimes \mathbf{H} \rightarrow \mathbf{K}_n^{\text{MW}}.$$

Using this and the fact that $\mathbb{A}^n/(\mathbb{A}^n - \{0\})$ is canonically isomorphic to $(\mathbb{A}^1/(\mathbb{A}^1 - \{0\}))^{\wedge n}$ as a \mathbb{G}_m -space, we deduce that the above morphism is exactly the morphism defining the \mathbf{H} -module structure on \mathbf{K}_n^{MW} corresponding to the n -th tensor product of the one on \mathbf{K}_1^{MW} . We now explicitly describe this \mathbf{H} -module structure on \mathbf{K}_n^{MW} .

Lemma 2.48.

(a) *The morphism*

$$\mathbf{K}_n^{\text{MW}} \otimes \mathbf{H} \rightarrow \mathbf{K}_n^{\text{MW}}$$

corresponding to the canonical \mathbf{H} -module structure on \mathbf{K}_n^{MW} is given as follows. Using the decomposition

$$\mathbf{K}_n^{\text{MW}} \otimes \mathbf{H} = (\mathbf{K}_n^{\text{MW}} \otimes \mathbb{Z}) \oplus (\mathbf{K}_n^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}}) = \mathbf{K}_n^{\text{MW}} \oplus \mathbf{K}_{n+1}^{\text{MW}},$$

on the factor $\mathbf{K}_n^{\text{MW}} \otimes \mathbb{Z} = \mathbf{K}_n^{\text{MW}}$ it is the identity and on the factor $\mathbf{K}_n^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} = \mathbf{K}_{n+1}^{\text{MW}}$ it is given by

$$n_\epsilon \cdot \eta : \mathbf{K}_{n+1}^{\text{MW}} \rightarrow \mathbf{K}_n^{\text{MW}}$$

where $n_\epsilon = \sum_{i=1}^n \langle (-1)^{i-1} \rangle \in \mathbf{K}_0^{\text{MW}}(k) = \text{GW}(k)$. In particular, this morphism is 0 if n is even, and is multiplication by η if n is odd.

(b) *The automorphism of $\mathbf{K}_{n+1}^{\text{MW}} \oplus \mathbf{K}_n^{\text{MW}}$ defined by the commutative diagram*

$$\begin{array}{ccc} \mathbf{K}_n^{\text{MW}} \otimes \mathbf{H} & \xrightarrow{\theta_n} & \mathbf{K}_n^{\text{MW}} \otimes \mathbf{H} \\ \downarrow = & & \downarrow = \\ \mathbf{K}_{n+1}^{\text{MW}} \oplus \mathbf{K}_n^{\text{MW}} & \longrightarrow & \mathbf{K}_{n+1}^{\text{MW}} \oplus \mathbf{K}_n^{\text{MW}} \end{array}$$

is given by multiplication by the matrix

$$\begin{pmatrix} \langle (-1)^n \rangle & n_\epsilon \cdot \eta \\ 0 & 1 \end{pmatrix}.$$

²it is conjectured [23, Conjecture 6.34] that $\mathbf{H}_n^{\mathbb{A}^1}(\mathbb{G}_m) = 0$ for $n \geq 1$

Proof. We only outline the argument and leave the details to the reader.

- (a) Recall from Remark 2.45 that the canonical \mathbf{H} -module structure on \mathbf{K}_1^{MW} is given by the morphism $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{H} \rightarrow \mathbf{K}_1^{\text{MW}}$, which on the factor $\mathbf{K}_1^{\text{MW}} \otimes \mathbb{Z}$ is the obvious homomorphism and is multiplication by η on the factor $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}}$. We will use the description of the diagonal $\Psi : \mathbf{H} \rightarrow \mathbf{H} \otimes \mathbf{H}$ of \mathbf{H} and the definition of the \mathbf{H} -module structure on a tensor product. The \mathbf{H} -module structure on \mathbf{K}_2^{MW} is given by the composition

$$\begin{aligned} \mathbf{K}_2^{\text{MW}} \otimes \mathbf{H} &\rightarrow \mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} \otimes \mathbf{H} \xrightarrow{\text{Id} \otimes \Psi} \mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} \otimes \mathbf{H} \otimes \mathbf{H} \\ &\cong (\mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}) \otimes (\mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}) \rightarrow \mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} = \mathbf{K}_2^{\text{MW}}, \end{aligned}$$

which is easily seen to be identity on the factor $\mathbf{K}_2^{\text{MW}} \otimes \mathbb{Z}$. On the factor $\mathbf{K}_2^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}}$, it is given by

$$\begin{aligned} (\lambda, \mu) \otimes (t) &\mapsto (\lambda) \otimes (\mu) \otimes ((t) \otimes 1 + 1 \otimes (t) + (-1) \otimes (t)) \\ &\mapsto \eta \cdot (\lambda) \cdot (t) \otimes (\mu) + (\lambda) \otimes \eta \cdot (\mu) \cdot (t) + \eta \cdot (\lambda) \cdot (-1) \otimes \eta \cdot (\mu) \cdot (t) \\ &\mapsto \epsilon \cdot \eta \cdot (\lambda) \cdot (\mu) \cdot (t) + \eta \cdot (\lambda) \cdot (\mu) \cdot (t) + \epsilon \cdot \eta^2 \cdot (-1) \cdot (\lambda) \cdot (\mu) \cdot (t) \\ &= \eta \cdot (\lambda) \cdot (\mu) \cdot (t) - \epsilon \cdot \eta \cdot (\lambda) \cdot (\mu) \cdot (t) = 2_\epsilon \cdot \eta \cdot (\lambda) \cdot (\mu) \cdot (t), \end{aligned}$$

where $\epsilon = -\langle -1 \rangle \in \mathbf{K}_0^{\text{MW}}(k)$ and we have used the ϵ -graded commutativity of Milnor-Witt K -theory and the defining relation $\eta^2 \cdot (-1) + 2\eta = 0$. An easy induction now finishes the proof in the general case.

- (b) We use the fact that θ_n is induced by the morphism $(\mathbb{A}^n - \{0\}) \times \mathbb{G}_m \cong (\mathbb{A}^n - \{0\}) \times \mathbb{G}_m$, which is the product of the morphism $(\mathbb{A}^n - \{0\}) \times \mathbb{G}_m \rightarrow \mathbb{A}^n - \{0\}$ given in (2.9) and of the projection $(\mathbb{A}^n - \{0\}) \times \mathbb{G}_m \rightarrow \mathbb{G}_m$. The former morphism on $\mathbf{H}_{n-1}^{\mathbb{A}^1}$ is just the \mathbf{H} -module structure on \mathbf{K}_n^{MW} determined in part (a) above. Now using the Künneth formula, the fact that the diagonal of the Hopf algebra $\mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) = \mathbf{K}_n^{\text{MW}}$ is given by the formula $x \mapsto x \otimes 1 \oplus 1 \otimes x$, and the description of the diagonal of $\mathbf{H}_0^{\mathbb{A}^1}(\mathbb{G}_m) = \mathbf{H}$, the desired automorphism can be explicitly computed. The details are left to the reader. □

Remark 2.49. The automorphism θ_n described in Lemma 2.48 is of order 2: $\theta_n \circ \theta_n = \text{Id}$. This is easy to check by using the identity

$$\langle (-1)^n \rangle \cdot n_\epsilon \cdot \eta + n_\epsilon \cdot \eta = 0.$$

Thus, θ_n is also the automorphism $\mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\} \times \mathbb{G}_m) \cong \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\} \times \mathbb{G}_m)$ induced by the morphism

$$(\mathbb{A}^n - \{0\}) \times \mathbb{G}_m \rightarrow (\mathbb{A}^n - \{0\}) \times \mathbb{G}_m; \quad ((\lambda_0, \dots, \lambda_{n-1}), t) \mapsto ((\lambda_0 t^{-1}, \dots, \lambda_{n-1} t^{-1}), t).$$

This fact will be used in the proof of Theorem 2.46 below.

Proof of Theorem 2.46. We proceed by induction on n . The case $n = 0$ is trivial. Now we assume that $n \geq 1$ and that the theorem is proven for $\mathbb{A}^n - \{0\}$.

With our notation described at the beginning of this subsection, we have $\tilde{\Omega}_{n-1}(\mathbb{A}^{n+1} - \{0\}) = (\mathbb{A}^n - \{0\}) \times \mathbb{A}^1$ and the filtration

$$\emptyset = \tilde{\Omega}_{-1}(\mathbb{A}^{n+1} - \{0\}) \subset \tilde{\Omega}_0(\mathbb{A}^{n+1} - \{0\}) \subset \dots \subset \tilde{\Omega}_{n-1}(\mathbb{A}^{n+1} - \{0\}) = (\mathbb{A}^n - \{0\}) \times \mathbb{A}^1$$

is exactly the product with \mathbb{A}^1 (on the right) of the filtration

$$\emptyset = \tilde{\Omega}_{-1}(\mathbb{A}^n - \{0\}) \subset \tilde{\Omega}_0(\mathbb{A}^n - \{0\}) \subset \cdots \subset \tilde{\Omega}_{n-1}(\mathbb{A}^n - \{0\}) = (\mathbb{A}^n - \{0\}).$$

Thus, by \mathbb{A}^1 -invariance of \mathbb{A}^1 -homology, the truncation $\tau_{\leq n-1} \tilde{C}_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$ in degrees $\leq n-1$ equals $\tilde{C}_*^{\text{cell}}(\mathbb{A}^n - \{0\})$. By induction, we thus already know the theorem for the oriented cellular \mathbb{A}^1 -chain complex of $\mathbb{A}^{n+1} - \{0\}$ in degree $\leq n-1$. So we only need to identify $\tilde{C}_n^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$ and the differential

$$(2.10) \quad \partial_n : \tilde{C}_n^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}) \rightarrow \tilde{C}_{n-1}^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}).$$

Now, $(\mathbb{A}^{n+1} - \{0\}) - \tilde{\Omega}_{n-1}(\mathbb{A}^{n+1} - \{0\}) = \{0\} \times \mathbb{G}_m \subset \mathbb{A}^n \times \mathbb{G}_m$, so by construction $C_n^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}) = \mathbf{H}_n^{\mathbb{A}^1}(Th(\nu_n))$, where ν_n is the normal bundle of the closed immersion

$$\{0\} \times \mathbb{G}_m \subset \mathbb{A}^{n+1} - \{0\}$$

Since the open subset $\mathbb{A}^n \times \mathbb{G}_m \subset \mathbb{A}^{n+1} - \{0\}$ contains $\{0\} \times \mathbb{G}_m$, it defines a trivialization of ν_n and produces an isomorphism of pointed spaces $Th(\nu_n) \cong (\mathbb{A}^n / (\mathbb{A}^n - \{0\})) \wedge (\mathbb{G}_m)_+$. Hence,

$$\begin{aligned} C_n^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}) &= \mathbf{H}_n^{\mathbb{A}^1}(Th(\nu_n)) = \mathbf{H}_n^{\mathbb{A}^1}(\mathbb{A}^n / (\mathbb{A}^n - \{0\})) \wedge (\mathbb{G}_m)_+ \\ &= \mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^n - \{0\}) \times \mathbb{G}_m), \end{aligned}$$

where the last equality follows from the fact that $\mathbb{A}^n - \{0\}$ is \mathbb{A}^1 -weak equivalent to $S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$. The desired differential ∂_n in (2.10) of the underlying cellular \mathbb{A}^1 -chain complex is, by construction, given by the composition

$$(2.11) \quad \begin{aligned} C_n^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}) &= \mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^n - \{0\}) \times \mathbb{G}_m) \rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}(\tilde{\Omega}_{n-1}) = \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \\ &\rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^n - \{0\}) / \tilde{\Omega}_{n-2}(\mathbb{A}^n - \{0\})) = C_{n-1}^{\text{cell}}(\mathbb{A}^n - \{0\}) = C_{n-1}^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}). \end{aligned}$$

This follows from the fact that $\tilde{\Omega}_{n-1} \cap (\mathbb{A}^n \times \mathbb{G}_m) = (\mathbb{A}^n - \{0\}) \times \mathbb{G}_m$ and from the induced commutative diagram

$$\begin{array}{ccccccc} (\mathbb{A}^n - \{0\}) \times \mathbb{G}_m & \rightarrow & \mathbb{A}^n \times \mathbb{G}_m & \rightarrow & \frac{\mathbb{A}^n}{\mathbb{A}^n - \{0\}} \wedge (\mathbb{G}_m)_+ & \rightarrow & S^1 \wedge ((\mathbb{A}^n - \{0\}) \times \mathbb{G}_m)_+ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{\Omega}_{n-1} & \rightarrow & \mathbb{A}^{n+1} - \{0\} & \rightarrow & Th(\nu_n) & \rightarrow & S^1 \wedge ((\tilde{\Omega}_{n-1})_+). \end{array}$$

Now, the composition $\mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^n - \{0\}) \times \mathbb{G}_m) \rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) = \mathbf{K}_n^{\text{MW}}$ of the first row in (2.11) is just induced by the projection $(\mathbb{A}^n - \{0\}) \times \mathbb{G}_m \rightarrow \mathbb{A}^n - \{0\}$.

However, in order to compute the differential $\partial_n : \tilde{C}_n^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}) \rightarrow \tilde{C}_{n-1}^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$ of the oriented cellular \mathbb{A}^1 -chain complex, recall that we need to use the induced trivialization from \mathbb{P}^n described in Example 2.19 and Lemma 2.20. If the X_i for $i \in \{0, \dots, n\}$ are the $n+1$ -coordinate functions on $\mathbb{A}^{n+1} - \{0\}$, then the normal bundle ν_n is oriented by the functions $\frac{X_i}{X_n}$ for $i \in \{0, \dots, n-1\}$ (which produces the isomorphism of pointed spaces $Th(\nu_n) \cong (\mathbb{A}^n / (\mathbb{A}^n - \{0\})) \wedge ((\mathbb{G}_m)_+)$). Thus, we have to compose the previous identifications with the isomorphism of vector bundles over \mathbb{G}_m

$$\mathbb{A}^n \times \mathbb{G}_m \rightarrow \mathbb{A}^n \times \mathbb{G}_m; \quad ((\lambda_0, \dots, \lambda_{n-1}), t) \mapsto (\lambda_0 t, \dots, \lambda_{n-1} t),$$

which takes the corrected trivialization given by the functions $\frac{X_i}{X_n}$ to the one coming from the functions X_i , $i \in \{0, \dots, n-1\}$. Therefore, the differential ∂_n in the oriented cellular \mathbb{A}^1 -chain complex (2.10) is the composition

$$\mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^n - \{0\}) \times \mathbb{G}_m) \xrightarrow{\cong} \mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^n - \{0\}) \times \mathbb{G}_m) \rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \rightarrow C_{n-1}^{\text{cell}}(\mathbb{A}^n - \{0\}),$$

where the leftmost isomorphism is induced by the above automorphism of $\mathbb{A}^n \times \mathbb{G}_m$. Now, the composition

$$\mathbf{K}_{n+1}^{\text{MW}} \otimes \mathbf{H} \rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^n - \{0\}) \times \mathbb{G}_m) \rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) = \mathbf{K}_n^{\text{MW}}$$

is precisely the morphism described in the Lemma 2.48, where the leftmost arrow is the one in (2.7). The canonical morphism $\mathbf{K}_n^{\text{MW}} = \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \rightarrow \tilde{C}_{n-1}^{\text{cell}}(\mathbb{A}^n - \{0\})$ is given as follows:

$$\mathbf{K}_n^{\text{MW}} \xrightarrow{\left(\langle -1 \rangle \quad \eta\right)} \mathbf{K}_n^{\text{MW}} \oplus \mathbf{K}_{n-1}^{\text{MW}},$$

if n is even and

$$\mathbf{K}_n^{\text{MW}} \xrightarrow{\left(\text{Id} \quad 0\right)} \mathbf{K}_n^{\text{MW}} \oplus \mathbf{K}_{n-1}^{\text{MW}},$$

if n is odd. To see this, first use the fact that $\mathbf{K}_n^{\text{MW}} = \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \rightarrow \tilde{C}_{n-1}^{\text{cell}}(\mathbb{A}^n - \{0\})$ is the composition

$$\mathbf{K}_n^{\text{MW}} = \mathbf{H}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}) \rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}(\text{Th}(\nu_{n-1})) \cong \mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^{n-1}/(\mathbb{A}^{n-1} - \{0\}) \wedge ((\mathbb{G}_m)_+)),$$

in which the rightmost isomorphism is induced by the trivialization coming from the functions $\frac{X_i}{X_n}$ for $i \in \{0, \dots, n-2\}$. If one uses the trivialization coming from the functions X_i instead, then it is easy to see that

$$(2.12) \quad \mathbf{K}_n^{\text{MW}} \rightarrow \mathbf{H}_{n-1}^{\mathbb{A}^1}((\mathbb{A}^{n-1}/(\mathbb{A}^{n-1} - \{0\}) \wedge ((\mathbb{G}_m)_+)) = \mathbf{K}_n^{\text{MW}} \oplus \mathbf{K}_{n-1}^{\text{MW}}$$

is the canonical inclusion. Thus, in order to compute the morphism in (2.12) for the trivialization coming from the functions $\frac{X_i}{X_n}$, we need to compose the morphism (2.12) with the automorphism

$$\mathbf{K}_n^{\text{MW}} \oplus \mathbf{K}_{n-1}^{\text{MW}} = \mathbf{H}_{n-2}^{\mathbb{A}^1}(\mathbb{A}^{n-1} - \{0\} \times \mathbb{G}_m) \cong \mathbf{H}_{n-2}^{\mathbb{A}^1}(\mathbb{A}^{n-1} - \{0\} \times \mathbb{G}_m) = \mathbf{K}_n^{\text{MW}} \oplus \mathbf{K}_{n-1}^{\text{MW}}$$

induced by $((\lambda_0, \dots, \lambda_{n-2}), t) \mapsto ((\lambda_0 t^{-1}, \dots, \lambda_{n-2} t^{-1}), t)$. This morphism is computed in Lemma 2.48 (b) and Remark 2.49 above. Using the induction, it is now easy to conclude the proof of the theorem. \square

Remark 2.50. If we let n go to infinity in the statement of Theorem 2.46, one obtains the oriented cellular \mathbb{A}^1 -chain complex of the oriented cellular (ind-)space $\mathbb{A}^\infty - \{0\}$, which is unbounded below with the differentials given by the same formula as in Theorem 2.46. It is periodic of period 2, and it is a free \mathbf{H} -resolution of \mathbb{Z} . This suggests that \mathbb{G}_m behaves like a ‘‘cyclic group of order 2’’. Indeed, consider the standard ‘‘small’’ free resolution of \mathbb{Z} by free $\mathbb{Z}[\mathbb{Z}/2]$ -modules of rank 1:

$$\dots \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{\partial_3} \mathbb{Z}[\mathbb{Z}/2] \rightarrow \dots \rightarrow \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{\partial_2} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{\partial_1} \mathbb{Z}[\mathbb{Z}/2] \rightarrow \mathbb{Z}$$

where $\mathbb{Z}[\mathbb{Z}/2]$ is the group ring of $\mathbb{Z}/2$, with σ the nontrivial element of $\mathbb{Z}/2$, and

$$\partial_n = \begin{cases} 1 - \sigma, & \text{if } n \text{ is odd; and} \\ 1 + \sigma, & \text{if } n \text{ is even.} \end{cases}$$

Here $\mathbb{Z}[\mathbb{Z}/2]$ replaces \mathbf{H} (in fact, it is the topological real realization of \mathbf{H}). The augmentation ideal $\mathbb{Z}(\mathbb{Z}/2) \subset \mathbb{Z}[\mathbb{Z}/2]$ of $\mathbb{Z}[\mathbb{Z}/2]$ is now the free subgroup with generator $[1] - [\sigma]$ (with obvious notations). The decomposition $\mathbb{Z}[\mathbb{Z}/2] = \mathbb{Z} \oplus \mathbb{Z}(\mathbb{Z}/2)$ is analogous to $\mathbf{H} = \mathbb{Z} \oplus \mathbf{K}_1^{\text{MW}}$ and the differential

$$\partial_1 : \mathbb{Z}(\mathbb{Z}/2) \oplus \mathbb{Z} \xrightarrow{1-\sigma} \mathbb{Z}(\mathbb{Z}/2) \oplus \mathbb{Z}$$

can be interpreted with respect to this analogy: the summand \mathbb{Z} is mapped isomorphically to $\mathbb{Z}(\mathbb{Z}/2)$ by $1 \mapsto [1] - [\sigma]$, and $\mathbb{Z}(\mathbb{Z}/2)$ maps to itself through the multiplication by 2 which replaces here η . The differential

$$\partial_2 : \mathbb{Z}(\mathbb{Z}/2) \oplus \mathbb{Z} \xrightarrow{1+\sigma} \mathbb{Z}(\mathbb{Z}/2) \oplus \mathbb{Z}$$

the factor $\mathbb{Z}(\mathbb{Z}/2)$ maps to 0 and since

$$(1 + \sigma)[1] = [1] + [\sigma] = 2 - ([1] - [\sigma]),$$

we see that $[1]$ maps to $-([1] - [\sigma]) + 2 \cdot [1]$. Thus, the topological real realization of the complex $\tilde{C}_*^{\text{cell}}(\mathbb{A}^\infty - \{0\})$ is precisely the $\mathbb{Z}[\mathbb{Z}/2]$ -resolution above (observe that the topological real realizations of the \mathbf{K}_n^{MW} 's appearing in Theorem 2.46 are \mathbb{Z} with trivial $\mathbb{Z}/2$ -action).

Of course, it is also possible to interpret the above computations through a complex embedding by replacing \mathbb{G}_m with S^1 and \mathbf{H} with the singular homology group $H_*^{\text{sing}}(S^1)$. We leave the details to the reader.

As a consequence of Theorem 2.46, we obtain the oriented cellular \mathbb{A}^1 -chain complex of the projective space \mathbb{P}^n with the canonical oriented cellular structure described above.

Corollary 2.51. *For any integer $n \geq 1$, the oriented cellular \mathbb{A}^1 -chain complex $\tilde{C}_*^{\text{cell}}(\mathbb{P}^n)$ has the form*

$$\mathbf{K}_n^{\text{MW}} \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{i+1}} \mathbf{K}_i^{\text{MW}} \xrightarrow{\partial_i} \dots \rightarrow \mathbf{K}_1^{\text{MW}} \xrightarrow{\partial_1} \mathbb{Z}$$

with

$$\partial_i = \begin{cases} 0, & \text{if } i \text{ is odd; and} \\ \eta, & \text{if } i \text{ is even.} \end{cases}$$

It follows that

$$\mathbf{H}_i^{\text{cell}}(\mathbb{P}^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbf{K}_i^{\text{MW}}/\eta, & \text{if } i < n \text{ is odd;} \\ \eta \mathbf{K}_i^{\text{MW}}, & \text{if } i < n \text{ is even;} \end{cases}$$

and

$$\mathbf{H}_n^{\text{cell}}(\mathbb{P}^n) = \begin{cases} \mathbf{K}_n^{\text{MW}}, & \text{if } n \text{ is odd;} \\ \eta \mathbf{K}_n^{\text{MW}}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Since the oriented cellular structure of $\mathbb{A}^{n+1} - \{0\}$ is obtained by pulling back that of \mathbb{P}^n (also see Remark 2.47), we observe that $\tilde{C}_*^{\text{cell}}(\mathbb{P}^n)$ is the quotient of $\tilde{C}_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$ by the action of \mathbf{H} :

$$\tilde{C}_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\}) \otimes_{\mathbf{H}} \mathbb{Z} \cong \tilde{C}_*^{\text{cell}}(\mathbb{P}^n).$$

The corollary now follows by the description of the differentials in $\tilde{C}_*^{\text{cell}}(\mathbb{A}^{n+1} - \{0\})$ determined in Theorem 2.46. \square

Remark 2.52. Clearly, the Suslin homology version of the cellular \mathbb{A}^1 -chain complex of \mathbb{P}^n is the complex with differential 0 everywhere:

$$C_*^{\text{Sus,cell}}(\mathbb{P}^n) : \mathbf{K}_n^{\text{M}} \xrightarrow{0} \cdots \rightarrow \mathbf{K}_i^{\text{M}} \xrightarrow{0} \cdots \rightarrow \mathbf{K}_1^{\text{M}} \xrightarrow{0} \mathbb{Z},$$

whose homology is $\mathbf{H}_i^{\text{Sus,cell}}(\mathbb{P}^n) = \mathbf{K}_i^{\text{M}}$ for every i . Observe that the cellular Suslin homology of \mathbb{P}^n is much simpler than the Suslin homology of \mathbb{P}^n (which is unknown in general).

We end this section by giving an application of the abstract characterization of cellular \mathbb{A}^1 -chain complexes discussed in Section 2.4 and the above computations to degree of morphisms from $\mathbb{P}^n \rightarrow \mathbb{P}^n$.

Let n be an odd integer. For any morphism $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ in the \mathbb{A}^1 -homotopy category $\mathcal{H}(k)$, we may define its degree $\deg(f) \in \text{GW}(k) = \mathbf{K}_0^{\text{MW}}(k)$ as follows. By Lemma 2.40, one may define a canonical homotopy class of morphism of cellular \mathbb{A}^1 -chain complexes $[f]_*^{\text{cell}} : C_*^{\text{cell}}(\mathbb{P}^n) \rightarrow C_*^{\text{cell}}(\mathbb{P}^n)$ and $\mathbf{H}_n^{\text{cell}}(f)$ is an endomorphism of sheaves of $\mathbf{H}_n^{\text{cell}}(\mathbb{P}^n)$. Since n is odd, $\mathbf{H}_n^{\text{cell}}(\mathbb{P}^n) = \mathbf{K}_n^{\text{MW}}$ by Corollary 2.51. Since $\text{Hom}_{\text{Ab}_{\mathbb{A}^1}(k)}(\mathbf{K}_n^{\text{MW}}, \mathbf{K}_n^{\text{MW}}) = \mathbf{K}_0^{\text{MW}}(k) = \text{GW}(k)$, it follows that $\mathbf{H}_n^{\text{cell}}(f)$ defines an element of $\text{GW}(k)$.

Definition 2.53. The element of $\text{GW}(k)$ defined above for any morphism $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ for odd n in the \mathbb{A}^1 -homotopy category $\mathcal{H}(k)$ is denoted by $\deg(f)$ and is called the *degree* of f .

Remark 2.54. The possibility of defining the degree of f as above is related to the fact that \mathbb{P}^n , for n odd, is orientable. This construction can be easily generalized to other examples of cellular orientable smooth projective k -schemes, for instance obtained by convenient blow-up of \mathbb{P}^n 's.

3. CELLULAR \mathbb{A}^1 -CHAIN COMPLEXES ASSOCIATED WITH THE BRUHAT DECOMPOSITION

Unless otherwise specified, we will fix and use the following notations for the rest of the article. The base field will be denoted by k and will be assumed to be perfect.

G	a reductive (connected) algebraic group over k
G_{der}	the derived group of G (it is a normal, semisimple subgroup scheme of G with trivial radical)
G_{sc}	the simply connected central cover of G_{der} in the sense of algebraic groups
T	a split maximal torus of G
B	a Borel subgroup of G containing T
W	the Weyl group $N_G(T)/T$ of G
w_0	the element of longest length in W
X	the character group $\text{Hom}(T, \mathbb{G}_m)$ of T
Y	the cocharacter group $\text{Hom}(\mathbb{G}_m, T)$ of T
$\Phi \subset X$	the set of roots
Φ^+	the set of positive roots corresponding to B
Φ^-	the complement of Φ^+ in Φ
$\Delta \subset \Phi^+$	the set of simple roots (or a basis) of Φ^+
$\alpha^\vee \in Y$	the coroot corresponding to a root α ; it satisfies $\langle \alpha, \alpha^\vee \rangle = 2$
$\Phi^\vee \subset Y$	the set of coroots
ϖ_α	the dual of α^\vee corresponding to the inner product $\langle \cdot, \cdot \rangle$ (that is, the fundamental weight corresponding to α)

s_α	the element of W corresponding to the reflection with respect to α
U_α	the root subgroup corresponding to α
\mathfrak{u}_α	the tangent space of U_α at the neutral element (or the Lie algebra of U_α). (it is non-canonically isomorphic to \mathbb{G}_a , on which T acts through α)
U	the product of the U_α 's for $\alpha \in \Phi^+$ (in any order); B is the product TU
S_α	the subgroup of G generated by $\alpha^\vee(\mathbb{G}_m)$, U_α and $U_{-\alpha}$ ($S_\alpha \simeq \mathbf{SL}_2$ or \mathbf{PGL}_2 ; it is always $\simeq \mathbf{SL}_2$ if G is simply connected)
P_λ	the parabolic subgroup of G corresponding to a cocharacter λ .

The aim of this section is to explicitly describe a cellular structure on any split reductive group G over k and give some information about the associated cellular \mathbb{A}^1 -chain complex. We will freely use the standard results on the associated root datum (X, Φ, Y, Φ^\vee) and Bruhat decomposition (see [27, Chapter 7 and 8], for example). Recall that $T \subset G$ is a fixed split torus and $B \subset G$ a Borel subgroup containing T . For any $w \in W$ we choose a lifting \dot{w} of w in $N_G(T)$. The Bruhat decomposition of G is the partition

$$G = \coprod_{w \in W} B\dot{w}B$$

Observe that $B\dot{w}B$ doesn't depend on the choice of \dot{w} . This induces the stratification by open subsets

$$(3.1) \quad \emptyset = \Omega_{-1} \subsetneq \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_\ell = G,$$

where ℓ is the length of the longest element w_0 in W and for every i , one has

$$\Omega_i - \Omega_{i-1} = \coprod_{\ell(w)=\ell-i} B\dot{w}B$$

Another way to define Ω_i is to observe that

$$\Omega_i = \bigcup_{w \in W; \ell(w) \geq \ell-i} \dot{w}(\Omega_0)$$

It is well known (see below for a precise recollection) that the k -schemes $B\dot{w}B$ are isomorphic to a product of an affine space and T . Thus, these are affine, smooth and cohomologically trivial and consequently, we obtain a cellular structure on G .

Definition 3.1. For a split reductive (connected) algebraic k -group G , with fixed split maximal torus T and Borel subgroup $T \subset B \subset G$, we call the above cellular structure the *canonical* or *Bruhat cellular structure* on G .

For our computations, we will need to orient this cellular structure. We first treat the case where the semi-simple rank of G is one.

3.1. The rank 1 case.

Let G be a split semisimple group of rank 1, T be a split maximal torus of G and B be a Borel subgroup of G containing T . In this case, the Weyl group $W = N_G(T)/T = \langle s_\alpha \rangle \simeq \mathbb{Z}/2$ has order 2, where the generating reflection s_α corresponds to the positive simple root $\alpha : T \rightarrow \mathbb{G}_m$. We denote the corresponding coroot as usual by $\alpha^\vee : \mathbb{G}_m \rightarrow T$. We also denote by U_α and $U_{-\alpha}$ the root subgroups. Let $\dot{s}_\alpha \in N_G(T)$ be an element whose class in W is s_α .

The Bruhat decomposition of G has the form $G = B\dot{s}_\alpha B \amalg B$ and induces the increasing filtration of G by open subsets of the form:

$$\Omega_0 \hookrightarrow \Omega_1 = G$$

with $\Omega_0 = B\dot{s}_\alpha B = U_\alpha\dot{s}_\alpha TU_\alpha$. Its complement $G - \Omega_0$ with the reduced induced subscheme structure is B . The morphisms induced by the product in G

$$U_\alpha \times T \times U_\alpha \rightarrow B\dot{s}_\alpha B; \quad (\lambda, t, \mu) \mapsto \lambda\dot{s}_\alpha t\mu$$

and

$$T \times U_\alpha \rightarrow B, (t, \mu) \mapsto t\mu$$

are isomorphisms of k -schemes. We then observe that the open subscheme

$$\dot{s}_\alpha(\Omega_0) = U_{-\alpha}TU_\alpha$$

contains B ; it is thus an open neighborhood of $B = TU_\alpha$. Clearly, the normal bundle of $B \subset G$ is $\mathfrak{u}_{-\alpha} \times B$, where we let $\mathfrak{u}_{-\alpha}$ denote the tangent space of $U_{-\alpha}$ at 0, which is a 1-dimensional k -vector space (in other words, $\mathfrak{u}_{-\alpha}$ is the Lie algebra of $U_{-\alpha}$). This identification is independent of the choice of \dot{s}_α .

Giving a trivialization of the normal bundle of B in G is thus equivalent to giving an isomorphism $\mathbb{G}_a \cong U_{-\alpha}$ of k -groups, or equivalently, a nonzero element of $\mathfrak{u}_{-\alpha}$. By [11, exp. XX, Théorème 2.1], the root groups U_α and $U_{-\alpha}$ are canonically in duality; and consequently, giving an isomorphism $u_\alpha : \mathbb{G}_a \cong U_{-\alpha}$ is equivalent to giving an isomorphism $u_\alpha : \mathbb{G}_a \cong U_\alpha$. This constitutes a “pinning” of G in the sense of [11, XXIII §1 Épinglages], [10]). As G is the union of the two open subschemes Ω_0 and $\dot{s}_\alpha(\Omega_0)$, we get a canonical commutative diagram of smooth k -schemes and spaces

$$(3.2) \quad \begin{array}{ccccccc} \Omega_0 & \longrightarrow & G & \longrightarrow & G/\Omega_0 & \longrightarrow & S^1 \wedge ((\Omega_0)_+) \\ \uparrow & & \uparrow & & & & \\ \Omega_0 \cap \dot{s}_\alpha(\Omega_0) & \longrightarrow & \dot{s}_\alpha(\Omega_0) & & & & \end{array}$$

in which the top row is a cofiber sequence. Observe that this diagram is canonical in the sense that it does not depend on the choice of $\dot{s}_\alpha \in N_G(T)$. By the motivic homotopy purity theorem [24, Theorem 2.23, page 115], G/Ω_0 is canonically \mathbb{A}^1 -weakly equivalent to the Thom space of the vector bundle $U_{-\alpha} \times B$ over B , that is, $U_{-\alpha}/(U_{-\alpha}^\times) \wedge (B_+)$. Using the pinning and the fact that $B \rightarrow T$ is an \mathbb{A}^1 -weak equivalence, we see that G/Ω_0 is canonically weakly \mathbb{A}^1 -equivalent to $\mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \wedge (T_+) \cong S^1 \wedge \mathbb{G}_m \wedge (T_+)$. The cellular \mathbb{A}^1 -chain complex of G for the chosen (strict) orientation thus has the form

$$(3.3) \quad C_*^{\text{cell}}(G) : \mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) \otimes \mathbf{Z}_{\mathbb{A}^1}[T] \xrightarrow{\partial} \mathbf{Z}_{\mathbb{A}^1}[\dot{s}_\alpha T].$$

Now we explicitly compute the differential. At this point, in order to identify $\mathbf{Z}_{\mathbb{A}^1}[\dot{s}_\alpha T]$ with $\mathbf{Z}_{\mathbb{A}^1}[T]$, we need to specify an explicit choice of lifting of s_α . By [11, XX, 3.1], given a pinning u_α (and the corresponding $u_{-\alpha}$), the element

$$(3.4) \quad w_\alpha(X) := u_\alpha(X)u_{-\alpha}(-X^{-1})u_\alpha(X)$$

belongs to $N_G(T)$ and is a lifting of s_α , for every unit X in k .

Convention 3.2. We will take $X = -1$, and $\dot{s}_\alpha = w_\alpha(-1)$ for our choice of the lift of $s_\alpha \in W$ to $N_G(T)$ in what follows.

Example 3.3. For $G = \mathbf{SL}_2$ and its standard pinning, we have $\dot{s}_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

With the choice of \dot{s}_α as in Convention 3.2, the isomorphism $\dot{s}_\alpha T \cong T$ of smooth k -schemes induces an isomorphism of sheaves between $\mathbf{Z}_{\mathbb{A}^1}[\dot{s}_\alpha T]$ and $\mathbf{Z}_{\mathbb{A}^1}[T]$. It is easy to see that $\Omega_0 \cap \dot{s}_\alpha(\Omega_0) \subset \dot{s}_\alpha(\Omega_0) = U_{-\alpha} T U_\alpha$ is exactly the open subscheme $U_{-\alpha}^\times T U_\alpha$.

Lemma 3.4. *In diagram (3.2), the vertical inclusion*

$$U_{-\alpha}^\times T U_\alpha \cong \Omega_0 \cap \dot{s}_\alpha(\Omega_0) \subset \Omega_0 = U_\alpha \dot{s}_\alpha T U_\alpha$$

is the right TU_α -equivariant morphism induced by the morphism:

$$U_{-\alpha}^\times \rightarrow \Omega_0; \quad \lambda \mapsto \lambda^{-1} \cdot \dot{s}_\alpha \cdot \alpha^\vee(\lambda) \cdot \lambda^{-1}.$$

Proof. By [11], there is a unique morphism of k -groups $\Phi_{u_\alpha} : \mathbf{SL}_2 \rightarrow G$ which is compatible with the pinnings on both sides. Recall that in \mathbf{SL}_2 , the root subgroup U_α is the subgroup of matrices of the form

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

and $U_{-\alpha}$ is that of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.$$

The induced morphism $\mathbb{G}_m \rightarrow T$ is then the coroot

$$\alpha^\vee : \mathbb{G}_m \rightarrow \mathbf{SL}_2; \quad u \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

and the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is sent precisely to \dot{s}_α by Φ_{u_α} . Thus, it suffices to prove the lemma for the standard pinning on \mathbf{SL}_2 . The proof is a straightforward computation and is left to the reader. \square

Remark 3.5.

- (1) The proof of Lemma 3.4 is sensitive to the choice of \dot{s}_α made in Convention 3.2. One may verify that choosing $w_\alpha(1)$ instead of $w_\alpha(-1)$ leads to an inclusion up to a sign.
- (2) Observe also that a change of the pinning of G amounts to multiplying u_α by a unit ρ (and $u_{-\alpha}$ by ρ^{-1}). Consequently, it amounts to multiplying $\dot{s}_\alpha = w_\alpha(-1)$ also by the image of ρ in T under $\alpha^\vee : \mathbb{G}_m \rightarrow T$.
- (3) Choosing a pinning u_α of G induces by the formula (3.4) for $X = -1$ a choice of a lift \dot{s}_α in $N_G(T)$. We observe that the converse holds. Choose a lift \dot{s}_α in $N_G(T)$. We see that the rational map $\dot{x} : G \dashrightarrow \mathbb{P}^1$ corresponding to the projection $\Omega_0 = U_\alpha \dot{s}_\alpha T U_\alpha \rightarrow T$ followed by the dual character $\varpi : T \rightarrow \mathbb{G}_m$ of α^\vee can be used to define a pinning using the rational map

$$\hat{\gamma} = \frac{\dot{x}}{\dot{x}(\dot{s}_\alpha)} : G \dashrightarrow \mathbb{P}^1,$$

which induces an isomorphism $U_\alpha \cong \mathbb{A}^1 \subset \mathbb{P}^1$. In the case of $G = \mathbf{SL}_2$, this rational map is given by $\frac{b}{a}$, where the elements of \mathbf{SL}_2 are written $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. One easily verifies that this defines a bijection between the sets

$$\{\text{Lifts of } s_\alpha \text{ in } N_G(T)\} \xrightarrow{\cong} \{\text{Pinnings of } G\}$$

with inverse sending u_α to \dot{s}_α . Observe also that both these sets admit an action of k^\times , which is preserved by the above bijection.

Using Lemma 3.4, it follows that the cellular \mathbb{A}^1 -chain complex (3.3) of G has the form

$$(3.5) \quad \mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) \otimes \mathbf{Z}_{\mathbb{A}^1}[T] \xrightarrow{\partial} \mathbf{Z}_{\mathbb{A}^1}[\dot{s}_\alpha T]$$

with the differential ∂ given by the composition

$$\mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) \otimes \mathbf{Z}_{\mathbb{A}^1}[T] \subset \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m \times T] \xrightarrow{\alpha^\vee \times \text{Id}_T} \mathbf{Z}_{\mathbb{A}^1}[T \times T] \xrightarrow{\mu} \mathbf{Z}_{\mathbb{A}^1}[T] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[\dot{s}_\alpha T],$$

where μ is induced by the product $\mu : T \times T \rightarrow T$ and the rightmost arrow is the isomorphism of sheaves induced by the product on the left with \dot{s}_α . We will use this formula several times.

Example 3.6. We now assume that G is simply connected; in other words, $\Phi_{u_\alpha} : \mathbf{SL}_2 \xrightarrow{\sim} G$ is an isomorphism. Then the coroot $\alpha^\vee : \mathbb{G}_m \cong T$ is also an isomorphism. Under the identifications $\mathbf{Z}_{\mathbb{A}^1}[T] = \mathbf{K}_1^{\text{MW}} \oplus \mathbb{Z}$ and $\mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) \otimes \mathbf{Z}_{\mathbb{A}^1}[T] = \mathbf{K}_2^{\text{MW}} \oplus \mathbf{K}_1^{\text{MW}}$, the cellular \mathbb{A}^1 -chain complex of G with respect to the pinning of G given by u_α is isomorphic to

$$(3.6) \quad C_*^{\text{cell}}(\mathbf{SL}_2) : \mathbf{K}_2^{\text{MW}} \oplus \mathbf{K}_1^{\text{MW}} \xrightarrow{\begin{pmatrix} \eta & 0 \\ \text{Id} & 0 \end{pmatrix}} \mathbf{K}_1^{\text{MW}} \oplus \mathbb{Z},$$

because μ is the direct sum of η and the identity. As a consequence, we obtain

$$\mathbf{H}_0^{\text{cell}}(G) = \mathbf{H}_0^{\mathbb{A}^1}(G) = \mathbb{Z}$$

and

$$\mathbf{H}_1^{\text{cell}}(G) = \mathbf{H}_1^{\mathbb{A}^1}(G) = \mathbf{K}_2^{\text{MW}},$$

where \mathbf{K}_2^{MW} is identified as the subsheaf of $\mathbf{K}_2^{\text{MW}} \oplus \mathbf{K}_1^{\text{MW}}$ given by the image of the morphism

$$\mathbf{K}_2^{\text{MW}} \rightarrow \mathbf{K}_2^{\text{MW}} \oplus \mathbf{K}_1^{\text{MW}}; \quad \alpha \mapsto (\alpha, -\eta\alpha).$$

Example 3.7. Now assume that G is not simply connected; in other words, $\Phi_{u_\alpha} : \mathbf{SL}_2 \rightarrow G$ has kernel μ_2 and induces an isomorphism $\mathbf{PGL}_2 \cong G$. Then the root $\alpha : T \cong \mathbb{G}_m$ is an isomorphism and the coroot $\alpha^\vee : \mathbb{G}_m \rightarrow T \cong \mathbb{G}_m$ is the squaring map. In this case, the cellular \mathbb{A}^1 -chain complex of G is given by

$$(3.7) \quad C_*^{\text{cell}}(\mathbf{SL}_2) : \mathbf{K}_2^{\text{MW}} \oplus \mathbf{K}_1^{\text{MW}} \xrightarrow{\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}} \mathbf{K}_1^{\text{MW}} \oplus \mathbb{Z},$$

because the morphism $\mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) \rightarrow \mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m)$ induced by the squaring map $\mathbb{G}_m \rightarrow \mathbb{G}_m$ is multiplication by $h = 1 + \langle -1 \rangle \in \mathbf{K}_0^{\text{MW}}(k)$, (see [23, Lemma 3.14 p. 55]) and $\eta h = 0$. Therefore, in this case we obtain

$$\mathbf{H}_0^{\text{cell}}(G) = \mathbf{H}_0^{\mathbb{A}^1}(G) = \mathbb{Z} \oplus \mathbf{K}_1^{\text{MW}}/h = \mathbf{K}_0^{\text{MW}},$$

since $\mathbf{K}_1^{\text{MW}}/h = \mathbf{I}$, the fundamental ideal sheaf in \mathbf{K}_0^{MW} and

$$\mathbf{H}_1^{\mathbb{A}^1}(G) \rightarrow \mathbf{H}_1^{\text{cell}}(G) = \mathbf{K}_2^{\text{MW}} \oplus {}_h\mathbf{K}_1^{\text{MW}}.$$

Observe that as $G \cong \mathbf{PGL}_2$ is not \mathbb{A}^1 -connected, $\mathbf{H}_1^{\mathbb{A}^1}(G)$ does not agree with $\pi_1^{\mathbb{A}^1}(G)$. Observe also that $\pi_0^{\mathbb{A}^1}(\mathbf{PGL}_2) = \mathbf{K}_1^{\text{M}}/2$ and that indeed $\mathbf{H}_0^{\mathbb{A}^1}(\mathbf{PGL}_2)$ is the free strictly \mathbb{A}^1 -invariant sheaf on the sheaf of sets $\mathbf{K}_1^{\text{M}}/2$ (see [23, Theorem 3.46 p. 77]).

Remark 3.8.

- (1) The above description of the oriented cellular \mathbb{A}^1 -chain complex of G depends on the choice of a pinning of G , since we used the explicit lift of s_α in $N_G(T)$ defined in Convention 3.2 to identify $\mathbf{H}_0^{\mathbb{A}^1}(\Omega_0)$ with $\mathbf{Z}_{\mathbb{A}^1}[\dot{s}_\alpha T]$. If we multiply the standard pinning of \mathbf{SL}_2 by $u \in k^\times$, the element w_α becomes $\begin{pmatrix} 0 & -u \\ u^{-1} & 0 \end{pmatrix}$ and the identification $\dot{s}_\alpha T$ with T has to be modified by multiplication by u . Now, an easy computation using the results of [23] shows that multiplication by a square on T induces the identity isomorphism on $\mathbf{Z}_{\mathbb{A}^1}[T]$. Thus, it follows that the above description of the oriented cellular \mathbb{A}^1 -chain complex of G in rank 1 only depends on the choice of a pinning “up to square”. This corresponds to an orientation of the normal bundle of $B \subset G$ as opposed to its trivialization.
- (2) However, if one chooses an orientation of the normal bundle of $B \subset G$ and a lifting of s_α in $N_G(T)$ which is not given by the above formula, the oriented cellular \mathbb{A}^1 -chain complex will a priori be different (although isomorphic).
- (3) The automorphisms of \mathbf{SL}_2 coming from the action of \mathbf{PGL}_2 are given, for every $\gamma \in k^\times$ by

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto \begin{pmatrix} a & \gamma^{-1}c \\ \gamma b & d \end{pmatrix}.$$

Such an automorphism only preserves the oriented cellular structure if γ is a square. In general, this automorphism is a cellular (auto)morphism and the induced automorphism of the oriented cellular \mathbb{A}^1 -chain complex is the multiplication by $\langle \gamma \rangle$ on $\mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) = \mathbf{K}_1^{\text{MW}}$ (coming from the orientation).

- (4) The above computations can be used to describe the cellular \mathbb{A}^1 -chain complex of a split reductive k -group of semisimple rank 1. Let $\text{rad}(G)$ denote the radical of G , which is a (split) subtorus of T contained in the center of G . The oriented cellular \mathbb{A}^1 -chain complex of G_{der} with the chosen pinning has the form

$$\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[T] \xrightarrow{\partial} \mathbf{Z}_{\mathbb{A}^1}[\dot{s}_\alpha T]$$

We know that the intersection $\text{rad}(G) \cap G_{\text{der}}$ is a finite subgroup of the center of G_{der} . If G_{der} is adjoint (that is, isomorphic to \mathbf{PGL}_2), then the center of G_{der} is trivial and G is the product group $\text{rad}(G) \times G_{\text{der}}$. If G_{der} is simply connected (that is, isomorphic to \mathbf{SL}_2), then there are two cases. If $\text{rad}(G) \cap G_{\text{der}}$ is trivial, then again $G = \text{rad}(G) \times G_{\text{der}}$ as a group. If $\text{rad}(G) \cap G_{\text{der}} = \mu_2 = Z(\mathbf{SL}_2)$, then G is isomorphic to $\text{rad}(G) \times_{\mu_2} G_{\text{der}}$ as a group. Observe that $\mathbb{G}_m \times_{\mu_2} \text{rad}(G) \rightarrow T$ is an isomorphism of groups, since μ_2 is also the intersection of \mathbb{G}_m (the maximal torus in G_{der} given by the coroot) and $\text{rad}(G)$. Thus $G_{\text{der}} \times_{\mathbb{G}_m} T \rightarrow G$ is an isomorphism of k -schemes with right and left T -action. Now, if one chooses a split torus T' such that $T = \mathbb{G}_m \times T'$ as group, then $G_{\text{der}} \times T' \rightarrow G$ is an isomorphism of k -schemes with right and left T -action. In any case, the cellular \mathbb{A}^1 -chain complex of G is just the tensor product of the cellular \mathbb{A}^1 -chain complex of G_{der} and the group ring of a split torus. In fact, one may assume only that G is a connected affine group with reductive quotient split of semi-simple rank 1 in order to determine its cellular \mathbb{A}^1 -chain complex.

- (5) The differential

$$\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[T] \xrightarrow{\partial} \mathbf{Z}_{\mathbb{A}^1}[\dot{s}T]$$

is right and left T -equivariant. The right action of T on the factor \mathbf{K}_1^{MW} is trivial, and is the obvious one on the other factor, and the left action of T on \mathbf{K}_1^{MW} is through

the character $\alpha : T \rightarrow \mathbb{G}_m$, and is the obvious one on $\mathbf{Z}_{\mathbb{A}^1}[\dot{s}T]$. If u is a unit (in T), then the action of u on \mathbf{K}_1^{MW} is the multiplication by $\langle u \rangle : \mathbf{K}_1^{\text{MW}} \rightarrow \mathbf{K}_1^{\text{MW}}$.

3.2. The general case.

In this section, we explain how a weak pinning (see Definition 3.17) gives an orientation of the cellular \mathbb{A}^1 -chain complex of a split, semisimple algebraic group. These results play an important role in orienting the cellular \mathbb{A}^1 -chain complex of a generalized flag variety, a detailed study of which will be taken up in the sequel (also see Section 5.3). Our proof of the main theorem will use Proposition 3.13 and Lemma 3.15, which rely on the differential calculus on G (Lemma 3.10).

Throughout this subsection, unless otherwise specified, we will assume that G is a split semisimple group of rank r over k . We fix a pinning $(T, B, \{u_\alpha\}_{\alpha \in \Delta})$ of G (see [11, XXIII §1 Épinglages], [10]), which consists of

- a maximal k -split torus T of G ;
- a Borel subgroup B of G containing T ; and
- an isomorphism $u_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ of the root groups, for every simple root $\alpha \in \Delta$,

where Δ is the basis of the root system of G determined by the choice of T and B . It will be convenient to fix an ordering $\{\alpha_1, \dots, \alpha_r\}$ of Δ . It follows from [11, XXII Théorème 1.1] that we get for each $\alpha \in \Delta$ an explicit lift \dot{s}_α in $N_G(T)$ of $s_\alpha \in W$ (also see Convention 3.2). We will also need, for every element $v \in W$, a choice of a reduced expression (as product of simple reflections s_α , where $\alpha \in \Delta$). We denote then by $\dot{v} \in N_G(T)$ the product in the same ordering of the lifts \dot{s}_α of the reflections appearing in the chosen reduced expression of v . For an element $v \in W$, we set

$$U_v = \prod_{\alpha \in \Phi_v} U_\alpha,$$

where $\Phi_v = \{\alpha \in \Phi^+ \mid v(\alpha) \in \Phi^-\}$. Observe that with our notation, U_{w_0} is the big cell Ω_0 of G . We also know that the product morphism

$$(3.8) \quad U_v U_{w_0 v} \cong U$$

is always an isomorphism of k -schemes [27, Lemma 8.3.5]. Recall from [27, Chapter 8 Section 3] that for an element $w \in W$, the Bruhat cell $B\dot{w}B$, which we will simply denote by BwB because of its independence on the choice of the lift $\dot{w} \in N_G(T)$ of $w \in W$, is isomorphic as a k -scheme through the product in G with

$$U_{w^{-1}} \times T \times U \xrightarrow{\sim} B\dot{w}B; \quad (u, t, u') \mapsto u\dot{w}tu'.$$

Hence, we will write $BwB = U_{w^{-1}}\dot{w}TU$. The above isomorphism shows that as a k -scheme, the Bruhat cell $B\dot{w}B$ is isomorphic to a product of an affine space and T ; thus, it is cohomologically trivial and the above filtration defines a cellular structure on G . In what follows, we need to compute the differential of the associated cellular \mathbb{A}^1 -chain complex in low degrees (≤ 2 to be precise). We will hence have to choose orientations of that cellular structure carefully in order to facilitate our cellular homology computations.

We need some recollection on “differential calculus” on G to describe the normal bundles involved in the cellular \mathbb{A}^1 -chain complex.

Convention 3.9. Given a vector bundle \mathcal{E} (that is, a locally free sheaf of \mathcal{O}_X -modules) on a k -scheme X , we will denote its underlying total space by $\mathbb{V}(\mathcal{E}) = \text{Spec}(\text{Sym}_k^*(\mathcal{E}^\vee))$. We will often abuse the notation and call $\mathbb{V}(\mathcal{E})$ a vector bundle on X , when the meaning is clear from the context.

Recall that the tangent bundle T_G (dual to $\Omega_{G/k}^1$) of G is trivial as G is a smooth algebraic group over k . Let $\mathfrak{g} := T_{G,e}$ be the tangent space of G at e (that is, the stalk of T_G at e ; this is the Lie algebra of G). Then the morphism

$$(3.9) \quad \theta : \mathbb{V}(\mathfrak{g}) \times G \rightarrow \mathbb{V}(T_G); \quad (v, g) \mapsto (d(? \times g)_e(v), g)$$

and the induced morphism $\mathfrak{g} \otimes_k \mathcal{O}_G \rightarrow T_G$ are isomorphisms of vector bundles over G . Here $? \times g : G \rightarrow G$ is the right multiplication by g and $d(? \times g)_e : \mathfrak{g} \rightarrow T_{G,g}$ denotes its differential at e . One easily checks that this isomorphism is G -right invariant, for the trivial G action on \mathfrak{g} on the left and the natural G -action on the right. However, θ is not left G -invariant. For $g \in G$, one may verify (using $(g \times ?) \circ (? \times h) = (? \times h) \circ (g \times ?)$, for $h \in G$) that $\theta^{-1} \circ d(g \times ?) \circ \theta$ is the constant automorphism of vector bundles $\mathbb{V}(\mathfrak{g}) \times G \xrightarrow{\cong} \mathbb{V}(\mathfrak{g}) \times G$ induced by the adjoint morphism

$$\text{Ad}(g) : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g},$$

which is the differential at e of the conjugation map $h \mapsto ghg^{-1}$. In this way one may easily prove the following fact, the proof of which is left to the reader.

Lemma 3.10. *Let $\mu : G \times G \rightarrow G$ be the product morphism. Using our identification $\mathbb{V}(T_G) = \mathbb{V}(\mathfrak{g}) \times G$, and thus $\mathbb{V}(T_{G \times G}) = \mathbb{V}(\mathfrak{g} \oplus \mathfrak{g}) \times G \times G$, the differential*

$$d(\mu) : \mathbb{V}(\mathfrak{g} \oplus \mathfrak{g}) \times G \times G \rightarrow \mathbb{V}(\mathfrak{g}) \times G$$

at $(h, g) \in G \times G$ (with obvious meaning) is the k -linear morphism

$$(\text{Id}_{\mathfrak{g}}, \text{Ad}(h)) : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathfrak{g}.$$

Let $w \in W$ and set $v := w_0 w \in W$; we then also have $w_0 v = w$, since w_0 is an element of order 2. We are going to define a canonical tubular neighborhood of BwB in G . We consider the open subset $\dot{v}(\Omega_0) \subset G$ obtained by left translation of Ω_0 by \dot{v} . Using (3.8), it follows that

$$\dot{v}(\Omega_0) = (\dot{v}U_v\dot{v}^{-1}) \cdot (\dot{v}U_{w_0v}\dot{v}^{-1}) \cdot \dot{v}\dot{w}_0TU.$$

Clearly, $\dot{v}\dot{w}_0$ is a lift of w in $N_G(T)$, so that $\dot{v}\dot{w}_0TU = \dot{w}TU$.

Lemma 3.11. *With the above assumptions and notation, we have:*

$$\begin{aligned} (a) \quad & \dot{v}U_{w_0v}\dot{v}^{-1} = U_{w^{-1}}; \\ (b) \quad & \dot{v}U_v\dot{v}^{-1} = \prod_{\beta \in \Phi^- | w(\beta) \in \Phi^-} U_\beta. \end{aligned}$$

Consequently,

$$(3.10) \quad \dot{v}(\Omega_0) = \left(\prod_{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-} U_\beta \right) \cdot (U_{w^{-1}}\dot{w}TU) = \left(\prod_{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-} U_\beta \right) \cdot BwB$$

is an open subset of G only depending on $w \in W$ and containing $BwB = U_{w^{-1}}\dot{w}TU$.

Proof.

(a) By [27, Exercise 8.1.12.(2)], for any root α , we have $\dot{v}U_\alpha\dot{v}^{-1} = U_{v(\alpha)}$. Thus, one has

$$\dot{v}U_{w_0v}\dot{v}^{-1} = \prod_{\alpha \in \Phi_{w_0v}} U_{v(\alpha)}.$$

Now, $\alpha \in \Phi_{w_0v}$ if and only if $\alpha \in \Phi^+$ and $w_0(v(\alpha)) \in \Phi^-$. Since $w_0(\Phi^-) = \Phi^+$, this is equivalent to $\alpha \in \Phi^+$ and $v(\alpha) \in \Phi^+$. But $\alpha = v^{-1}(v(\alpha)) = w_0w^{-1}(v(\alpha)) \in \Phi^+$. Thus, such α 's are precisely those in Φ^+ such that $v(\alpha) \in \Phi^+$ and $w^{-1}(v(\alpha)) \in \Phi^-$. This means that $v(\alpha) \in \Phi_{w^-}$. But the cardinality of Φ_{w_0v} is the length of $w_0v = w$, and so also of w^{-1} . This proves (a).

(b) Since $\dot{v}U_v\dot{v}^{-1} = \prod_{\alpha \in \Phi^+ | v(\alpha) \in \Phi^-} U_{v(\alpha)}$, setting $\beta = v(\alpha)$ one easily obtains

$$\{\alpha \in \Phi^+ | v(\alpha) \in \Phi^-\} = v^{-1}\{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-\},$$

using $v^{-1} = w^{-1}w_0$ and (b) follows.

The formula (3.10) is a straightforward consequence of (a) and (b). \square

Remark 3.12.

(1) Let $\alpha \in \Phi$. We denote by $\mathfrak{g}_\alpha \subset \mathfrak{g}$ the image of the differential at $1 \in G$ of the morphism $U_\alpha \rightarrow G$. This differential induces an isomorphism of 1-dimensional k -vector spaces $\mathfrak{u}_\alpha \cong \mathfrak{g}_\alpha$ and it is well known that $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the rank one weight k -vector subspace of \mathfrak{g} corresponding to the character α . This leads to the decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \oplus \mathfrak{h}$, where \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} (which is the Lie algebra or the tangent space of T).

(2) The normal bundle ν_w of BwB in G is constant in the sense that it is of the form $\nu_w = \mathfrak{v}_w \otimes_k \mathcal{O}_G$. But observe that the canonical epimorphism

$$T_G|_{BwB} \rightarrow \nu_w$$

is not constant! It can be made explicit at any point of BwB by using Lemma 3.10 and the standard formulas for the adjoint morphism acting on \mathfrak{g} .

Proposition 3.13. *We keep the above notation and assumptions. Let $w \in W$ be an element of the Weyl group of G .*

(a) *If ν_w denotes the normal bundle of BwB in G , then the composition*

$$\bigoplus_{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-} \mathfrak{u}_\beta \hookrightarrow \mathfrak{g}|_{BwB} \rightarrow \mathfrak{v}_w$$

is an isomorphism.

(b) *The right B -action on G induces through the above isomorphism the trivial action on the normal bundle of BwB in G (and the obvious right action on the base).*

(c) *The left B -action on G induces the following action on the normal bundle of BwB in G : the adjoint action of T (through $B \rightarrow T$) on the k -vector space $\bigoplus_{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-} \mathfrak{u}_\beta$ (and the obvious left action on the base).*

Proof. Consider the open subset

$$\prod_{\alpha \in \Phi^+ | v(\alpha) \in \Phi^-} U_{v(\alpha)} \times BwB$$

as a tubular neighborhood of $B\dot{w}B$ and observe that the morphism of smooth k -schemes induced by taking the product in G :

$$\left(\prod_{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-} U_\beta \right) \times B\dot{w}B \rightarrow \left(\prod_{\alpha \in \Phi^+ | v(\alpha) \in \Phi^-} U_{v(\alpha)} \right) \times B\dot{w}B$$

is an isomorphism. For every point $x \in B\dot{w}B$, the above tubular neighborhood induces, by taking the differential at the point $(0, x)$, the isomorphism

$$\left(\bigoplus_{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-} \mathfrak{u}_\beta \right) \oplus (T_G|_{B\dot{w}B})_x \cong \mathfrak{g}|_{B\dot{w}B},$$

given by the sum of the canonical inclusions (the canonical inclusion of the first summand being given by $\bigoplus_{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-} \mathfrak{u}_\beta \cong \bigoplus_{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-} \mathfrak{g}_\beta \subset \mathfrak{g}$). This proves (a) and the parts (b), (c) are easy consequences of (a). \square

3.2.1. Orientations in codimension 1.

Let $w \in W$ be an element of length $\ell - 1$, where ℓ is the length of the longest element $w_0 \in W$. There is a unique $\alpha_i \in \Delta$ with $w = w_0 s_i$. If α'_i denotes the simple root $w_0(-\alpha_i) \in \Delta$, then we have $w = s'_i w_0$, where $s'_i \in W$ is the reflection corresponding to α'_i . We claim that

$$(3.11) \quad s'_i = w_0 s_i w_0^{-1} = w_0 s_i w_0.$$

Set $\sigma_i := w_0 s_i w_0^{-1} = w_0 s_i w_0$. Indeed, for $w \in W$, the set $\Phi_w = \{\alpha \in \Phi^+ | w(\alpha) \in \Phi^-\}$ has $\ell(w)$ elements, by [27, Lemma 8.3.2 (ii)]. If $\beta \in \Phi^+ - \{\alpha'_i\}$, then $w_0(\beta) \in \Phi^- - \{-\alpha_i\}$ and $s_i w_0(\beta) \in \Phi^-$ so that $\sigma_i(\beta) \in \Phi^+$. If $\beta = \alpha'_i$, then $w_0(\beta) = -\alpha_i$ and $\sigma_i(\beta) = -\alpha'_i$, so that $\Phi_{\sigma_i} = \{\alpha'_i\}$. Hence, $\sigma_i = s'_i$ as claimed.

For $i \in \{1, \dots, r\}$, let us denote by w_i the product $w_0 s_i$ and $Y_i := B\dot{w}_i B$ the corresponding codimension 1-cell. In that case, the open neighborhood of Y_i obtained in Lemma 3.11 is $U_{-\alpha'_i} Y_i$. It follows that we get an isomorphism

$$\left(\mathfrak{u}_{-\alpha'_i} \otimes \mathcal{O}_{Y_i} \right) \oplus T_{Y_i} \cong T_G|_{Y_i},$$

which is determined by the isomorphism $\mathfrak{u}_{-\alpha'_i} \cong \mathfrak{g}_{-\alpha'_i}$ on the first factor (followed by the canonical inclusion) and the canonical inclusion $T_{Y_i, y} \subset \mathfrak{g}$ for each $y \in Y_i$ on the second factor. This isomorphism induces the above isomorphism

$$(3.12) \quad \nu_{Y_i} \xrightarrow{\cong} \mathfrak{u}_{-\alpha'_i} \otimes_k \mathcal{O}_{Y_i}.$$

Observe here that one may also consider the product $Y_i U_{-\alpha_i}$ for an open neighborhood of Y_i . We get in that case an isomorphism of vector bundles

$$T_{Y_i} \oplus (\mathfrak{u}_{-\alpha_i} \otimes \mathcal{O}_{Y_i}) \cong T_G|_{Y_i},$$

which, for every $y \in Y_i$, is the canonical inclusion $T_{Y_i, y} \subset \mathfrak{g}$ on the first factor and induced by the morphism $Ad(y) : \mathfrak{u}_{-\alpha_i} \subset \mathfrak{g}$. This isomorphism induces also an isomorphism

$$(3.13) \quad \nu_{Y_i} \xrightarrow{\cong} \mathfrak{u}_{-\alpha_i} \otimes_k \mathcal{O}_{Y_i},$$

which does not agree in general with (3.12).

Remark 3.14. In fact one may generalize the above observations and prove the analogues of Lemma 3.11 and Proposition 3.13 for multiplying on the right of the Bruhat cell $B\dot{w}B$. In that case, one would see that the left B -action induces the trivial action on the normal bundle of the cell.

Comparing the isomorphisms (3.12) and (3.13) above, we get an isomorphism

$$(3.14) \quad \mathbf{u}_{-\alpha_i} \xrightarrow{\cong} \mathbf{u}_{-\alpha'_i}$$

of k -vector spaces inducing the isomorphism of line bundles $\mathbf{u}_{-\alpha'_i} \otimes_k \mathcal{O}_{Y_i} \cong \mathbf{u}_{-\alpha_i} \otimes_k \mathcal{O}_{Y_i}$ over Y_i which we now describe. Observe first that this isomorphism is canonical and does not depend on any choices, except α_i , which also determines $\alpha'_i = w_0(-\alpha_i)$.

Lemma 3.15. *With the above assumptions and notation, let $y_i : Y_i \rightarrow \mathbb{G}_m$ denote the invertible regular function obtained by composing the canonical projection $Y_i \rightarrow T$ and the character $\alpha_i : T \rightarrow \mathbb{G}_m$. Then the isomorphism (3.14) is the composition of the multiplication $\mathbf{u}_{-\alpha_i} \xrightarrow{\cong} \mathbf{u}_{-\alpha_i}$ by $(y_i)^{-1}$, the isomorphism $Ad(\dot{s}_i) : \mathbf{u}_{-\alpha_i} \xrightarrow{\cong} \mathbf{u}_{\alpha_i}$ and $Ad(\dot{w}_0) : \mathbf{u}_{\alpha_i} \xrightarrow{\cong} \mathbf{u}_{-\alpha'_i}$.*

Proof. The automorphism group of a line bundle over Y_i is the group of units $\mathcal{O}(Y_i)^\times$ of Y_i . We need to compute the unit in $\mathcal{O}(Y_i)^\times$ corresponding to the automorphism (3.12)⁻¹ \circ (3.13) of ν_{Y_i} . Observe that the closed immersion $\dot{w}_0\dot{s}_iT \subset Y_i$ induces an isomorphism on the group of invertible functions. Now, we explicitly compare the two isomorphisms $T_{Y_i} \oplus (\mathbf{u}_{-\alpha_i} \otimes_k \mathcal{O}_{Y_i}) \cong T_G|_{Y_i}$ and $(\mathbf{u}_{-\alpha'_i} \otimes_k \mathcal{O}_{Y_i}) \oplus T_{Y_i} \cong T_G|_{Y_i}$, corresponding to (3.12) and (3.13) respectively, on restriction to $\dot{w}_0\dot{s}_iT$. On the summand T_{Y_i} , both are given by the obvious inclusion, as well as on the summand $\mathbf{u}_{-\alpha'_i}$. On the other hand, on the summand $\mathbf{u}_{-\alpha_i}$ it is given by $Ad(y)$, with $y \in Y_i$ the point at which it is computed. Setting $y = \dot{w}_0\dot{s}_it \in \dot{w}_0\dot{s}_iT$, we get $Ad(y) = Ad(\dot{w}_0) \circ Ad(\dot{s}_i) \circ (\alpha_i(t))^{-1}$ and the lemma follows. \square

Remark 3.16.

- (1) Observe that the isomorphism $Ad(\dot{s}_i) : \mathbf{u}_{-\alpha_i} \xrightarrow{\cong} \mathbf{u}_{\alpha_i}$ can be explicitly described. Indeed, the choice of the pinning of G uniquely determines an isomorphism $u_\alpha : \mathbb{G}_a \cong U_\alpha$. Now by [11, XX, Théorème 2.1], we have a canonical duality between U_α and $U_{-\alpha}$, so that in that case we get an isomorphism $U_{-\alpha} \xrightarrow{\cong} U_\alpha$ and in [11, XXIII, §1.2], it is proved that $Ad(\dot{s}_i)$ is exactly the inverse to this isomorphism.
- (2) Observe also that, although the isomorphism (3.14) between $\mathbf{u}_{-\alpha_i}$ and $\mathbf{u}_{-\alpha'_i}$ is canonical, its description in Lemma 3.15 does depend on the pinning and the choices of reduced expressions made at the beginning! Indeed, the invertible function $x_{\alpha_i} : Y_i \rightarrow \mathbb{G}_m$ on $Y_i = U\dot{w}_0\dot{s}_iT U$ obtained by pulling back $\alpha_i : T \rightarrow \mathbb{G}_m$ by the natural morphism $Y_i \rightarrow \dot{w}_iT \xrightarrow{\cong} T$ depends on the choices of the lifting of $w_i = w_0s_i$, which is precisely cancelled by $Ad(\dot{w}_0) \circ Ad(\dot{s}_i)$.

3.2.2. Orientations for a general Bruhat cell.

It follows from the previous discussion, that to define an orientation of the cellular structure on G in the sense of Definition 2.11, is equivalent to give, for each $w \in W$, an orientation of the k -vector space

$$(3.15) \quad \mathbf{v}_w := \bigoplus_{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-} \mathbf{u}_\beta.$$

In codimension one, as we have seen in the previous example, it exactly amounts to give an orientation of each of the $\mathbf{u}_{-\alpha}$, $\alpha \in \Delta$. Now by [11, XXII Théorème 1.1] \mathbf{u}_α and $\mathbf{u}_{-\alpha}$ are canonically dual to each other, for $\alpha \in \Phi$. Thus, an orientation for the Bruhat cellular structure on G in codimension 1 is the same thing as a pinning of G up to the multiplication of each u_α by squares of units (for each α).

Definition 3.17. Consider the equivalence relation on the set of pinnings of G defined as follows: two pinnings $(T, B, \{u_\alpha\}_{\alpha \in \Delta})$ and $(T', B', \{u'_\alpha\}_{\alpha \in \Delta'})$ are equivalent if and only if $T = T'$, $B = B'$ and u_α is a multiple of u'_α by an element of $k^{\times 2}$ for every $\alpha \in \Delta = \Delta'$. We will call an equivalence class of pinnings a *weak pinning* of G .

Given a pinning $(T, B, \{u_\alpha\}_\alpha)$ of G , one has the canonical lift \dot{s}_α of s_α in $N_G(T)$ for each simple root α (see Convention 3.2). Using the formula 3.4, one sees that multiplying the u_α by squares in k^\times multiplies the element \dot{s}_α also by a square in T (by which we mean an element of $\prod_{i=1}^r k^{\times 2}$). Now, fix for each $v \in W$ a reduced expression

$$v = \sigma_1 \circ \cdots \circ \sigma_{\ell(v)}$$

of v as a product of reflections σ_i associated to simple roots in Δ . Then the product (in the given ordering) $\prod_i \dot{\sigma}_i$ is a lift of v in $N_G(T)$, which we denote by \dot{v} .

Let $w \in W$, and consider $v := ww_0$ so that we have $w = vw_0$. Clearly,

$$\{\beta \in \Phi^- | w^{-1}(\beta) \in \Phi^-\} = \{\beta \in \Phi^- | v^{-1}(\beta) \in \Phi^+\} = -\Phi_{v^{-1}}$$

Thus, for any w we can rewrite the isomorphism (3.15) concerning the normal bundle of $B\dot{w}B$ as

$$\mathbf{v}_w \cong \bigoplus_{\alpha \in \Phi_{v^{-1}}} \mathbf{u}_{-\alpha}.$$

By [27, Chapter 8 Section 3], if $v = \sigma_1 \circ \cdots \circ \sigma_\lambda$ is a reduced expression of v and if β_j is the simple root corresponding to σ_j , then

$$\Phi_{v^{-1}} = \{\beta_1, \sigma_1(\beta_2), \dots, \sigma_1 \circ \cdots \circ \sigma_{\ell(v)-2}(\beta_{\ell(v)})\}.$$

Thus with our assumptions and choices, we get an constant isomorphism of constant vector bundles

$$\bigoplus_{i=1}^{\ell(v)} \mathbb{G}_a \cong \nu_w \cong \bigoplus_{\alpha \in \Phi_{v^{-1}}} \mathbf{u}_{-\alpha} \otimes_k \mathcal{O}_{B\dot{w}B}$$

induced by the direct sum of the isomorphisms of vector spaces

$$k \xrightarrow{u_{-\alpha_i}} \mathbf{u}_{-\alpha_i} \xrightarrow{\sim} \mathbf{u}_{-\sigma_1 \circ \cdots \circ \sigma_{i-1}(\alpha_i)}$$

given by the pinning followed by the conjugation by $\dot{\sigma}_1 \circ \cdots \circ \dot{\sigma}_{i-1}$. The associated orientation is easily seen to be only dependent on the weak pinning of G and the reduced expression of v . This shows that a weak pinning of G and a choice of a reduced expression of v for each $v \in W$ define an orientation of the Bruhat cellular structure of G . We will call this orientation *standard orientation* associated to the weak pinning and the choice of reduced expressions. Observe that an element of W of length ≤ 2 admits a unique reduced expression and consequently, there are no choices involved in that case. We assume henceforth that a choice of a weak pinning of G and for each $v \in W$ a choice of a reduced expression of v is made.

Let $w \in W$. The Bruhat cell $B\dot{w}B$ does not depend on any of the above choices, but in order to make computations with the oriented cellular \mathbb{A}^1 -chain complex of G we need to describe the corresponding summand in $\tilde{C}_*^{\text{cell}}(G)$ explicitly. The normal bundle is already oriented by our above choices; this summand in $\tilde{C}_i^{\text{cell}}(G)$ where $i = \ell(w_0) - \ell(w)$ is the codimension of $B\dot{w}B$ in G is

$$\mathbf{K}_i^{\text{MW}} \otimes \mathbf{H}_0^{\mathbb{A}^1}(B\dot{w}B)$$

Assume now that we are given a reduced expression of w , which is not necessarily the one previously used to orient the normal bundles of the Bruhat cells. We then get a lift $\tilde{w} \in N_G(T)$ of w and have

$$B\dot{w}B = B\tilde{w}B = U\tilde{w}TU$$

so that $\mathbf{H}_0^{\mathbb{A}^1}(B\dot{w}B) = \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}T]$. We will use this freedom later in the proof of the main theorem, to get, at least in degree ≤ 2 a canonical description of $\tilde{C}_*^{\text{cell}}(G)$ in degree ≤ 2 depending only on a weak pinning of G and a choice of a reduced expression of w_0 .

Remark 3.18. The obvious isomorphism of sheaves induced by the multiplication on the left by \tilde{w}

$$\mathbf{Z}_{\mathbb{A}^1}[T] \cong \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}T]$$

only depends on both the chosen reduced expression of w and the weak pinning. Indeed, $\mathbf{Z}_{\mathbb{A}^1}[T]$ is isomorphic to $\mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] \otimes_{\mathbb{A}^1} \cdots \otimes_{\mathbb{A}^1} \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m]$ (r factors), and (as in the rank one case above), multiplication by squares (in \mathbb{G}_m) induces the identity on each factor of the tensor product. Observe that our choice of the lift \tilde{w} does depend (up to squares) on the weak pinning as the lifts \tilde{s}_i for the simple roots do, and of course also on the chosen reduced expression of w .

For each $i \in \{1, \dots, \ell\}$ the above choices and conventions thus define an isomorphism:

$$\tilde{C}_i^{\text{cell}}(G) \cong \bigoplus_{w \in W; \ell(w) = \ell - i} \mathbf{K}_i^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}T]$$

where the factor $\mathbf{K}_i^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}T]$ corresponds to the normal bundle of $BwB := B\dot{w}B = B\tilde{w}B$. Again, one should notice that the identification of the factor $\mathbf{K}_i^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}T]$ not only depends on the orientation of the normal bundle of BwB , which uses a reduced expression of $v := w_0w$, but also on the choice of a reduced expression of w itself, giving the lifting \tilde{w} . In dimension 0 for instance, observe that the writing $\tilde{C}_0^{\text{cell}}(G) \cong \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_0T]$ depends on a choice of a reduced expression for w_0 and the weak pinning. On the other hand, note that the identification $\tilde{C}_\ell^{\text{cell}}(G) = \mathbf{K}_\ell^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[B]$ corresponding to the unique cell of maximal codimension ℓ in G (that is, the Borel subgroup B), one uses another reduced expression for w_0 .

Remark 3.19. In order to explicitly describe the oriented cellular \mathbb{A}^1 -chain complex $\tilde{C}_*^{\text{cell}}(G)$, we need a weak pinning of G and for any element of W two choices of reduced expressions: the “normal expression” and the “horizontal expression”. For $w \in W$, one gets an orientation of the normal bundle of the cell BwB by using a the normal reduced expression for $v = w_0w$ and the identification $\mathbf{H}_0^{\mathbb{A}^1}(BwB) = \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}T]$ using the horizontal reduced expression of w . The lift of $w \in W$ in $N_G(T)$ corresponding to the normal reduced expression of w will be denoted by \dot{w} and the one corresponding to the horizontal reduced expression will be denoted by \tilde{w} .

Remark 3.20.

- (1) Observe that the structure of a right $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module on $\tilde{C}_i^{\text{cell}}(G)$ is the obvious one. The right action of T on the factors \mathbf{K}_i^{MW} is trivial, and is the obvious one on the other factors. The structure of left $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module is the obvious one on $\mathbf{Z}_{\mathbb{A}^1}[\dot{w}T]$ and is given on \mathbf{K}_i^{MW} as the tensor product of the characters $\alpha : T \rightarrow \mathbb{G}_m$ of T corresponding to the roots involved in the reduced expression of the v 's used; this follows from Proposition 3.13.
- (2) The cellular \mathbb{A}^1 -chain complex $C_*^{\text{cell}}(G)$ is also a complex of left and right $\mathbf{Z}_{\mathbb{A}^1}[B]$ -modules, but observe that in fact $\mathbf{Z}_{\mathbb{A}^1}[B] = \mathbf{Z}_{\mathbb{A}^1}[T]$. Given a weak pinning of G and choice of reduced expressions for elements of W , the oriented cellular structure on G is B -right invariant by (1) and induces an oriented cellular structure on G/B (which is in fact strictly cellular) as well. The morphism $G \rightarrow G/B$ is thus automatically oriented cellular in the obvious way and induces an isomorphism of chain complexes $\tilde{C}_*^{\text{cell}}(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \cong \tilde{C}_*^{\text{cell}}(G/T)$.
- (3) It is clear by construction that the differential in the complex $\tilde{C}_*^{\text{cell}}(G)$ restricted on the summand $\mathbf{K}_i^{\text{MW}} \otimes_{\mathbf{Z}_{\mathbb{A}^1}}[\dot{w}T] \subset \tilde{C}_i^{\text{cell}}(G)$ corresponding to BwB (where $\ell(w) = \ell - i$) has its image contained in the direct sum

$$\bigoplus_{w < v; \ell(v) = \ell - i + 1} \mathbf{K}_{i-1}^{\text{MW}} \otimes_{\mathbf{Z}_{\mathbb{A}^1}}[\dot{w}T],$$

where \leq denotes the Bruhat ordering, with $w \leq v$ meaning $BwB \subset \overline{BvB}$ (see, for example, [27, §8.5.4]).

- (4) It is possible to choose a pinning of G so that the lift of an element $w \in W$ defined as above using a reduced expression of w does not depend on the reduced expression. However it seems impossible to find a (weak) pinning such that the orientation of each normal bundle of the Bruhat cell BwB , as defined above, only depends on w and not on the chosen reduced expression! This may be verified explicitly for SL_3 for the normal bundle of B .

Given a weak pinning of G and a choice of normal and horizontal reduced expressions for elements of W , there is a canonical description of the oriented cellular \mathbb{A}^1 -chain complex of G . We will have to use it explicitly in low degrees. In degree 0 one has:

$$\tilde{C}_0^{\text{cell}}(G) = \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_0T]$$

It is known that codimension 1 cells BwB correspond exactly to words $w \in W$ of length $\ell - 1$, which are exactly of the form $w_i := w_0s_i = s'_i w_0$ for some $i \in \{1, \dots, r\}$. Thus, one has

$$\tilde{C}_1^{\text{cell}}(G) = \bigoplus_{i=0}^r \mathbf{K}_1^{\text{MW}} \otimes_{\mathbf{Z}_{\mathbb{A}^1}}[\tilde{w}_iT]$$

Let us analyze now the oriented chain complex of G in degree 2, which corresponds to the codimension 2 Bruhat cells. Each word $w \in W$ of length $\ell - 2$ can be written as $w_0s_i s_j$ for some $i \neq j$ as $\ell(w_0w) = 2$ by [3, Proposition 1.77]. We will assume here for simplicity that the Dynkin diagram of G is connected.

Notation 3.21. For a pair of indices (i, j) let us write $|i - j|$ for the distance between α_i and α_j in the Dynkin diagram of G . If $|i - j| \geq 2$ (that is, there is no edge connecting α_i and α_j in the Dynkin diagram), then $w_0s_i s_j = w_0s_j s_i$ (as s_i and s_j commute with each other). If $|i - j| = 1$, then necessarily $w_0s_i s_j \neq w_0s_j s_i$ (recall that we assume that the Dynkin diagram is connected). We will write $[i, j]$ for the class of (i, j) modulo the following equivalence relation

on $\{1, \dots, r\}^2$: $(i, j) \equiv (j, i)$ if and only if $|i - j| \geq 2$ or $i = j$. We will also denote by $w_{[i,j]}$ the word $w_0 s_i s_j$.

With the above notation, we get

$$\tilde{C}_2^{\text{cell}}(G) = \bigoplus_{[i,j]} \mathbf{K}_2^{\text{MW}} \otimes \mathbb{Z}[\tilde{w}_{[i,j]}T]$$

The right $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module structure of $\tilde{C}_2^{\text{cell}}(G)$ is the obvious one on $\mathbb{Z}[\tilde{w}_{[i,j]}T]$ and trivial on each of the \mathbf{K}_2^{MW} factors. The left $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module structure is, as described in Remark 3.20, the obvious one on the factor $\mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{[i,j]}T]$, taking into account the formula for $t \in T$: $t \cdot \dot{w} = \dot{w}w^{-1}(t)$, and the left action of T on the factor \mathbf{K}_*^{MW} is through the tensor product of the characters, in the given ordering, associated to the simple roots appearing in writing the reduced expression of $v = ww_0$, which here has length ≤ 2 .

3.3. The free strictly \mathbb{A}^1 -invariant sheaf on a split torus.

We end this section by describing the structure of the sheaf $\mathbf{Z}_{\mathbb{A}^1}[T]$ for a split torus T over k , understanding of which is clearly central to all our computations.

Convention 3.22. For u a unit (in some field extension F of k), the associated symbol in $\mathbf{K}_1^{\text{MW}}(F)$ is denoted by $(u) \in \mathbf{K}_1^{\text{MW}}(F)$. By abuse of notation, we will often drop F from the notation and simply write $(u) \in \mathbf{K}_1^{\text{MW}}$ etc. Recall that T is k -split torus of rank r and is pointed by 1. For $t \in T$, we will denote by $[t] \in \mathbf{Z}_{\mathbb{A}^1}[T]$ the image of the basis element of $\mathbb{Z}[T]$ through the morphism $\mathbb{Z}[T] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[T]$. We will denote by $(t) \in \mathbf{Z}_{\mathbb{A}^1}[T]$ the element $[t] - [1]$. Observe that if $T = \mathbb{G}_m$, then $\mathbf{Z}_{\mathbb{A}^1}[T] = \mathbb{Z} \oplus \mathbf{K}_1^{\text{MW}}$ and for $t \in T$ the element (t) just defined is in \mathbf{K}_1^{MW} and equals the corresponding symbol, making the two notations compatible.

Choose an isomorphism

$$\prod_{i=1}^r \alpha_i^\vee : \prod_{i=1}^r \mathbb{G}_m \xrightarrow{\sim} T.$$

In the situation of our main theorem, the above isomorphism is given by the simple coroots, since T is a maximal k -split torus of G , which is simply connected. For $t \in T$, we let $t_i \in \mathbb{G}_m$ be its i -th component through the above isomorphism; so in other words,

$$t_i = \varpi_i(t)$$

where the $\{\varpi_i\}_i$ denote the fundamental weights, or the basis of $\text{Hom}(T, \mathbb{G}_m)$ dual to $\{\alpha_i\}_i$.

Notice that given the group homomorphism $\alpha_i^\vee : \mathbb{G}_m \rightarrow T$, we have $\alpha_i^\vee(u) \in T$ for a unit u , which corresponds to the element $[\alpha_i^\vee(u)]$ of $\mathbf{Z}_{\mathbb{A}^1}[T]$. The symbol $(\alpha_i^\vee(u))$ denotes the element $[\alpha_i^\vee(u)] - [1]$ of $\mathbf{Z}_{\mathbb{A}^1}[T]$. The morphism

$$(\alpha_i^\vee(-)) : \mathbb{G}_m \rightarrow \mathbf{Z}_{\mathbb{A}^1}[T]; \quad u \mapsto (\alpha_i^\vee(u)) \in \mathbf{Z}_{\mathbb{A}^1}[T]$$

extends automatically to a morphism of sheaves $\alpha_i^\vee(-) : \mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) = \mathbf{K}_1^{\text{MW}} \rightarrow \mathbf{Z}_{\mathbb{A}^1}[T]$ because $\mathbf{Z}_{\mathbb{A}^1}[T]$ is strictly \mathbb{A}^1 -invariant.

Notation 3.23. For the sake of brevity, we simply denote the morphism $\mathbf{K}_1^{\text{MW}} \rightarrow \mathbf{Z}_{\mathbb{A}^1}[T]$ induced by α_i^\vee by

$$u \mapsto (u)_i := (\alpha_i^\vee(u)) = [\alpha_i^\vee(u)] - [1],$$

where u is a unit (in a field extension F of k). For $t \in T$, we let $[t]_i \in \mathbf{Z}_{\mathbb{A}^1}[T]$ denote the image of $[\varpi_i(t)] = 1 + (\varpi_i(t)) \in \mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) = \mathbb{Z} \oplus \mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m)$ through the morphism $\mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) \rightarrow \mathbf{Z}_{\mathbb{A}^1}[T]$ induced by the coroot α_i^\vee .

Since the ring $\mathbf{Z}_{\mathbb{A}^1}[T]$ is commutative, so for v a unit (in the field extension F) and $i \neq j$, one has in $\mathbf{Z}_{\mathbb{A}^1}[T](F)$:

$$(u)_i(v)_j = (v)_j(u)_i.$$

In case $i = j$ we have further:

$$(u)_i(v)_i = (\eta(u)(v))_i$$

where $\eta : \mathbf{K}_2^{\text{MW}} \rightarrow \mathbf{K}_1^{\text{MW}}$ is the multiplication by η [23]. This follows easily from the fact that in the ring $\mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] = \mathbb{Z} \oplus \mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m) = \mathbb{Z} \oplus \mathbf{K}_1^{\text{MW}}$ the product of (u) and (v) is given by $\eta(u)(v)$, which follows from the very definition of η . The isomorphism

$$(3.16) \quad \bigotimes_{i=1}^r \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] \cong \mathbf{Z}_{\mathbb{A}^1}[T]$$

as a sheaf of rings is induced by taking the product in $\mathbf{Z}_{\mathbb{A}^1}[T]$

$$(3.17) \quad \mathbb{Z} \oplus \left(\bigoplus_{1 \leq i_1 < \dots < i_s \leq r} \mathbf{K}_s^{\text{MW}} \right) \xrightarrow{\cong} \mathbf{Z}_{\mathbb{A}^1}[T]$$

$$(u_1) \cdots (u_s) \mapsto (u_1)_{i_1} \cdots (u_s)_{i_s}$$

is an isomorphism of sheaves. If $t = (u_1, \dots, u_r) \in T = \prod_{i=1}^r \mathbb{G}_m$, then the following equality holds in $\mathbf{Z}_{\mathbb{A}^1}[T]$ through the above isomorphism:

$$(t) = [t_1, \dots, t_r] - [1] = \bigoplus_{1 \leq i_1 < \dots < i_s \leq r} (t_{i_1}) \cdots (t_{i_s}).$$

For example, in the case $r = 2$, $\mathbb{G}_m \times \mathbb{G}_m = T$ and if $t = (u, v) \in \mathbb{G}_m \times \mathbb{G}_m$, then we have

$$(t) = (u)_1 + (v)_2 + (u)_1(v)_2.$$

Set $\mathbf{H}_T := \mathbf{Z}_{\mathbb{A}^1}[T]$ and for each i , let $\mathbf{H}_i \cong \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m]$ denote the sub-Hopf algebra of \mathbf{H}_T given by the image of the morphism $\mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] \rightarrow \mathbf{H}_T$ induced by $\alpha_i^\vee : \mathbb{G}_m \rightarrow T$. Thus $\mathbf{H}_T = \bigotimes_{i=1}^r \mathbf{H}_i$ as a Hopf algebra. Now, define

$$(3.18) \quad \phi_i : \mathbf{K}_2^{\text{MW}} \rightarrow \mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}_i; \quad (u, v) \mapsto (u) \otimes (v)_i - \eta(u)(v) \otimes [1],$$

where $(u)_i \in \mathbf{H}_i$ denotes the element $[u] - [1]$ of $\mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] \cong \mathbf{H}_i$. Thus for each i , we have the obvious exact sequence

$$(E_i) : \quad 0 \rightarrow \mathbf{K}_2^{\text{MW}} \xrightarrow{\phi_i} \mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}_i \rightarrow \mathbf{H}_i \rightarrow \mathbb{Z} \rightarrow 0,$$

where \cdot is (induced by) the product. This exact sequence, considered as a chain complex $(E_i)_*$ with \mathbb{Z} in degree -1 , is contractible as each of its terms is a projective object in $Ab_{\mathbb{A}^1}(k)$. It follows that the tensor product $C_* := \bigotimes_{i=1}^r (E_i)_*$ is also contractible and hence, acyclic. So we get an exact sequence

$$(3.19) \quad C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow \mathbb{Z} \rightarrow 0,$$

where $C_{-1} = \mathbb{Z}$, $C_0 = \bigotimes_{i=1}^r \mathbf{H}_i = \mathbf{H}$,

$$C_1 \cong \left(\bigoplus_{i=1}^r \mathbf{K}_1^{\text{MW}} \right) \otimes \left(\bigotimes_{i=1}^r \mathbf{H}_i \right) \cong \bigoplus_{i=1}^r \mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}$$

and the differential $\tilde{\partial}_1 : C_1 \rightarrow C_0$ on restriction to the i th summand is given by the \mathbf{H} -module homomorphism

$$\mathbf{K}_1^{\text{MW}} \otimes \mathbf{H} \rightarrow \mathbf{H}; \quad (u) \otimes [t] \mapsto [\alpha_i^\vee(u)t],$$

for every i . Thus, the previous contractible chain complex produces a presentation of $\tilde{Z}_1 := \text{Ker } \tilde{\partial}_1$ (by right \mathbf{H} -modules):

$$C_3 \rightarrow C_2 \rightarrow \tilde{Z}_1 \rightarrow 0$$

Thus, it follows that the induced exact sequence

$$C_3 \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \rightarrow C_2 \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \rightarrow \tilde{Z}_1 \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \rightarrow 0$$

is a presentation of $\tilde{Z}_1 \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$. Since the complexes (E_i) are exact sequences of \mathbf{H}_i -modules, we have

$$C_* \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \cong \bigotimes_{i=1}^r (E_i \otimes_{\mathbf{H}_i} \mathbb{Z})_*.$$

Now, for every i , the complex

$$(E_i)_* \otimes_{\mathbf{H}_i} \mathbb{Z} : 0 \rightarrow \mathbf{K}_2^{\text{MW}} \xrightarrow{-\eta} \mathbf{K}_1^{\text{MW}} \xrightarrow{0} \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$$

is (isomorphic to) the augmented cellular \mathbb{A}^1 -chain complex of \mathbb{P}^2 . Consequently, $C_* \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$ is (isomorphic to) the cellular \mathbb{A}^1 -chain complex of $(\mathbb{P}^2)^r$. It follows easily that the morphism

$$C_3 \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \rightarrow C_2 \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$$

is the morphism

$$\left(\bigoplus_{(i,j,m); i < j < m} \mathbf{K}_3^{\text{MW}} \right) \oplus \left(\bigoplus_{(i,j); i \neq j} \mathbf{K}_3^{\text{MW}} \right) \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & \bigoplus_{(i,j)} (\pm\eta) \end{pmatrix}} \left(\bigoplus_i \mathbf{K}_2^{\text{MW}} \right) \oplus \left(\bigoplus_{(i,j); i < j} \mathbf{K}_2^{\text{MW}} \right),$$

where i, j, s run through $\{1, \dots, r\}$. Going through the previous construction, it is easy to identify the isomorphism just constructed:

Lemma 3.24. *The isomorphism*

$$\left(\bigoplus_i \mathbf{K}_2^{\text{MW}} \right) \oplus \left(\bigoplus_{(i,j); i < j} \mathbf{K}_2^{\text{M}} \right) \xrightarrow{\cong} \tilde{Z}_1 \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$$

is induced by the following morphisms: for each i , its restriction to the i -th \mathbf{K}_2^{MW} is ϕ_i from (3.18) composed with the inclusions $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}_i \subset \mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}_j \subset C_1$. For each (i, j) with $i \neq j$, its restriction to the \mathbf{K}_2^{M} corresponding to (i, j) is

$$(u, v) \mapsto (u) \otimes (v)_i - (v) \otimes (u)_j \in (\mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}_i) \oplus (\mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}_j) \subset C_1.$$

4. THE ORIENTED CELLULAR \mathbb{A}^1 -CHAIN COMPLEX OF A SPLIT, SEMISIMPLE, SIMPLY CONNECTED ALGEBRAIC GROUP IN LOW DEGREES

In this section we prove our main theorem, which determines the sheaf $\mathbf{H}_1^{\text{cell}}(G)$, where G is a split, semisimple, almost simple, simply connected algebraic group over k of rank r . Thus, the Dynkin diagram of G is reduced irreducible with r vertices. We choose an ordering $\{\alpha_1, \dots, \alpha_r\}$ of the set Δ of simple roots. For each i , we write $w_i := w_0 s_i$, and for $i \neq j$ we write $w_{[ij]} := w_0 s_i s_j$. Recall also that for each i we let s'_i be the reflection (corresponding to the simple root $w_0(-\alpha_i)$) such that $s'_i w_0 = w_i = w_0 s_i$.

We briefly recall our conventions that will be used throughout this section. We fix the choice of a weak pinning of G and for each $w \in W$ a reduced expression as a product of the s_α 's, where $\alpha \in \Delta$ runs through the set of simple roots. For $w \in W$ we denote by \dot{w} the lift

of w in $N_G(T)$ given by the above formula using the elements \dot{s}_α given by the (weak)-pinning and using the chosen reduced expression of w . Usually, we will simply write BwB instead of $B\dot{w}B$ and $\mathbf{Z}_{\mathbb{A}^1}[wT]$ instead of $\mathbf{Z}_{\mathbb{A}^1}[\dot{w}T]$. Recall that $C_*^{\text{cell}}(G)$ is a complex of right and left $\mathbf{Z}_{\mathbb{A}^1}[T]$ -modules (see Remark 3.20).

Since G is simply connected, the right (and left) action of $\mathbf{Z}_{\mathbb{A}^1}[T]$ on $\mathbf{H}_*^{\text{cell}}(G)$ is trivial, as it extends to an action of $\mathbf{H}_0^{\text{cell}}(G) = \mathbb{Z}$ as G is \mathbb{A}^1 -connected. The main idea of our main computation is that the exact sequence

$$C_2^{\text{cell}}(G) \rightarrow Z_1(G) \rightarrow \mathbf{H}_1^{\text{cell}}(G) \rightarrow 0$$

induces an exact sequence

$$(4.1) \quad C_2^{\text{cell}}(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \rightarrow Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \rightarrow \mathbf{H}_1^{\text{cell}}(G) \rightarrow 0,$$

by the above observation.

In the Section 4.1 we describe the sheaf $Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$ and in the Section 4.2 we describe the morphism $C_2^{\text{cell}}(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \rightarrow Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$ induced by the differential. The main theorem will then follow directly from these computations.

4.1. The differential in degree 1.

We fix a reduced expression of the longest element

$$(4.2) \quad w_0 = \sigma_\ell \cdots \sigma_1$$

in the Weyl group of G as a product of simple reflections. Recall that (see Remark 3.19) in order to make computations with the oriented cellular \mathbb{A}^1 -chain complex of G , we need to choose a “normal” reduced expression \dot{w} and a “horizontal” reduced expression \tilde{w} for every element $w \in W$. For every $w \in W$, we have already fixed a choice of the normal reduced expression \dot{w} at the beginning of this section. We need to fix the horizontal reduced expressions. We begin by recalling a few known facts about the Weyl group of G , which will be used.

Lemma 4.1. *Let $w \in W$ and let $\sigma_q \cdots \sigma_1$ be a reduced expression of w . Let $v \in W$ with $v < w$ for the Bruhat order and such that $\ell(v) = \ell(w) - 1 = q - 1$. Then there is a unique $\lambda \in \{1, \dots, \ell(w)\}$ such that the product*

$$\sigma_q \cdots \widehat{\sigma}_\lambda \cdots \sigma_1 := \sigma_q \cdots \sigma_{\lambda+1} \sigma_{\lambda-1} \cdots \sigma_1$$

is a reduced expression of v .

Proof. The existence of λ follows from the exchange condition [3, (E) p.79 and 3.59, p.137]. If $\mu \in \{1, \dots, q\}$ were such that $\sigma_\ell \cdots \widehat{\sigma}_\lambda \cdots \sigma_1 = \sigma_q \cdots \widehat{\sigma}_\mu \cdots \sigma_1$, with say $\lambda > \mu$, then it follows after simplification and multiplication on the right with σ_μ that

$$\sigma_{\lambda-1} \cdots \sigma_{\mu+1} = \sigma_\lambda \cdots \sigma_\mu$$

contradicting the minimality of the reduced expression we started with. \square

Remark 4.2. There is no generalization of Lemma 4.1 for the case where $\ell(v) = \ell(w) - 2$. For example, in the Weyl group of type A_2 (so $G \cong \mathbf{SL}_3$), the longest word is $w_0 = s_1 s_2 s_1$ (also equal to $s_2 s_1 s_2$) and there are two different ways to get s_1 by removing two simple reflections. For the Weyl group of type C_2 (so $G \cong \mathbf{Sp}_4$), the longest word has the form $w_0 = s_2 s_1 s_2 s_1$ (also equal to $s_1 s_2 s_1 s_2$) and there are two different ways to get $s_2 s_1$ by removing two reflections.

Lemma 4.3. *Let $w \in W$ be of length q and let $s, s' \in W$ be simple reflections such that $s'w = ws$ and $\ell(ws) = q + 1$. Let α and α' the simple roots corresponding to s and s' , respectively. Then*

$$w(\alpha) = \alpha'.$$

Proof. Let $\sigma_q \cdots \sigma_1$ be a reduced expression of w . The equation $s'w = ws$, that is, $s' = wsw^{-1}$ shows that $w(\alpha) = \pm\alpha'$. If $w(\alpha) = -\alpha'$, then $\alpha \in \Phi_w$. From [27, Chapter 8 Section 3] we have

$$\Phi_w = \{\beta_1, \sigma_1(\beta_2), \dots, \sigma_1 \circ \cdots \circ \sigma_{q-1}(\beta_q)\},$$

where β_1, \dots, β_q are the simple roots corresponding to the $\sigma_1, \dots, \sigma_q$, respectively. Thus, there exists $j \in \{0, \dots, q-1\}$ with $\alpha = \sigma_1 \circ \cdots \circ \sigma_j(\beta_{j+1})$. This implies that in W

$$s = \sigma_1 \cdots \sigma_j \sigma_{j+1} (\sigma_1 \cdots \sigma_j)^{-1}$$

but then

$$ws = \sigma_q \cdots \sigma_{j+2} \sigma_j \cdots \sigma_1$$

which contradicts that the length of ws is $q + 1$. Therefore, we must have $w(\alpha) = \alpha'$. \square

Remark 4.4.

- (1) Observe that Lemma 4.3 (and its proof) is still valid if one replaces s by a reflection in W , that is to say of the form vsv^{-1} for some $v \in W$ and a simple reflection s such that the root $v(\alpha)$ is positive (with α the simple root corresponding to s).
- (2) Observe that Lemma 4.3 does not contradict the equation $w_0(-\alpha_i) = \alpha'_i$ that was already used in Section 3. Indeed, $w_0 s_i = s'_i w_0$; however, $\ell(s'_i w_0) \neq \ell(w_0) + 1$.

We now fix the horizontal reduced expressions for Bruhat cells in codimension ≤ 1 .

Notation 4.5. For the chosen reduced expression $w_0 = \sigma_\ell \cdots \sigma_1$ of w_0 as in (4.2), we set

$$\dot{w}_0 = \tilde{w}_0 = \dot{\sigma}_\ell \cdots \dot{\sigma}_1 \in N_G(T)$$

using the chosen weak pinning of G . We denote by β_μ the simple root corresponding to σ_μ . For $i \in \{1, \dots, r\}$, let $\lambda_i \in \{1, \dots, \ell\}$ be given by Lemma 4.1 applied to $w_i := w_0 s_i$. Define $w''_i := \sigma_\ell \cdots \sigma_{\lambda_i+1}$ and $w'_i := \sigma_{\lambda_i-1} \cdots \sigma_1$ so that the chosen reduced expression (4.2) of w_0 has the form

$$w_0 = w''_i \sigma_{\lambda_i} w'_i.$$

We then take $w''_i w'_i$ for the horizontal reduced expression of $w_i = w_0 s_i$ and set

$$\tilde{w}_i := \dot{w}''_i \dot{w}'_i$$

to be the corresponding lift to $N_G(T)$.

In particular, for the cellular \mathbb{A}^1 -chain complex of G with the orientation described in Section 3.2, we will use the following identifications: $\tilde{C}_0^{\text{cell}}(G) = \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_0 T]$ and $\tilde{C}_1^{\text{cell}}(G) = \bigoplus_{i=1}^r \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T]$.

We now determine the differential $\partial_1 : \tilde{C}_1^{\text{cell}}(G) \rightarrow \tilde{C}_0^{\text{cell}}(G)$. We will continue to use the notation used in Section 3 (more precisely, Section 3.2.1 regarding the Bruhat cellular structure on G).

We use the above horizontal reduced expressions to identify the factor $\mathbf{H}_1^{\mathbb{A}^1}(Th(\nu_{Y_i}))$ of $C_1^{\text{cell}}(G)$ corresponding to Y_i in the following way. We set $\bar{\Omega}_{0i} := \Omega_0 \cup Y_i$, where $\Omega_0 = Bw_0B$

and $Y_i := Bw_iB$. It is easy to see that $\overline{\Omega}_{0i}$ is an open subset in G and the product in G induces an isomorphism of smooth k -schemes

$$\Omega_\ell \times_B \cdots \times_B \Omega_{\lambda_{i+1}} \times_B P_{\lambda_i} \times_B \Omega_{\lambda_{i-1}} \times_B \cdots \times_B \Omega_1 \xrightarrow{\cong} \overline{\Omega}_{0i},$$

where P_{λ_i} is the parabolic subgroup corresponding to the root β_{λ_i} . Indeed, P_{λ_i} is the union of its big cell Ω_{λ_i} and its translate $\dot{\sigma}_{\lambda_i}(\Omega_{\lambda_i})$, so that the above iterated fiber product is the union of the two open subschemes $\Omega_\ell \times_B \cdots \times_B \Omega_{\lambda_{i+1}} \times_B \Omega_{\lambda_i} \times_B \Omega_{\lambda_{i-1}} \times_B \cdots \times_B \Omega_1$ and $\Omega_\ell \times_B \cdots \times_B \Omega_{\lambda_{i+1}} \times_B \dot{\sigma}_{\lambda_i}(\Omega_{\lambda_i}) \times_B \Omega_{\lambda_{i-1}} \times_B \cdots \times_B \Omega_1$ and the morphism induced by the product is clearly a bijection onto $\overline{\Omega}_{0i}$ (using the Bruhat decomposition) and its restriction on each of the previous open subschemes is an isomorphism onto the an open subscheme of $\overline{\Omega}_{0i}$: its big cell $\Omega_\ell \dots \Omega_{\lambda_i} \dots \Omega_1$ and its translate $\dot{\Sigma} \cdot \Omega_\ell \dots \Omega_{\lambda_i} \dots \Omega_1$ with $\dot{\Sigma} = (\dot{\sigma}_\ell \cdots \dot{\sigma}_{\lambda_{i+1}}) \cdot \dot{\sigma}_{\lambda_i} \cdot (\dot{\sigma}_\ell \cdots \dot{\sigma}_{\lambda_{i+1}})^{-1}$. For every $\mu \in \{1, \dots, \ell\}$, the big cell in the parabolic subgroup

$$P_\mu = B\sigma_\mu B \amalg B$$

corresponding to β_μ is given by

$$\Omega_\mu = U_{\beta_\mu} \sigma_\mu B = B\sigma_\mu B = B\sigma_\mu U_{\beta_\mu}.$$

Using these descriptions, we get an isomorphism

$$\overline{\Omega}_{0i} \cong U_{\beta_\ell} \dot{\sigma}_\ell \cdots U_{\beta_{\lambda_{i+1}}} \dot{\sigma}_{\lambda_{i+1}} P_{\lambda_i} \dot{\sigma}_{\lambda_{i-1}} U_{\beta_{\lambda_{i-1}}} \cdots \dot{\sigma}_1 U_{\beta_1}$$

as well as the isomorphisms

$$\Omega_0 = \Omega_\ell \times_B \cdots \times_B \Omega_1 \cong U_{\beta_\ell} \dot{\sigma}_\ell \cdots U_{\beta_1} \dot{\sigma}_1 TU$$

and

$$\begin{aligned} Y_i &= \Omega_\ell \times_B \cdots \times_B \Omega_{\lambda_{i+1}} \times_B B \times_B \Omega_{\lambda_{i-1}} \times_B \cdots \times_B \Omega_1 \\ &\cong U_{\beta_\ell} \dot{\sigma}_\ell \cdots U_{\beta_{\lambda_{i+1}}} \dot{\sigma}_{\lambda_{i+1}} B \dot{\sigma}_{\lambda_{i-1}} U_{\beta_{\lambda_{i-1}}} \cdots \dot{\sigma}_1 U_{\beta_1}, \end{aligned}$$

for every i . Let $\tilde{\gamma}_i : G \dashrightarrow \mathbb{A}^1$ be the rational map on G defined on Ω_0 by the formula

$$(4.3) \quad \tilde{\gamma}_i(g_{\beta_\ell} \dot{\sigma}_\ell \cdots g_{\beta_1} \dot{\sigma}_1 tu) := (g_{\beta_{\lambda_i}})^{-1},$$

for all $g_{\beta_\mu} \in U_{\beta_\mu}$, $t \in T$ and $u \in U$. Observe that this definition depends upon the choice of the pinning of G . Using the above description of Y_i we see that the open subscheme

$$(4.4) \quad \mathcal{V}_i := U_{\beta_\ell} \dot{\sigma}_\ell \cdots U_{\beta_{\lambda_{i+1}}} \dot{\sigma}_{\lambda_{i+1}} U_{-\beta_{\lambda_i}} TU \dot{\sigma}_{\lambda_{i-1}} U_{\beta_{\lambda_{i-1}}} \cdots \dot{\sigma}_1 U_{\beta_1}$$

of $\overline{\Omega}_{0i}$ can be considered as an open tubular neighborhood of Y_i . Through the pinning $u_{-\beta_{\lambda_i}} : \mathbb{G}_a \xrightarrow{\cong} U_{-\beta_{\lambda_i}}$ we get a regular function on \mathcal{V}_i , whose zero locus is exactly Y_i . The corresponding rational function on G will be denoted by $\check{\gamma}_i$.

In a dual way, we observe that the open subscheme

$$(4.5) \quad U_{\beta_\ell} \dot{\sigma}_\ell \cdots U_{\beta_{\lambda_{i+1}}} \dot{\sigma}_{\lambda_{i+1}} U T U_{-\beta_{\lambda_i}} \dot{\sigma}_{\lambda_{i-1}} U_{\beta_{\lambda_{i-1}}} \cdots \dot{\sigma}_1 U_{\beta_1}$$

of $\overline{\Omega}_{0i}$ can also be considered as an open tubular neighborhood of Y_i , and gives rise to another regular function on this tubular neighborhood whose zero locus is exactly Y_i . The corresponding rational function on G is denoted by ${}^t\tilde{\gamma}_i$.

Lemma 4.6. *Using the above notations, we have the following.*

(a) One has $\tilde{\gamma}_i = \tilde{\gamma}_i$. Moreover, for any $t \in T$ and any $x \in \Omega_0$, one has

$$\tilde{\gamma}_i(xt) = \tilde{\gamma}_i(x)$$

and

$$\tilde{\gamma}_i(tx) = \alpha'_i(t) \cdot \tilde{\gamma}_i(x),$$

where $\alpha'_i = w_0(-\alpha_i)$.

(b) For any $t \in T$ and any $x \in \Omega_0$, one has

$${}^t\tilde{\gamma}_i(tx) = {}^t\tilde{\gamma}_i(x)$$

and

$${}^t\tilde{\gamma}_i(xt) = \alpha_i(t)^{-1} \cdot {}^t\tilde{\gamma}_i(x).$$

Proof. The first statement in (a) follows from Lemma 3.4 and an easy computation. The equality $\tilde{\gamma}_i(xt) = \tilde{\gamma}_i(x)$ is clear by definition of $\tilde{\gamma}_i$. For the last statement, note that for all $g_{\beta_\mu} \in U_{\beta_\mu}$, $t, t' \in T$ and $u \in U$, we have

$$tg_{\beta_\ell}\dot{\sigma}_\ell \cdots g_{\beta_1}\dot{\sigma}_1 t'u = g'_{\beta_\ell}\dot{\sigma}_\ell \cdots g'_{\beta_1}\dot{\sigma}_1 tt'u,$$

where $g'_{\beta_i} = \sigma_{i+1} \circ \cdots \circ \sigma_\ell(t) \cdot g_{\beta_i} \cdot (\sigma_{i+1} \circ \cdots \circ \sigma_\ell(t))^{-1}$, for every i . Thus, in particular, we have $g'_{\beta_{\lambda_i}} = (w''_i)^{-1}(\beta_{\lambda_i})(t) \cdot g_{\beta_{\lambda_i}}$ and consequently, for $x = g_{\beta_\ell}\dot{\sigma}_\ell \cdots g_{\beta_1}\dot{\sigma}_1 t'u$, we have $\tilde{\gamma}_i(tx) = w''_i(\beta_{\lambda_i})(t) \cdot \tilde{\gamma}_i(x)$. Now, it remains to show that $w''_i(\beta_{\lambda_i}) = \alpha'_i$. Since

$$\sigma_\ell \cdots \sigma_1 = w_0 = s'_i s'_i w_0 = s'_i \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \sigma_1,$$

we have $s'_i \sigma_\ell \cdots \sigma_{\lambda_i+1} = \sigma_\ell \cdots \sigma_{\lambda_i}$. By Lemma 4.3 applied to $w = w''_i = \sigma_{\lambda_\ell} \cdots \sigma_{\lambda_i+1}$, it follows that $w''_i(\beta_{\lambda_i}) = \alpha'_i$. This completes the proof of (a). The proof of (b) can be obtained in a similar way using the relation $(w'_i)^{-1}(\beta_{\lambda_i}) = \alpha_i$ and is left to the reader. \square

The following lemma compares the orientation θ_i and ${}^t\theta_i$ of the normal bundle ν_{Y_i} of Y_i in G obtained from $\tilde{\gamma}_i$ and ${}^t\tilde{\gamma}_i$, respectively.

Lemma 4.7. *We have the following equality of trivializations as well as orientations of ν_{Y_i} (on Y_i):*

$${}^t\theta_i = x_{-\beta_{\lambda_i}} \cdot \theta_i$$

where $x_{-\beta_{\lambda_i}} : Y_i \rightarrow \mathbb{G}_m$ is the invertible function obtained by composing the projection $Y_i = B\tilde{w}_i T U \rightarrow T$ and the character associated with the root $-\beta_{\lambda_i}$.

Proof. The assertion follows from the comparison of the tubular neighborhoods and the fact that $\mathcal{O}(Y_i)^\times = \mathcal{O}(\tilde{w}_i T)^\times$. Then one may explicitly compute this unit on elements of the form $\tilde{w}_i \cdot t$ with $t \in T$ and use the fact that the adjoint automorphism $Ad_t : \mathfrak{u}_{-\beta_{\lambda_i}} \cong \mathfrak{u}_{-\beta_{\lambda_i}}$ is the multiplication by $\beta_{\lambda_i}(t)^{-1}$. \square

Unless otherwise stated, we will always use for ν_{Y_i} the right and left T -equivariant trivialization θ_i (which depends upon $\tilde{\gamma}_i$). The factor $\mathbf{H}_1^{\mathbb{A}^1}(Th(\nu_i))$ of $C_1^{\text{cell}}(G)$ corresponding to Y_i can now be described as the sheaf

$$\mathbf{H}_1^{\mathbb{A}^1}(Th(\nu_i)) = \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_i] \cong \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T].$$

This description depends on the weak pinning of G and the choice of the reduced expression (4.2) of w_0 . Observe that the above tensor product is a tensor product of left and right

$\mathbf{Z}_{\mathbb{A}^1}[T]$ -modules over \mathbb{Z} . The left and right $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module structure is as usual obtained by using the diagonal morphism

$$\Psi_T : \mathbf{Z}_{\mathbb{A}^1}[T] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[T] \otimes \mathbf{Z}_{\mathbb{A}^1}[T]; \quad [t] \mapsto [t] \otimes [t] = [t, t] \in \mathbf{Z}_{\mathbb{A}^1}[T] \otimes \mathbf{Z}_{\mathbb{A}^1}[T] = \mathbf{Z}_{\mathbb{A}^1}[T \times T].$$

Remark 4.8. Let M be a right and left $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module. Let σ be an automorphism of T . We define the left twist

$$\mathbb{Z}[\sigma] \otimes M$$

of M by σ to be the right and left $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module $\mathbb{Z}[\sigma] \otimes M$ with the obvious right $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module structure, and with the left $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module structure \odot induced by:

$$t \odot ([\sigma] \otimes m) = \sigma^{-1}(t) \cdot m.$$

We analogously define the right twist of M by σ :

$$M \otimes \mathbb{Z}[\sigma]$$

with

$$(m \otimes [\sigma]) \odot t = (m \cdot \sigma(t)) \otimes [\sigma]$$

giving the right $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module structure.

In order to describe the differential $\partial_1 : \tilde{C}_1^{\text{cell}}(G) \rightarrow \tilde{C}_0^{\text{cell}}(G)$, it will be convenient to use the canonical isomorphism of right and left $\mathbf{Z}_{\mathbb{A}^1}[T]$ -modules

$$\begin{aligned} \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T] &\xrightarrow{\cong} \mathbb{Z}[\tilde{w}_i''] \otimes (\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[T]) \otimes \mathbb{Z}[\tilde{w}_i']; \\ (u) \otimes [\tilde{w}_i \cdot t] &\mapsto \tilde{w}_i'' \otimes ((u) \otimes [\tilde{w}_i'(t)]) \otimes \tilde{w}_i', \end{aligned}$$

with the right and left $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module structures given in Remark 3.20 and Remark 4.8.

Lemma 4.9. *The restriction of $\partial_1 : \tilde{C}_1^{\text{cell}}(G) \rightarrow \tilde{C}_0^{\text{cell}}(G)$ to the factor $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T] \subset \tilde{C}_1^{\text{cell}}(G)$ is the morphism (of right and left $\mathbf{Z}_{\mathbb{A}^1}[T]$ -modules):*

$$\partial_1 : \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_0 T]$$

induced by

$$(u) \otimes [\tilde{w}_i \cdot 1] \mapsto (\tilde{w}_0 \cdot \alpha_i^\vee(u)) \in \mathbf{Z}_{\mathbb{A}^1}(\tilde{w}_0 T) \subset \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_0 T].$$

Proof. From the discussion preceding Lemma 4.6, it follows that the restriction of ∂_1 to the factor $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T]$ is precisely the differential in the complex

$$\mathbb{Z}[\tilde{w}_i''] \otimes \tilde{C}_*^{\text{cell}}(P_{\lambda_i}) \otimes \mathbb{Z}[\tilde{w}_i'].$$

The formula is then immediately follows from the rank 1 case already treated in Section 3.1, see (3.5):

$$\begin{aligned} \partial_1 : (u) \otimes [\tilde{w}_i \cdot 1] &\mapsto \tilde{w}_i'' \cdot \dot{\sigma}_{\lambda_i} \cdot (\beta_{\lambda_i}^\vee(u)) \cdot \tilde{w}_i' \\ &= \tilde{w}_i'' \cdot \dot{\sigma}_{\lambda_i} \cdot \tilde{w}_i' \cdot ((\tilde{w}_i')^{-1} \beta_{\lambda_i}^\vee(u)) \\ &= \tilde{w}_0 \cdot (\alpha_i^\vee(u)) = (\tilde{w}_0 \cdot \alpha_i^\vee(u)), \end{aligned}$$

where we have used the fact that $(\tilde{w}_i')^{-1}(\beta_{\lambda_i}^\vee) = \alpha_i^\vee$, which follows from Lemma 4.3. \square

Thus, we obtain a complete description of the first differential $\partial_1 : \widetilde{C}_1^{\text{cell}}(G) \rightarrow \widetilde{C}_0^{\text{cell}}(G)$ in $\widetilde{C}_*^{\text{cell}}(G)$. Using this and our computations in Section 3.3, we will now describe the sheaf $Z_1(G) := \text{Ker}(\partial_1)$ and its quotient $Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$. Let

$$\theta_0 : \mathbf{Z}_{\mathbb{A}^1}[T] \xrightarrow{\cong} \mathbf{Z}_{\mathbb{A}^1}[\widetilde{w}_0 T]$$

denote the obvious isomorphism $[t] \mapsto [\widetilde{w}_0 t]$ and

$$\theta_1 : \bigoplus_{i=1}^r \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[T] \xrightarrow{\cong} \bigoplus_{i=1}^r \mathbb{Z}[\widetilde{w}'_i] \otimes \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\widetilde{w}'_i T] = \widetilde{C}_1^{\text{cell}}(G)$$

be the obvious isomorphism, induced by $(u) \otimes [t] \mapsto \widetilde{w}'_i \otimes (u) \otimes [\widetilde{w}'_i \cdot t]$ on the direct summand indexed by i . The composition $\theta_0^{-1} \circ \partial_1 \circ \theta_1$ is the morphism

$$\widetilde{\partial}_1 : \bigoplus_{i=1}^r \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[T] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[T]$$

induced by $(u) \otimes [t] \mapsto (\widetilde{w}_0 \cdot \alpha_i^\vee(u))$ on the direct summand indexed by i . Observe that the morphism $\widetilde{\partial}_1$ only depends on the isomorphism given by the “decomposition” of T into coroots:

$$\prod_{i=1}^r \alpha_i^\vee : \prod_{i=1}^r \mathbb{G}_m \xrightarrow{\cong} T.$$

Note further that the differential $\widetilde{\partial}_1$ precisely agrees with the first differential in the complex C_* (see (3.19) in Section 3.3). As a direct consequence of Lemma 3.24, we get the following description of the quotient $Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$ of $Z_1(G)$.

Lemma 4.10. *The morphism of sheaves*

$$\left(\bigoplus_i \mathbf{K}_2^{\text{MW}} \right) \oplus \left(\bigoplus_{(i,j); i < j} \mathbf{K}_2^{\text{M}} \right) \rightarrow Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$$

induced for each i by

$$(u, v) \mapsto \phi_i(u, v) := (u) \otimes (\widetilde{w}_i \cdot \alpha_i^\vee(v)) - (\eta(u)(v)) \otimes [\widetilde{w}_i \cdot 1_T]$$

(in $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\widetilde{w}_i T]$) and for each (i, j) with $i \neq j$ by

$$(u, v) \mapsto \delta_{ij}(u, v) := (u) \otimes (\widetilde{w}_i \cdot \alpha_j^\vee(v)) - (v) \otimes (\widetilde{w}_j \cdot \alpha_i^\vee(u))$$

is an isomorphism.

Remark 4.11. The above isomorphism in Lemma 4.10 depends only on the weak-pinning of G fixed at the beginning. We will see below that its dependence on the choices of reduced expressions is very mild (see 4.37). In fact, its dependence on the weak pinning itself is also very mild. Indeed, when one changes the weak pinning, one gets a multiplication by a unit of the form $\langle \pi \rangle$ with $\pi \in k^\times$ on each \mathbf{K}_2^{MW} factor, but the identity on the sum of Milnor-K-theory factors, as units in $\text{GW}(k)$ act trivially on \mathbf{K}_2^{M} .

4.2. The differential in degree 2.

In this section, we determine the morphism

$$\widetilde{C}_2^{\text{cell}}(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \rightarrow Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$$

whose cokernel is $\mathbf{H}_1^{\text{cell}}(G)$ by (4.1). With our conventions and notation fixed earlier in this section and Notation 3.21, we have

$$\tilde{C}_2^{\text{cell}}(G) = \bigoplus_{[i,j]} \mathbf{K}_2^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{[i,j]}T].$$

We use the reduced expression (4.2) of w_0 . Let $w \in W$ be an element of length $\ell - 2$. There is a pair $(i, j) \in \{1, \dots, r\}^2$ with $w = w_0 s_i s_j = w_{[i,j]}$. The pair (i, j) is unique except in the case s_i and s_j commute, in which case (j, i) is the only other pair satisfying $w = w_{[i,j]}$.

Notation 4.12. For every i , let $\lambda_i \in \{1, \dots, \ell\}$ be the index given by Lemma 4.1 so that $\sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \sigma_1$ is a reduced expression of $w_i = w_0 s_i$. By Lemma 4.1 again, it follows that there is a unique index $\mu_j \in \{1, \dots, \ell\} - \{\lambda_i\}$ such that a reduced expression of $w_{[i,j]}$ can be obtained by removing σ_{μ_j} in the above reduced expression of w_i .

Lemma 4.13. *With the above notation, the following conditions are equivalent:*

- (a) $\lambda_i > \mu_j$;
- (b) $\lambda_i > \lambda_j$.

Moreover, if both the conditions hold, then $\mu_j = \lambda_j$.

Proof. (a) \Rightarrow (b): Suppose that $\lambda_i > \mu_j$, so that $w_0 s_i s_j = \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \hat{\sigma}_{\mu_j} \cdots \sigma_1$. Since $w_0 s_i = (w_0 s_i s_j) s_j$, it follows after cancelling on the left up to σ_{μ_j+1} that:

$$\sigma_{\mu_j} \cdots \sigma_1 = \sigma_{\mu_j-1} \cdots \sigma_1 s_j,$$

which implies that $\sigma_\ell \cdots \sigma_1 = \sigma_\ell \cdots \hat{\sigma}_{\mu_j} \cdots \sigma_1 s_j$. Hence, we obtain

$$\sigma_\ell \cdots \hat{\sigma}_{\lambda_j} \cdots \sigma_1 = w_0 s_j = \sigma_\ell \cdots \sigma_1 s_j = \sigma_\ell \cdots \hat{\sigma}_{\mu_j} \cdots \sigma_1$$

and this implies that $\mu_j = \lambda_j$ by the uniqueness assertion in Lemma 4.1.

(b) \Rightarrow (a) : Suppose that $\lambda_i > \lambda_j$. From $w_0 s_j = \sigma_\ell \cdots \hat{\sigma}_{\lambda_j} \cdots \sigma_1$, we get $\sigma_{\lambda_j-1} \cdots \sigma_1 = \sigma_{\lambda_j} \cdots \sigma_1 s_j$, after cancelling on the left. Now, using $\lambda_i > \lambda_j$ and multiplying on the left by $\sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \sigma_{\lambda_j+1}$, we obtain $\sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \hat{\sigma}_{\lambda_j} \cdots \sigma_1 = \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \sigma_1 s_j$. Thus,

$$\sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \sigma_1 = w_0 s_i = (w_0 s_i s_j) s_j = \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \hat{\sigma}_{\lambda_j} \cdots \sigma_1 s_j = \sigma_\ell \cdots \hat{\sigma}_{\lambda_j} \cdots \sigma_1,$$

which implies that $\lambda_j = \mu_j$, again by the uniqueness assertion in Lemma 4.1 applied to $w_0 s_i = \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \sigma_1$. \square

Remark 4.14.

- (1) In a similar way, Lemma 4.1 implies that there is a unique index $\mu_i \in \{1, \dots, \ell\} - \{\lambda_j\}$ so that the expression obtained from the above one for w_j is a reduced expression of $w_{[i,j]}$. One may prove the analogue of the previous lemma; that is, $\mu_i > \lambda_j$ if and only if $\lambda_i > \lambda_j$ and if both the conditions hold, then $\mu_i = \lambda_i$.
- (2) Observe that if $\lambda_i > \lambda_j$, then $\mu_i = \lambda_i$ and $\mu_j = \lambda_j$. In fact, the following conditions are equivalent:
 - (i) $\mu_i = \lambda_i$;
 - (ii) $\mu_j = \lambda_j$;
 - (iii) $\lambda_i > \lambda_j$ or α_i and α_j are not adjacent in the Dynkin diagram of G .

Indeed, if $\lambda_i < \lambda_j$ then $w_{ji} = \sigma_\ell \cdots \hat{\sigma}_{\lambda_j} \cdots \hat{\sigma}_{\lambda_i} \cdots \sigma_1$, which is also w_{ij} by (i) or (ii). Thus, we get $s_i s_j = s_j s_i$.

Let $Z_{ij} := Bw_{[i,j]}B$ denote the Bruhat cell corresponding to $w_{[i,j]}$, and recall that Y_i and Y_j denote the Bruhat cells Bw_iB and Bw_jB corresponding to w_i and w_j , respectively. Observe that Y_i and Y_j are exactly the two cells of codimension 1 in G whose closure contains Z_{ij} . It follows, as we already observed above, that the differential

$$\partial_2 : C_2^{\text{cell}}(G) \rightarrow C_1^{\text{cell}}(G)$$

restricted to the summand of $C_2^{\text{cell}}(G)$ corresponding to Z_{ij} has its image contained in the direct sum of the two summands of $C_1^{\text{cell}}(G)$ corresponding to Y_i and Y_j . We will denote by $\partial_2^{(i)} : \tilde{\mathbf{H}}_2^{\mathbb{A}^1}(\text{Th}(\nu_{ij})) \rightarrow \tilde{\mathbf{H}}_1^{\mathbb{A}^1}(\text{Th}(\nu_i))$ and $\partial_2^{(j)} : \tilde{\mathbf{H}}_2^{\mathbb{A}^1}(\text{Th}(\nu_{ij})) \rightarrow \tilde{\mathbf{H}}_1^{\mathbb{A}^1}(\text{Th}(\nu_j))$ the corresponding components, where for all $i \neq j$, we denote by $\nu_i := \nu_{Y_i}$ the normal bundle of the immersion $\iota_i : Y_i \hookrightarrow G$ (it is a locally free sheaf of rank 1 over \bar{Y}_i) and ν_{ij} denotes the normal bundle of the immersion $Z_{ij} \hookrightarrow G$. We will denote by $\bar{\nu}_i$ the normal bundle of the immersion $\bar{Y}_i \hookrightarrow G$, for every i . Let \bar{Y}_i be the union $Y_i \amalg Z_{ij}$ and \bar{Y}_j be the union $Y_j \amalg Z_{ij}$, for all $i \neq j$. These are all smooth k -subschemes of G . Indeed, the multiplication (in G) induces an isomorphism

$$(4.6) \quad \bar{Y}_i \cong \Omega_\ell \times_B \cdots \times_B P_{\mu_j} \times_B \cdots \times_B B \times_B \cdots \times_B \Omega_1$$

with B at the λ_i -th place as well as

$$(4.7) \quad \bar{Y}_j \cong \Omega_\ell \times_B \cdots \times_B P_{\mu_i} \times_B \cdots \times_B B \times_B \cdots \times_B \Omega_1$$

with B at the λ_j -th place (note that (4.6) depends upon whether $\lambda_i > \mu_j$ or otherwise and hence, B may appear to the left of P_{μ_j} ; similarly for (4.7)).

Let $\tilde{X}_{ij} := \Omega_0 \amalg Y_i \amalg Y_j \amalg Z_{ij}$, which is an open subscheme of G . Let $\nu_{ij/i}$ denote the normal bundle of the closed immersion $Z_{ij} \hookrightarrow \bar{Y}_i$ and $\nu_{ij/j}$ denote that of the closed immersion $Z_{ij} \hookrightarrow \bar{Y}_j$. In the cartesian square of closed immersions

$$\begin{array}{ccc} Z_{ij} & \subset & \bar{Y}_i \\ \cap & & \cap \\ \bar{Y}_j & \subset & \tilde{X}_{ij} \end{array}$$

the closed immersions $\bar{Y}_i \subset \tilde{X}_{ij}$ and $\bar{Y}_j \subset \tilde{X}_{ij}$ are transversal. We thus obtain the identifications

$$\nu_{ij/i} = (\bar{\nu}_j)|_{Z_{ij}}$$

and

$$\nu_{ij/j} = (\bar{\nu}_i)|_{Z_{ij}}$$

as well as a canonical isomorphism

$$(4.8) \quad (\bar{\nu}_i)|_{Z_{ij}} \oplus (\bar{\nu}_j)|_{Z_{ij}} \cong \nu_{ij}.$$

We will use the following well-known generalization of \mathbb{A}^1 -homotopy purity:

Lemma 4.15. *There is a canonical \mathbb{A}^1 -cofiber sequence of the form:*

$$\text{Th}(\nu_i) \rightarrow \text{Th}(\bar{\nu}_i) \rightarrow \text{Th}(\nu_{ij}).$$

Proof. One considers the flag of open subsets

$$\tilde{X}_{ij} - \bar{Y}_i \subset \tilde{X}_{ij} - Z_{ij} \subset \tilde{X}_{ij}$$

in \tilde{X}_{ij} and obtains a cofibration sequence

$$(\tilde{X}_{ij} - Z_{ij})/(\tilde{X}_{ij} - \bar{Y}_i) \subset \tilde{X}_{ij}/(\tilde{X}_{ij} - \bar{Y}_i) \rightarrow \tilde{X}_{ij}/(\tilde{X}_{ij} - Z_{ij})$$

and applies the \mathbb{A}^1 -homotopy purity theorem [24, Theorem 2.23, page 115]. \square

It follows from the very construction of the cellular \mathbb{A}^1 -chain complex described in Section 2.3 that the connecting homomorphism in the \mathbb{A}^1 -homology sequence of the previous \mathbb{A}^1 -cofibration:

$$(4.9) \quad \tilde{\mathbf{H}}_2^{\mathbb{A}^1}(Th(\nu_{ij})) \rightarrow \tilde{\mathbf{H}}_1^{\mathbb{A}^1}(Th(\nu_i))$$

is the restriction of the differential $\partial_2^{(i)}$ to the factor $\tilde{\mathbf{H}}_2^{\mathbb{A}^1}(Th(\nu_{ij})) \subset C_2^{\text{cell}}(G)$, and analogously for $\partial_2^{(j)}$. The following property uses the fact that G is simply connected in the sense of algebraic groups.

Lemma 4.16. *The group $\text{Pic}(\bar{Y}_i)$ is trivial (and so is $\text{Pic}(\bar{Y}_j)$).*

Proof. The cofiber sequence of Lemma 4.15 gives rise to the exact sequence

$$\mathcal{O}(Y_i)^\times \rightarrow H^1(Th(\nu_{ij/i}; \mathbb{G}_m) \rightarrow \text{Pic}(\bar{Y}_i) \rightarrow \text{Pic}(Y_i).$$

Now, $\text{Pic}(Y_i) = 0$, since Y_i is isomorphic to the product of T with an affine space and $H^1(Th(\nu_{ij/i}; \mathbb{G}_m) = \mathbb{Z}$. The morphism $\mathcal{O}(Y_i)^\times \rightarrow H^1(Th(\nu_{ij/i}; \mathbb{G}_m) = \mathbb{Z}$ is the valuation at Z_{ij} . Since G is simply connected, the pairing $X \otimes Y \rightarrow \mathbb{Z}$ induces an isomorphism $X \rightarrow Y^*$ so that the fundamental weights for a basis of X . Now, [8, Lemma 4.2] implies that the valuation $\mathcal{O}(Y_i)^\times \rightarrow \mathbb{Z}$ is surjective. \square

Remark 4.17. Note that [8, Lemma 4.2] is not proved in [8], and is referred to [10] (where it is shown by reduction to the case of \mathbf{SL}_2). In fact, by reduction to the case of \mathbf{SL}_2 , it is easy to prove this lemma using Lemma 3.4.

It follows from Lemma 4.16 that every trivialization of $\bar{\nu}_i$ induces a unique trivialization of ν_i . Let $\bar{\theta}_i$ be a right and left T -equivariant trivialization of $\bar{\nu}_i$ and let θ_i denote the induced trivialization of ν_i . Then the above \mathbb{A}^1 -cofiber sequence

$$Th(\nu_i) \rightarrow Th(\bar{\nu}_i) \rightarrow Th(\nu_{ij})$$

becomes canonically and right and left T -equivariantly isomorphic to

$$(4.10) \quad \Sigma(\mathbb{G}_m) \wedge ((Y_i)_+) \rightarrow \Sigma(\mathbb{G}_m) \wedge ((\bar{Y}_i)_+) \rightarrow \Sigma(\mathbb{G}_m) \wedge Th(\nu_{ij/i})$$

where the factor $\Sigma(\mathbb{G}_m)$ on the rightmost term corresponds to θ_i (note that $Th(\nu_{ij})$ is the direct sum of $Th(\nu_{ij/i})$ and a rank one trivial bundle). Observe that the axioms of symmetric monoidal triangulated categories implies that the connecting morphism

$$\Sigma(\mathbb{G}_m) \wedge Th(\nu_{ij/i}) \rightarrow \Sigma(\Sigma(\mathbb{G}_m) \wedge ((Y_i)_+))$$

of the triangle (4.10) is the composite of the smash product with $\Sigma(\mathbb{G}_m)$ of the connecting morphism $\partial_{ij/i} : Th(\nu_{ij/i}) \rightarrow \Sigma((Y_i)_+)$ of the triangle $(Y_i)_+ \rightarrow (\bar{Y}_i)_+ \rightarrow Th(\nu_{ij/i})$ and the permutation $\Sigma(\mathbb{G}_m) \wedge \Sigma((Y_i)_+) \cong \Sigma(\Sigma(\mathbb{G}_m) \wedge ((Y_i)_+))$.

Thus, the restriction $\partial_2^{(i)}$ of the differential ∂_2 to the factor of $C_2^{\text{cell}}(G)$ corresponding to Z_{ij} is the morphism of sheaves

$$\begin{aligned} \mathbf{K}_1^{\text{MW}} \otimes \mathbf{H}_1^{\mathbb{A}^1}(Th(\nu_{ij/i})) &\xrightarrow{\cong} \mathbf{H}_2^{\mathbb{A}^1}(\Sigma(\mathbb{G}_m) \wedge Th(\nu_{ij/i})) \rightarrow \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_i] \\ (u) \otimes x &\mapsto -\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i})(x), \end{aligned}$$

where u stands for the above factor \mathbb{G}_m itself coming from the restriction to ν_i of $\bar{\theta}_i$. The sign comes from the permutation morphism on the smash product of two 1-dimensional simplicial spheres. Observe that by Lemma 2.23, a change of the trivialization preserving the orientation acts trivially on the \mathbb{A}^1 -cofibration $\Sigma(\mathbb{G}_m) \wedge ((Y_i)_+) \rightarrow \Sigma(\mathbb{G}_m) \wedge ((\bar{Y}_i)_+)$, up to \mathbb{A}^1 -homotopy. Thus the previous computations are only dependent on the orientations induced by $\bar{\theta}_i$.

Now, we need to conveniently choose orientations (or trivializations) of both $\bar{\nu}_i$ and $\bar{\nu}_j$ so as to facilitate our computation of ∂_2 .

Lemma 4.18. *In the case $\lambda_i > \lambda_j$, the morphism of k -schemes*

$$(4.11) \quad \Omega_\ell \times_B \cdots \times_B \Omega_{\lambda_{i+1}} \times_B P_{\beta_{\lambda_i}} \times_B \cdots \times_B P_{\beta_{\lambda_j}} \times_B \cdots \times_B \Omega_1 \rightarrow \tilde{X}_{ij}$$

induced by taking the product in G is an isomorphism of cellular k -schemes.

Proof. We proceed in the same way as in Section 4.1 to describe $\bar{\Omega}_{0i}$ as a fiber product. From a combinatorial point of view, the Bruhat decomposition implies that this morphism induced a bijection onto \tilde{X}_{ij} , which is open in G (its complement in G being a union of Bruhat cells stable under the Bruhat ordering). Now the big cell $\Omega_\ell \times_B \cdots \times_B \Omega_{\lambda_{i+1}} \times_B \Omega_{\beta_{\lambda_i}} \times_B \cdots \times_B \Omega_{\beta_{\lambda_j}} \times_B \cdots \times_B \Omega_1$ and its four left translates, obtained by replacing the big cell in $P_{\beta_{\lambda_j}}$ and $P_{\beta_{\lambda_i}}$ by their left translate through $\dot{\sigma}_{\lambda_i}$ and or $\dot{\sigma}_{\lambda_j}$ respectively, cover both left and right hand side of the morphism (4.11). The restriction of (4.11) on each of these open sets is clearly an isomorphism of schemes. \square

In view of Lemma 4.18, one may compute $C_*^{\text{cell}}(\tilde{X}_{ij})$ using the rank one case and the Künneth formula in the case $\lambda_i > \lambda_j$. However, we take a more direct approach.

Lemma 4.19. *We will use the notation of Lemma 4.6. If $\lambda_i > \lambda_j$, then $\tilde{\gamma}_i$ is defined on a neighborhood of \bar{Y}_i so that it defines a right and left T -equivariant trivialization $\bar{\theta}_i$ of $\bar{\nu}_i$ and ${}^t\tilde{\gamma}_j$ is defined on a neighborhood of \bar{Y}_j so that it defines a right and left T -equivariant trivialization ${}^t\bar{\theta}_j$ of $\bar{\nu}_j$.*

Proof. This follows easily from Lemma 4.18. For instance, one sees that

$$U_{\beta_\ell} \dot{\sigma}_\ell \cdots U_{\beta_{\lambda_i+1}} \dot{\sigma}_{\lambda_i+1} U_{-\beta_{\lambda_i}} U_{\beta_{\lambda_i-1}} \dot{\sigma}_{\lambda_i-1} \cdots U_{\beta_{\lambda_j+1}} \dot{\sigma}_{\lambda_j+1} P_{\lambda_j} U_{\beta_{\lambda_j-1}} \dot{\sigma}_{\lambda_j-1} \cdots \dot{\sigma}_1 U_1$$

is an open neighborhood of $\bar{Y}_i = U_{\beta_\ell} \dot{\sigma}_\ell \cdot \widehat{U}_{-\beta_{\lambda_i}} \cdots P_{\lambda_j} \cdots \dot{\sigma}_1 U_1$ on which $\tilde{\gamma}_i$ is defined and clearly defines \bar{Y}_i as its zero locus. For ${}^t\tilde{\gamma}_j$ one uses the open neighborhood

$$U_{\beta_\ell} \dot{\sigma}_\ell \cdots U_{\beta_{\lambda_i+1}} \dot{\sigma}_{\lambda_i+1} P_{\beta_{\lambda_i}} U_{\beta_{\lambda_i-1}} \dot{\sigma}_{\lambda_i-1} \cdots U_{\beta_{\lambda_j+1}} \dot{\sigma}_{\lambda_j+1} U_{-\lambda_j} U_{\beta_{\lambda_j-1}} \dot{\sigma}_{\lambda_j-1} \cdots \dot{\sigma}_1 U_1$$

of \bar{Y}_j and proceeds similarly. \square

Remark 4.20.

- (1) One may show that under the assumptions of Lemma 4.19, $\tilde{\gamma}_j$ does not extend to \bar{Y}_j .

- (2) In the case $\lambda_i < \lambda_j$, there is no analogue of Lemma 4.18. In this case, neither $t\tilde{\gamma}_i$ nor $\tilde{\gamma}_i$ is defined on an open neighborhood of \bar{Y}_i . Thus, we have to choose a right and left T -equivariant trivialization of $\bar{\nu}_i$ in another way, see Section 4.4.

In view of Lemma 4.19, we will always use the following orientation of ν_{ij} in what follows. By (4.8), we have $\Lambda^2(\nu_{ij}) = \bar{\nu}_i|_{Z_{ij}} \otimes \bar{\nu}_j|_{Z_{ij}}$. So in order to orient ν_{ij} , it suffices to choose an orientation θ'_i of $\bar{\nu}_i$ and θ'_j of $\bar{\nu}_j$ and then take the tensor product of these. In the case $\lambda_i > \lambda_j$, we take for θ'_i the orientation induced by $\tilde{\gamma}_i$ and for θ'_j the one induced by $t\tilde{\gamma}_j$.

We thus get an identification of sheaves

$$\mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Z_{ij}] \cong \tilde{\mathbf{H}}_2^{\mathbb{A}^1}(Th(\nu_{ij}))$$

with the first factor \mathbf{K}_1^{MW} corresponding to θ'_i and the second one corresponding to θ'_j . The corresponding variables in \mathbf{K}_1^{MW} in the computations below will be denoted by u for θ'_i and v for θ'_j . We also get identifications

$$\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Z_{ij}] \cong \tilde{\mathbf{H}}_1^{\mathbb{A}^1}(Th(\nu_{ij/i}))$$

using $\theta'_j|_{Z_{ij}}$ as the orientation of $\nu_{ij/i}$ and similarly

$$\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Z_{ij}] \cong \tilde{\mathbf{H}}_1^{\mathbb{A}^1}(Th(\nu_{ij/j}))$$

using $\theta'_i|_{Z_{ij}}$ as the orientation of $\nu_{ij/j}$. Each of these isomorphisms is left and right T -equivariant, where each factor is equipped with the corresponding left and right $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module structure. For instance, the factor \mathbf{K}_1^{MW} corresponding to θ'_i is endowed with the left and right $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module structures coming from the equivariant properties of the function used to define the trivialization (see Lemma 4.6). If we use these orientations θ'_i of $\bar{\nu}_i$ and θ'_j of $\bar{\nu}_j$ to identify $\tilde{\mathbf{H}}_1^{\mathbb{A}^1}(Th(\nu_i))$ with $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_i]$ and $\tilde{\mathbf{H}}_1^{\mathbb{A}^1}(Th(\nu_j))$ with $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_j]$, then we obtain the following fact from the above discussion.

Lemma 4.21. *The differential*

$$\partial_2^{(i)} : \mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Z_{ij}] \cong \tilde{\mathbf{H}}_2^{\mathbb{A}^1}(Th(\nu_{ij})) \rightarrow \tilde{\mathbf{H}}_1^{\mathbb{A}^1}(Th(\nu_i)) \cong \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_i]$$

in $\tilde{C}_*^{\text{cell}}(G)$ is given by

$$(u) \otimes (v) \otimes x \mapsto -(u) \otimes \mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i})((v) \otimes x)$$

and the differential

$$\partial_2^{(j)} : \mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Z_{ij}] \cong \tilde{\mathbf{H}}_2^{\mathbb{A}^1}(Th(\nu_{ij})) \rightarrow \tilde{\mathbf{H}}_1^{\mathbb{A}^1}(Th(\nu_j)) \cong \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Y_j]$$

is given by

$$(u) \otimes (v) \otimes x \mapsto (v) \otimes \mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/j})((u) \otimes x).$$

Remark 4.22. The sign in the formula for $\partial_2^{(i)}$ in Lemma 4.21 has already been explained in the paragraph preceding Lemma 4.6. Observe that the sign disappears in the formula for $\partial_2^{(j)}$ for the following reason. We have to first use the permutation $\Sigma(\mathbb{G}_m) \wedge \Sigma(\mathbb{G}_m) \cong \Sigma(\mathbb{G}_m) \wedge \Sigma(\mathbb{G}_m)$, which induces after applying $\mathbf{H}_2^{\mathbb{A}^1}$ the automorphism

$$\mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} \cong \mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}}; \quad (u) \otimes (v) \mapsto -(v) \otimes (u)$$

and then apply the formula analogous to that of $\partial_2^{(i)}$. In the computation of the differential, we use again a permutation of the two simplicial circles as explained above, and this cancels the first sign.

However, in Section 4.1, we made the convention to orient ν_i always with $\tilde{\gamma}_i$ (see the paragraph after Lemma 4.7). This was made to be able to explicitly compute $Z_1(G)$ and $Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$. We have to take this into account in our explicit computations as the restriction of θ'_i need not be θ_i (Remark 4.20).

We also need to determine the degree 1 differentials

$$\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i}) : \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Z_{ij}] \cong \mathbf{H}_1^{\mathbb{A}^1}(\text{Th}(\nu_{ij/i})) \rightarrow \mathbf{H}_0^{\mathbb{A}^1}(Y_i) = \mathbf{Z}_{\mathbb{A}^1}[Y_i]$$

and

$$\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/j}) : \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Z_{ij}] \cong \mathbf{H}_1^{\mathbb{A}^1}(\text{Th}(\nu_{ij/j})) \rightarrow \mathbf{H}_0^{\mathbb{A}^1}(Y_j) = \mathbf{Z}_{\mathbb{A}^1}[Y_j].$$

The problem here is the following: the restriction of θ'_j on Y_i is a local parameter for the closed immersion $Z_{ij} \subset Y_i$ and consequently, defines the orientation of $\nu_{ij/i}$ which is used in the determination of the differential. However, there is another local parameter, $\tilde{\gamma}_{ij/i}$, for the closed immersion $Z_{ij} \subset Y_i$, for which the morphism $\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i})$ is “easy” to compute, and it is not always the restriction of θ'_j . We now describe $\tilde{\gamma}_{ij/i}$.

The union $\bar{Y}_{ij} := Y_i \amalg Z_{ij}$ is a smooth k -scheme isomorphic to

$$\Omega_\ell \times_B \cdots \times_B P_{\mu_j} \times_B \cdots \times_B B \times_B \cdots \Omega_1$$

(by taking the product in G), where P_{μ_j} is in the μ_j th place and B is in the λ_i th place. Thus,

$$\bar{Y}_{ij} \cong U_{\beta_\ell} \dot{\sigma}_\ell \cdots P_{\mu_j} \dot{\sigma}_{\mu_j-1} U_{\beta_{\mu_j-1}} \cdots \hat{\sigma}_{\lambda_i} \hat{U}_{\lambda_i} \cdots \dot{\sigma}_1 U_{\beta_1}$$

and

$$Z_{ij} \cong U_{\beta_\ell} \dot{\sigma}_\ell \cdots B_{\mu_j} \dot{\sigma}_{\mu_j-1} U_{\beta_{\mu_j-1}} \cdots \hat{\sigma}_{\lambda_i} \hat{U}_{\lambda_i} \cdots \dot{\sigma}_1 U_{\beta_1}.$$

Thus, it is easy to compute the morphism $\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i})$, if one chooses the local parameter defining B in P_{μ_j} through the open neighborhood $U_{-\beta_{\mu_j}} \times B \subset P_{\mu_j}$ for the orientation as it was done in the determination of the rank one case in Section 3.1. We denote by $\tilde{\gamma}_{ij/i}$ this local parameter.

Lemma 4.23. *If $\lambda_i > \lambda_j$, then $\tilde{\gamma}_i|_{Y_j} = \tilde{\gamma}_{ij/j}$ and ${}^t\tilde{\gamma}_j|_{Y_i} = {}^t\tilde{\gamma}_{ij/i}$.*

Proof. This follows in the same way as the proof of Lemma 4.19 and we leave the details to the reader. \square

Remark 4.24.

- (1) One may prove an analogous formula to Lemma 4.7 comparing the trivializations $\theta_{ij/i}$ and ${}^t\theta_{ij/i}$ of $\nu_{ij/i}$ induced by $\tilde{\gamma}_{ij/i}$ and ${}^t\tilde{\gamma}_{ij/i}$:

$${}^t\theta_{ij/i} = \tilde{y}_{\beta_{\lambda_j}} \cdot \theta_{ij/i},$$

where $\tilde{y}_{\beta_{\lambda_j}} : Z_{ij} \rightarrow \mathbb{G}_m$ is the invertible function obtained by composing the projection $Z_{ij} = B\tilde{w}_{ij}TU \rightarrow T$ and the character β_{λ_j} .

- (2) If $\lambda_i < \lambda_j$ and if α_i and α_j are adjacent in the Dynkin diagram, then it can be shown that ${}^t\tilde{\gamma}_i|_{Y_j}$ does not agree with ${}^t\tilde{\gamma}_{ij/j}$ and $\tilde{\gamma}_j|_{Y_i}$ does not agree with $\tilde{\gamma}_{ij/i}$ as well.

- (3) One may verify that for $G = \mathbf{SL}_3$ with the standard pinning, with $w_0 = s_2 s_1 s_2$ the chosen reduced expression of the longest word in W , we are in the case $\lambda_1 = 3 > 1 = \lambda_2$, and that the restriction of $\tilde{\gamma}_2$ to Y_1 is different from $\tilde{\gamma}_{12/1}$.

Now, it remains to put together the previous computations, but at this point it is convenient to distinguish the case $\lambda_i > \lambda_j$ from the case $\lambda_i < \lambda_j$.

4.3. Image of the degree 2 differential: the case $\lambda_i > \lambda_j$.

In this case, we have $\lambda_j = \mu_j$ by Lemma 4.13 and this yields a canonical reduced expression of w_{ij} obtained by removing two simple reflections in the chosen reduced expression (4.2) of w_0 :

$$w_{[i,j]} := w_0 s_i s_j = \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \hat{\sigma}_{\lambda_j} \cdots \sigma_1.$$

We write this expression in an obvious way as $w_{[i,j]} = w'''_{[i,j]} \hat{\sigma}_{\lambda_i} w''_{[i,j]} \hat{\sigma}_{\lambda_j} w'_{[i,j]}$, thereby defining $w'_{[i,j]}$, $w''_{[i,j]}$ and $w'''_{[i,j]}$. Thus,

$$w_i = w_0 s_i = w'''_{[i,j]} \hat{\sigma}_{\lambda_i} w''_{[i,j]} \sigma_{\lambda_j} w'_{[i,j]}$$

is the reduced expression for w_i obtained by removing σ_{λ_i} from that of w_0 . Consequently, $w''_i = w'''_{[i,j]}$ and $w''_{[i,j]} \sigma_{\lambda_j} w'_{[i,j]} = w'_i$ (see Notation 4.5). Also,

$$w_j = w_0 s_j = w'''_{[i,j]} \sigma_{\lambda_i} w''_{[i,j]} \hat{\sigma}_{\lambda_j} w'_{[i,j]}$$

is the reduced expression for w_j obtained by removing λ_j from that of w_0 and hence, we have $w''_j = w'''_{[i,j]} \sigma_{\lambda_i} w''_{[i,j]}$ and $w'_j = w'_{[i,j]}$. We now fix the horizontal reduced expressions for Bruhat cells in codimension 2.

Notation 4.25. With the above notation, we set

$$\tilde{w}_{[i,j]} := \dot{w}'''_{[i,j]} \dot{w}''_{[i,j]} \dot{w}'_{[i,j]} = \dot{\sigma}_\ell \cdots \dot{\sigma}_{\lambda_i+1} \dot{\sigma}_{\lambda_i-1} \cdots \dot{\sigma}_{\lambda_j+1} \dot{\sigma}_{\lambda_j-1} \cdots \dot{\sigma}_1 \in N_G(T).$$

When there is no confusion, we will simply write w_{ij}, w'_j etc for $w_{[i,j]}, w'_{[i,j]}$ etc for the sake of brevity.

We will use the identification $\mathbf{Z}_{\mathbb{A}^1}[Z_{ij}] \cong \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{ij}T]$. In that case, the restriction of θ'_i to ν_i is θ_i and it follows that

$$(4.12) \quad \partial_2^{(i)}((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot t]) = -(u) \otimes \mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i})((v) \otimes [\tilde{w}_{ij} \cdot t]),$$

for all $t \in T$. Now, we want to compute $\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i})((v) \otimes [\tilde{w}_{ij} \cdot t])$. By the very definition, $\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i})$ is the connecting morphism

$$\mathbf{H}_1^{\mathbb{A}^1}(Th(\nu_{ij/i})) \rightarrow \mathbf{H}_0^{\mathbb{A}^1}(Y_i) = \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T]$$

coming from the cofibration sequence $Y_i \subset \bar{Y}_i \rightarrow Th(\nu_{ij/i})$. If we use the orientation of $\nu_{ij/i}$ induced by $\tilde{\gamma}_{ij/i}$, we get an identification

$$\mathbf{H}_1^{\mathbb{A}^1}(Th(\nu_{ij/i})) \cong \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[Z_{ij}] = \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{ij}T],$$

and $\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i})$ is given by the formula for rank 1 case determined in Lemma 4.9. Our conventions say that v in this case comes the orientation θ'_j induced by ${}^t\tilde{\gamma}_j$, which restricts on Y_i exactly to ${}^t\tilde{\gamma}_{ij/i}$ by Lemma 4.23. Thus, in order to compute

$$\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i}) : \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{ij}T] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T]$$

using our orientations, we first need to change the orientation of $\nu_{ij/i}$ to that induced by $\tilde{\gamma}_{ij/i}$, and then apply Lemma 4.9, together with an appropriate twist. More precisely, with the previous orientations and conventions, the cellular \mathbb{A}^1 -chain complex of \bar{Y}_i is given by

$$\tilde{C}^{\text{cell}}(\bar{Y}_i) = \mathbb{Z}[\dot{w}_{ij}''\dot{w}_{ij}''] \otimes \tilde{C}_*^{\text{cell}}(P_{\lambda_j}) \otimes \mathbb{Z}[\dot{w}_{ij}'],$$

with $\tilde{C}_*^{\text{cell}}(P_{\lambda_j})$ being the chain complex:

$$\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[T] \xrightarrow{\partial} \mathbf{Z}_{\mathbb{A}^1}[\dot{\sigma}_{\lambda_j} T]; \quad \partial((v) \otimes [t]) \mapsto [\dot{\sigma}_{\lambda_j} \cdot (\beta_{\lambda_j}^{\vee}(v) \cdot t)],$$

by Remark 3.8 (4). We will use Remark 4.24 and the following straightforward lemma.

Lemma 4.26. *Let $\zeta \in \mathcal{O}_X^\times$ be an invertible function on X . The automorphism of*

$$\mathbf{H}_1^{\mathbb{A}^1}(\text{Th}(\mathcal{O}_X)) = \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[X]$$

induced by the automorphism of the line bundle \mathcal{O}_X given by multiplication by ζ is given at every point $x \in X$ by multiplication by $\langle \zeta(x) \rangle$ on the factor \mathbf{K}_1^{MW} .

Therefore, since $\tilde{w}_i = \dot{w}_{ij}''\dot{w}_{ij}''\dot{\sigma}_{\lambda_j}\dot{w}_{ij}'$, it can be verified that

$$\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/i}) : \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{ij}T] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_iT]$$

is given by

$$(4.13) \quad (v) \otimes [\tilde{w}_{ij} \cdot t] \mapsto \tilde{w}_i \cdot (\langle \beta_{\lambda_j}(t) \rangle \cdot (v)_j)[t]$$

as $(\dot{w}_{ij}')^{-1}(\beta_{\lambda_j}^{\vee}) = \alpha_j^{\vee}$ by Lemma 4.3 (as seen in the proof of Lemma 4.6).

Over a field extension F of k , the sections of the sheaf $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{ij}T]$ are generated as a right $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module by elements of the form $(v) \otimes [\tilde{w}_{ij} \cdot 1]$ and the sections of the sheaf $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{ij}T]$ are generated as a right $\mathbf{Z}_{\mathbb{A}^1}[T]$ -module by elements of the form $(u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]$, where $u, v \in F^\times$ and 1 denotes the neutral element of $T(F)$. So it suffices to determine ∂_2 on these symbols. Substituting (4.13) into (4.12), we obtain

$$(4.14) \quad \partial_2^{(i)}((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = -(u) \otimes ((v)_j \cdot [\tilde{w}_i \cdot 1])$$

in $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_iT]$. We now compute $\partial_2^{(j)}((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1])$. In that case the restriction of θ_j^i to ν_j is $\beta_{\lambda_j}^{-1} \cdot \theta_j$ and the restriction of $\tilde{\gamma}_i$ to Y_j is $\tilde{\gamma}_{ij/j}$ by Lemma 4.23. Now, the chain complex of \bar{Y}_j in this case is:

$$\tilde{C}^{\text{cell}}(\bar{Y}_i) = \mathbb{Z}[\dot{w}_{ij}'''] \otimes \tilde{C}_*^{\text{cell}}(P_{\lambda_i}) \otimes \mathbb{Z}[\dot{w}_{ij}''\dot{w}_{ij}']$$

with $\tilde{C}_*^{\text{cell}}(P_{\lambda_i})$ being the chain complex (by Remark 3.8 (4)):

$$\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[T] \xrightarrow{\partial} \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_jT], \quad \partial((u) \otimes [t]) = [\dot{\sigma}_{\lambda_i} \cdot (\beta_{\lambda_i}^{\vee}(u) \cdot t)]$$

Now, since $w_0s_i s_j = \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \hat{\sigma}_{\lambda_j} \dot{\sigma}_1 s_i s_j$, we have $\dot{w}_{ij}''\dot{w}_{ij}' = \dot{\sigma}_{\lambda_i} \cdots \dot{\sigma}_1 \dot{s}_i \dot{s}_j$. We rewrite this as

$$\begin{aligned} \dot{\sigma}_{\lambda_i} \tilde{w}_{ij}'' \tilde{w}_{ij}' &= \dot{\sigma}_{\lambda_i-1} \cdots \dot{\sigma}_1 \dot{s}_i \dot{s}_j \\ &= \dot{\sigma}_{\lambda_i-1} \cdots \hat{\sigma}_{\lambda_j} \cdots \dot{\sigma}_1 \dot{s}_j \dot{s}_i \dot{s}_j, \end{aligned}$$

which follows from Lemma 4.3 in case s is the reflection $s_j s_i s_j$, as observed in Remark 4.4, as $s_j(\alpha_i) \in \Phi^+$. Consequently, it follows that

$$(\dot{w}_{ij}')^{-1} \circ (\dot{w}_{ij}'')^{-1}(\beta_{\lambda_i}^{\vee}) = s_j(\alpha_i^{\vee}).$$

Thus, $\mathbf{H}_1^{\mathbb{A}^1}(\partial_{ij/j}) : \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{ij}T] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_jT]$ is given by

$$(4.15) \quad (v) \otimes [\tilde{w}_{ij} \cdot t] \mapsto \tilde{w}_j \cdot (\langle \beta_{\lambda_i}(t) \rangle \cdot s_j((v)_i)) [t].$$

However, note that the formula (4.15) is with respect to the orientation θ_j . With respect to the orientation θ_j , the formula for $\partial_2^{(j)}$ is given by Lemma 4.21 as follows:

$$\partial_2^{(j)}((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = (v) \otimes (\tilde{w}_j \cdot s_j \cdot (u)_i).$$

According to our choice of orientations (made in the paragraph above Lemma 4.21), in order to get the correct formula for the differential $\partial_2^{(j)}$, we need to compute the image of $(v) \otimes (\tilde{w}_j \cdot s_j \cdot (u)_i)$ under the automorphism of $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_jT]$ given by the change of orientation from θ_j to ${}^t\theta_j$. By Lemma 4.7, ${}^t\theta_j$ differs from θ_j by the character associated with the root $-\beta_{\lambda_j}$. We will use the following observation in the determination of $\partial_2^{(j)}$.

Lemma 4.27. *For any $v \in F^\times$, $w \in W$ with a lift $\tilde{w} \in N_G(T)$ and $t \in T$, if T acts on \mathbf{K}_1^{MW} on the left through the character $\beta : T \rightarrow \mathbb{G}_m$, then one has*

$$(t) \odot ((v) \otimes [\tilde{w} \cdot 1]) = (\eta(\beta(t))(v)) \otimes [\tilde{w} \cdot 1] + \langle \beta(t) \rangle (v) \otimes [\tilde{w} \cdot w^{-1}(t)]$$

in $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}T]$, where \odot denotes the $\mathbf{Z}_{\mathbb{A}^1}[T]$ -action on $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}T]$.

Proof. One first reduces to the case $T = \mathbb{G}_m$ and $\beta : \mathbb{G}_m \rightarrow \mathbb{G}_m$ a group homomorphism. If $\Psi : \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] \rightarrow \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] \otimes \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m]$ is the diagonal, we have

$$\Psi(t) = (t)_1 + (t)_2 + (t)_1(t)_2,$$

where we identify $\mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m] \otimes \mathbf{Z}_{\mathbb{A}^1}[\mathbb{G}_m]$ with $\mathbb{Z} \oplus \mathbf{K}_1^{\text{MW}} \oplus \mathbf{K}_1^{\text{MW}} \oplus (\mathbf{K}_1^{\text{MW}} \otimes \mathbf{K}_1^{\text{MW}})$ and use the conventions of Section 3.3. Clearly, $(t)_1(t)_2 = (-1)(t)$. Moreover, we have $(t)_i(t')_i = (\eta(t)(t'))_i$, for $i = 1, 2$. Thus, (t) acts through $(v) \mapsto \eta(\beta(t))(v)$ on \mathbf{K}_1^{MW} and using the formula for $\Psi(t)$ and the relation $\langle v \rangle = \eta(v) + 1$, we get

$$\begin{aligned} (t) \odot ((v) \otimes [\tilde{w} \cdot 1]) &= (\eta(\beta(t))(v)) \otimes [\tilde{w} \cdot 1] + (v) \otimes (t)[\tilde{w} \cdot 1] + (\eta(\beta(t))(v)) \otimes (t)[\tilde{w} \cdot 1] \\ &= (\eta(\beta(t))(v)) \otimes [\tilde{w} \cdot 1] + \langle \beta(t) \rangle (v) \otimes (t)[\tilde{w} \cdot 1] \\ &= (\eta(\beta(t))(v)) \otimes [\tilde{w} \cdot 1] + \langle \beta(t) \rangle (v) \otimes [\tilde{w} \cdot w^{-1}(t)]. \end{aligned}$$

□

Given any field extension F of k and $u \in F^\times$, we define

$$(4.16) \quad \rho_{ij} := \beta_{\lambda_j}(s_j(\alpha_i^\vee(u)))^{-1} \in F^\times,$$

for every $i, j \in \{1, \dots, r\}$ with $i \neq j$. Now, putting everything together, it follows from Lemmas 4.7, 4.26 and 4.27 that we have

$$(4.17) \quad \partial_2^{(j)}((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = \langle \rho_{ij} \rangle (v) \otimes \tilde{w}_j \cdot s_j(u)_i + \eta(\rho_{ij})(v) \otimes [\tilde{w}_j \cdot 1]$$

in $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_jT]$.

To summarize, we have obtained the following from the above discussion and formulas (4.14) and (4.17).

Proposition 4.28. *Let F be a finitely generated field extension of k and let $u, v \in F^\times$. Let $i, j \in \{1, \dots, r\}$ be such that $i \neq j$ and let the notations be as above. Then there is a unit ρ_{ij} in F depending only upon u (and given by (4.16)) such that:*

$$\partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = \langle \rho_{ij} \rangle (v) \otimes \tilde{w}_j \cdot s_j(u)_i + \eta(\rho_{ij})(v) \otimes [\tilde{w}_j \cdot 1] - (u) \otimes \tilde{w}_i \cdot (v)_j.$$

We now determine the image of

$$\partial_2 : \tilde{C}_2^{\text{cell}}(G) = \bigoplus_{[i,j]} \mathbf{K}_2^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{[i,j]}T] \rightarrow Z_1(G).$$

We will use the notation consistent with Lemma 4.10. For the sake of brevity, we will simply write w_{ij} for $w_{[i,j]}$. Let F be a finitely generated field extension of k and let $u, v \in F^\times$. For each i , we denote by

$$\phi_i : \mathbf{K}_2^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{ij}T] \rightarrow \bigoplus_{i=1}^r \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T]$$

the morphism defined by

$$(4.18) \quad \begin{aligned} (u, v) \otimes [\tilde{w}_{ij} \cdot 1] &\mapsto \phi_i(u, v) := (u) \otimes \tilde{w}_i \cdot (\alpha_i^\vee(v)) - \eta(u)(v) \otimes [\tilde{w}_i \cdot 1] \\ &= (u) \otimes \tilde{w}_i \cdot (v)_i - \eta(u)(v) \otimes [\tilde{w}_i \cdot 1] \end{aligned}$$

and for each (i, j) with $i \neq j$, we denote by

$$\delta_{ij} : \mathbf{K}_2^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_{ij}T] \rightarrow \bigoplus_{i=1}^r \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T]$$

the morphism defined by

$$(4.19) \quad \begin{aligned} (u, v) &\mapsto \delta_{ij}(u, v) := (u) \otimes \tilde{w}_i \cdot (\alpha_j^\vee(v)) - (v) \otimes \tilde{w}_j \cdot (\alpha_i^\vee(u)) \\ &= (u) \otimes \tilde{w}_i \cdot (v)_j - (v) \otimes \tilde{w}_j \cdot (u)_i. \end{aligned}$$

Note that the image of ϕ_i lies in the summand $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T]$ of $\tilde{C}_1^{\text{cell}}(G)$, whereas the image of δ_{ij} lies in $(\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T]) \oplus (\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_j T])$.

For $i \neq j$, the roots α_i and α_j are related in the group of cocharacters of T by

$$s_j(\alpha_i^\vee) = \alpha_i^\vee + n_{ji}\alpha_j^\vee,$$

where $n_{ji} = -\langle \alpha_j, \alpha_i^\vee \rangle$ is the corresponding coefficient of the Cartan matrix of G [7, Chapter VI, §1, page 144], which is known to belong to the set $\{0, 1, 2, 3\}$. One has $n_{ji} = 0$ if and only if α_i and α_j are not adjacent in the Dynkin diagram of G . It follows that in $\mathbf{Z}_{\mathbb{A}^1}[T]$, with our conventions, we have

$$(4.20) \quad s_j(u)_i = (\alpha_i^\vee(u)\alpha_j^\vee(u^{n_{ji}})) = (u)_i + (u^{n_{ji}})_j + (u)_i(u^{n_{ji}})_j.$$

Theorem 4.29. *Let G be a split, semisimple, almost simple, simply connected algebraic group over k and let F be a finitely generated field extension of k . Let $i, j \in \{1, \dots, r\}$ be such that $i \neq j$. For $u \in F^\times$, let ρ_{ij} be defined by (4.16).*

- (a) *Suppose that α_i and α_j are not adjacent in the Dynkin diagram of G . Then $\langle \rho_{ij} \rangle = 1$ and for any $u, v \in F^\times$, we have*

$$\partial_2((u) \otimes (v) \otimes [\tilde{w}_{[i,j]} \cdot 1]) = \delta_{ji}(v, u).$$

- (b) *Suppose that α_i and α_j are adjacent in the Dynkin diagram of G with $\lambda_i > \lambda_j$ and n_{ji} even; that is, $n_{ji} = 2$. Then $\langle \rho_{ij} \rangle = 1$ and for any $u, v \in F^\times$, we have*

$$\partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = \delta_{ji}(v, u) + \phi_j(v, u^2)(1 + (-1)_i).$$

- (c) *Suppose that α_i and α_j are adjacent in the Dynkin diagram of G with $\lambda_i > \lambda_j$ and n_{ji} odd. Then $\langle \rho_{ij} \rangle = \langle u \rangle$ and we have*

$$\partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = \delta_{ji}(v, u) + \phi_j(\langle u \rangle(v)(u^n))(1 + (u)_i),$$

where $n = n_{ij}$.

Proof. We first prove (a). If α_i and α_j are not adjacent in the Dynkin diagram of G , then $s_i s_j = s_j s_i$ and we have $s_j(\alpha_i^\vee(u)) = (\alpha_i^\vee(u))$. Since $s_i s_j = s_j s_i$, we have $w_{[i,j]} = w_{[j,i]}$ and we can always assume $\lambda_i > \lambda_j$, after permuting the indices if required. Since $\partial_1 \circ \partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = 0$, applying Proposition 4.28 and Lemma 4.9, we obtain

$$\begin{aligned} 0 &= \partial_1 \circ \partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) \\ &= \partial_1(\langle \rho_{ij} \rangle(v) \otimes \tilde{w}_j \cdot s_j(u)_i + \eta(\rho_{ij})(v) \otimes [\tilde{w}_j \cdot 1] - (u) \otimes \tilde{w}_i \cdot (v)_j) \\ &= \tilde{w}_0 \cdot \langle \rho_{ij} \rangle(v)_j \cdot (u)_i + \tilde{w}_0 \cdot (\eta(\rho_{ij})(v))_j - \tilde{w}_0 \cdot (u)_i \cdot (v)_j \end{aligned}$$

in $\mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_0 T]$. By the structure of the commutative algebra $\mathbf{Z}_{\mathbb{A}^1}[T]$ (see (3.16)), it follows that $\eta(\rho_{ij})(v) = 0$ in $\mathbf{K}_1^{\text{MW}}(F)$ and that $\langle \rho_{ij} \rangle(v)_j \cdot (u)_i - (u)_i \cdot (v)_j = 0$. Since $\eta(\rho_{ij})(v) = 0$ for every $v \in F^\times$, it follows from a standard argument using restriction to the purely transcendental field extension $F(t)$ and applying the residue homomorphism at the place t that $\eta(\rho_{ij}) = 0$. Consequently, $\langle \rho_{ij} \rangle = 1$ in $\text{GW}(F)$. Thus, we have

$$\partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = (v) \otimes \tilde{w}_j \cdot (u)_i - (u) \otimes \tilde{w}_i \cdot (v)_j = -\delta_{ij}(u, v) = \delta_{ji}(v, u),$$

proving (a). We now prove (b). Since $n_{ji} = 2$, we have

$$s_j(u)_i = (u)_i + (u^2)_j + (u)_i(u^2)_j.$$

As in the proof of part (a), the relation $\partial_1 \circ \partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = 0$ after applying Proposition 4.28 and Lemma 4.9 yields

$$0 = \tilde{w}_0 \cdot \langle \rho_{ij} \rangle(v)_j \cdot s_j(u)_i + \tilde{w}_0 \cdot (\eta(\rho_{ij})(v))_j - \tilde{w}_0 \cdot (u)_i \cdot (v)_j \in \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_0 T].$$

Thus, after cancelling \tilde{w}_0 , we have the following equality in $\mathbf{Z}_{\mathbb{A}^1}[T]$:

$$\begin{aligned} 0 &= (\langle \rho_{ij} \rangle(v))_j \cdot s_j(u)_i + (\eta[\rho_{ij}](v))_j - (u)_i(v)_j \\ &= (\langle \rho_{ij} \rangle(v))_j(u)_i + (\langle \rho_{ij} \rangle(v))_j(u^2)_j + (\langle \rho_{ij} \rangle(v))_j(u)_i(u^2)_j + (\eta(\rho_{ij})(v))_j - (u)_i(v)_j \end{aligned}$$

As in the proof of (a), it follows from the structure of $\mathbf{Z}_{\mathbb{A}^1}[T]$ that $\eta(\rho_{ij}) = 0$, whence $\langle \rho_{ij} \rangle = 1$ in $\text{GW}(F)$. Now, by Proposition 4.28, we have

$$\begin{aligned} \partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) &= (v) \otimes \tilde{w}_j \cdot s_j(u)_i - (u) \otimes \tilde{w}_i \cdot (v)_j \\ &= (v) \otimes \tilde{w}_j \cdot (u)_i + (v) \otimes \tilde{w}_j \cdot (u^2)_j + (v) \otimes \tilde{w}_j \cdot (u)_i(u^2)_j \\ &\quad - (u) \otimes \tilde{w}_i \cdot (v)_j \\ &= \delta_{ji}(v, u) + (v) \otimes \tilde{w}_j \cdot (u^2)_j + (v) \otimes \tilde{w}_j \cdot (u)_i(u^2)_j \end{aligned}$$

From the relations $(u)(u) = (-1)(u)$, $(u^2) = h(u)$ and $\eta h = 0$ (where $h = 2\epsilon = 1 + \langle -1 \rangle$) in Milnor-Witt K -theory, it follows that

$$(u)_i(u^2)_j = (u)_i(u)_j h = (-1)_i(u)_j h = (-1)_i(u^2)_j$$

and that $\phi_j(v, u^2) = (v) \otimes \tilde{w}_j \cdot (u^2)_j$. Therefore, we obtain

$$\partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = \delta_{ji}(v, u) + \phi_j(v, u^2)(1 + (-1)_i),$$

as desired.

In order to prove (c), we proceed exactly as above to expand and simplify the relation $\partial_1 \circ \partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) = 0$ using Proposition 4.28, Lemma 4.9 and (4.20) to obtain

$$\begin{aligned} 0 &= (\langle \rho_{ij} \rangle(v))_j \cdot s_j(u)_i + (\eta(\rho_{ij})(v))_j - (u)_i(v)_j \\ &= (\langle \rho_{ij} \rangle(v))_j(u)_i + (\langle \rho_{ij} \rangle(v))_j(u^n)_j + (\langle \rho_{ij} \rangle(v))_j(u)_i(u^n)_j + (\eta(\rho_{ij})(v))_j - (u)_i(v)_j \\ &= (\langle \rho_{ij} \rangle(v))_j(u)_i + (\eta(\rho_{ij})(v)(u^n))_j + (u)_i(\eta(\rho_{ij})(v)(u^n))_j + (\eta(\rho_{ij})(v))_j - (u)_i(v)_j \end{aligned}$$

in $\mathbf{Z}_{\mathbb{A}^1}[T]$. By the structure of $\mathbf{Z}_{\mathbb{A}^1}[T]$ (3.16), we obtain

$$\eta(\rho_{ij})(v)(u^n) + \eta(\rho_{ij})(v) = 0$$

and

$$(\langle \rho_{ij} \rangle(v))_j(u)_i + (u)_i(\eta(\rho_{ij})(v)(u^n))_j - (u)_i(v)_j = 0,$$

for any $u, v \in F^\times$. Since n is odd, we have $\eta(u^n) = n_\epsilon \cdot \eta(u) = \eta(u)$. Using the relation $(\rho_{ij}u) = \langle \rho_{ij} \rangle(u) + (\rho_{ij})$ (see [23, Lemma 3.5]), we obtain

$$0 = \eta(\rho_{ij})(v)(u^n) + \eta(\rho_{ij})(v) = \eta(\rho_{ij}u) \cdot (v),$$

for every $v \in F^\times$. Again, using the restriction to $F(t)$ followed by the residue at the place t , it follows that

$$\eta(\rho_{ij}u) = \langle \rho_{ij}u \rangle - 1 = 0 \in \text{GW}(F).$$

Hence, $\langle \rho_{ij} \rangle = \langle u \rangle$ and also $\eta(\rho_{ij}) = \eta(u)$. Since

$$0 = \eta(\rho_{ij}u) = \eta(\rho_{ij}) + \eta(\rho_{ij})(u) = \eta(u) + \eta(u)(u),$$

we also have $-\eta(u)(u) = \eta(u)$. Therefore,

$$\begin{aligned} \partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) &= \langle \rho_{ij} \rangle(v) \otimes \tilde{w}_j \cdot s_j(u)_i + \eta(\rho_{ij})(v) \otimes [\tilde{w}_j \cdot 1] - (u) \otimes \tilde{w}_i \cdot (v)_j \\ &= \langle u \rangle(v) \otimes \tilde{w}_j \cdot (u)_i + \langle u \rangle(v) \otimes \tilde{w}_j \cdot (u^n)_j + \langle u \rangle(v) \otimes \tilde{w}_j \cdot (u)_i(u^n)_j \\ &\quad + \eta(u)(v) \otimes [\tilde{w}_j \cdot 1] - (u) \otimes \tilde{w}_i \cdot (v)_j \\ &= \delta_{ji}(v, u) + \eta(u)(v) \otimes \tilde{w}_j \cdot (u)_i + \langle u \rangle(v) \otimes \tilde{w}_j \cdot (u^n)_j \\ &\quad + \langle u \rangle(v) \otimes \tilde{w}_j \cdot (u)_i(u^n)_j + \eta(u)(v) \otimes [\tilde{w}_j \cdot 1] \\ &= \delta_{ji}(v, u) + (\langle u \rangle(v) \otimes \tilde{w}_j \cdot (u^n)_j + \eta(u)(v) \otimes [\tilde{w}_j \cdot 1]) (1 + (u)_i) \\ &= \delta_{ji}(v, u) + \phi_j(\langle u \rangle(v)(u^n))(1 + (u)_i). \end{aligned}$$

□

In particular, considering these computations modulo T (that is, after tensoring with \mathbb{Z} over $\mathbf{Z}_{\mathbb{A}^1}[T]$) and using Lemma 4.10 we can conclude:

Theorem 4.30. *Let G and F be as in Theorem 4.29. Let i and j be distinct indices in $\{1, \dots, r\}$ and assume $\lambda_i > \lambda_j$ for our chosen reduced expression of w_0 . For any $u, v \in F^\times$, we denote by $\bar{\partial}_{ij}((u)(v))$ the class of*

$$\partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) \in Z_1(G)$$

in $Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$. Then for any $u, v \in F^\times$, we have (with obvious notations):

(a) Suppose that α_i and α_j are not adjacent in the Dynkin diagram of G . Then

$$\bar{\partial}_{ij}((u)(v)) = \bar{\delta}_{ji}(v, u) \in (\mathbf{K}_2^M)_{ji} \subset Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}.$$

(b) Suppose that α_i and α_j are adjacent in the Dynkin diagram of G with n_{ji} even; that is, $n_{ji} = 2$. Then

$$\bar{\partial}_{ij}((u)(v)) = \bar{\delta}_{ji}(v, u) + h \cdot \bar{\phi}_j(v, u) \in (\mathbf{K}_2^{\text{M}})_{ji} \oplus (\mathbf{K}_2^{\text{MW}})_j \subset Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}.$$

(c) Suppose that α_i and α_j are adjacent in the Dynkin diagram of G with n_{ji} odd. Then

$$\bar{\partial}_{ij}((u)(v)) = \bar{\delta}_{ji}(v, u) + (n_{ji})_\epsilon \cdot \bar{\phi}_j(v, u) \in (\mathbf{K}_2^{\text{M}})_{ji} \oplus (\mathbf{K}_2^{\text{MW}})_j \subset Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}.$$

4.4. Image of the degree 2 differential: the case $\lambda_i < \lambda_j$.

Let $[i, j]$ be a pair of indices such that $\lambda_i < \lambda_j$ in the sense of Notation 4.5. Recall that λ_i and λ_j depend upon the fixed reduced expression (4.2) of w_0 . We will show that there always exists a reduced expression

$$w_0 = \sigma'_\ell \cdots \sigma'_1$$

as a product of simple reflections, possibly different from 4.2, such that $\lambda'_i > \lambda'_j$, with obvious notation: λ'_i denotes the unique index such that

$$\sigma'_\ell \cdots \widehat{\sigma'_{\lambda'_i}} \cdots \sigma'_1$$

is a reduced expression of $w_i = w_0 s_i$, for every i .

Notation 4.31. We write $\check{w}_0 = \dot{\sigma}'_\ell \cdots \dot{\sigma}'_1 \in N_G(T)$, and \check{w}_i for the lift of w_i obtained by removing $\dot{\sigma}'_{\lambda'_i}$ from it. These choices define another orientation $\check{\theta}_i$ of ν_i and another identification

$$\mathbf{Z}_{\mathbb{A}^1}(Y_i) \cong \mathbb{Z}[\check{w}_i T].$$

We clearly have an isomorphism of sheaves

$$(4.21) \quad \Upsilon_i : \mathbf{K}_1^{\text{MW}} \otimes \mathbb{Z}[\tilde{w}_i T] \xrightarrow{\cong} \mathbf{K}_1^{\text{MW}} \otimes \mathbb{Z}[\check{w}_i T].$$

We denote for every $i \in \{0, 1, \dots, r\}$

$$\varsigma_i := (\check{w}_i)^{-1} \tilde{w}_i,$$

which is an element of T . The element ς_0 is related to the elements ς_i for $1 \leq i \leq r$ by the following relation.

Lemma 4.32. *For any unit u (in a finitely generated field extension of k), we have*

$$[\varsigma_0](u)_i = [\varsigma_i](\langle \tau_i \rangle(u))_i \in \mathbf{Z}_{\mathbb{A}^1}[T].$$

Proof. The above equality is equivalent to showing that

$$([\varsigma_0] - [\varsigma_i][\tau_i]_i) \cdot (u)_i = 0.$$

This is obtained by applying the formula of the first differential $\partial_1^{(i)}$ from Lemma 4.9, the comparison isomorphism (4.21) and its analogue in degree 0. \square

Proposition 4.33. *Let $\tilde{\theta}_i$ denote the orientation of the normal bundle ν_i of Y_i in G determined by the choice of the reduced expressions in Notation 4.5.*

- (a) Let $\tau_i \in k^\times$ denote the unit corresponding to the comparison of the orientations $\tilde{\theta}_i$ and $\check{\theta}_i$ of ν_i ; in other words, the automorphism

$$\check{\theta}_i^{-1} \circ \tilde{\theta}_i : \mathcal{O}_{Y_i}^1 \rightarrow \mathcal{O}_{Y_i}^1.$$

The composition

$$k \xrightarrow{u_{-\beta_{\lambda_i}}} \mathfrak{u}_{-\beta_{\lambda_i}} \xrightarrow{\text{Ad}_{\tilde{w}_i}} \mathfrak{u}_{-\alpha'_i} \xrightarrow{\text{Ad}_{(\tilde{w}_i^{-1})}} \mathfrak{u}_{-\beta_{\lambda'_i}} \xrightarrow{u_{-\beta_{\lambda'_i}}^{-1}} k$$

through the chosen weak pinning of G corresponds to multiplication by an element of k^\times , which agrees with τ_i up to square classes.

- (b) The isomorphism Υ_i of (4.21) satisfies the following formula, for every units u and v in a finitely generated field extension of k (with obvious notations):

$$\Upsilon_i(\tilde{\phi}_i(u, v)) = \check{\phi}_i(\langle \tau_i \rangle(u)(v)) \cdot [c_i] \in \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\check{w}_i T].$$

Proof. Part (a) follows directly from Lemma 4.26, Lemma 4.32 and the obvious variants of Proposition 3.13 and Lemma 3.15. Part (b) follows immediately from the fact that the isomorphism $\mathbf{K}_1^{\text{MW}} \otimes \mathbb{Z}[\tilde{w}_i T] \xrightarrow{\cong} \mathbf{K}_1^{\text{MW}} \otimes \mathbb{Z}[\check{w}_i T]$ is induced by

$$(u) \otimes [\tilde{w}_i \cdot t] \mapsto \langle \tau_i \rangle(u) \otimes [\check{w}_i \cdot (c_i t)].$$

□

Corollary 4.34. *For any $j \neq i$, we have*

$$\varpi_j(\varsigma_0) = \varpi_j(\varsigma_i)$$

and for $j = i$, we have

$$\varpi_i(\varsigma_0) \equiv \varpi_i(\varsigma_i) \pmod{k^{\times 2}}.$$

In other words, there exists $\tau'_i \in k^\times$ such that

$$\varsigma_0 = \varsigma_i \cdot \alpha_i^\vee(\tau'_i) \in T$$

with $\langle \tau_i \rangle = \langle \tau'_i \rangle$.

Proof. One expands the equation $([c_0] - [c_i][\tau_i]_i) \cdot (u)_i = 0$ in $\mathbf{Z}_{\mathbb{A}^1}[T]$ using the decomposition (3.17). For $t = (t_1, \dots, t_r) \in T$, with t_i in the image of the coroot α_i^\vee , the component of $[t](u)_i$ in the summand \mathbf{K}_1^{MW} corresponding to the index j in (3.17) is, 0 for each $j \neq i$, and is $(u) + \eta(t_i)(u) = \langle t_i \rangle(u)$ for $j = i$. The component of $[t](u)_i$ in the summand \mathbf{K}_2^{MW} corresponding to the index (i, j) in (3.17) for $i \neq j$ is $(t_j)(u)$ (up to permutation, depending on whether $i < j$ or $j < i$). Thus, for a given i , the equation $([c_0] - [c_i][\tau_i]_i) \cdot (u)_i = 0$ for every $u \in F^\times$ implies that for each $j \neq i$, we have

$$\varpi_j(\varsigma_0)(u) = \varpi_j(\varsigma_i)(u) \in \mathbf{K}_2^{\text{MW}},$$

for every $u \in F^\times$. By restricting to a purely transcendental extension of F and taking a suitable residue, it follows that $(\varpi_j(\varsigma_0)) = (\varpi_j(\varsigma_i))$ in \mathbf{K}_1^{MW} and hence, $\varpi_j(\varsigma_0) = \varpi_j(\varsigma_i)$ in \mathbb{G}_m . On the other hand, for $j = i$, we have

$$\langle \varpi_j(\varsigma_0) \rangle(u) = \langle \varpi_j(\varsigma_i) \rangle \langle \tau_i \rangle(u) \in \mathbf{K}_1^{\text{MW}},$$

for every $u \in F^\times$ and consequently, in the same way as above, we have $\langle \varpi_j(\varsigma_0) \rangle(u) = \langle \varpi_j(\varsigma_i) \rangle \tau_i$ in $\text{GW}(k)$. This proves the claim. □

Remark 4.35. As one sees above, τ_i is only well-defined up to squares (since it depends upon the weak pinning). So in fact we may always choose $\tau_i \in k^\times$ so that we have $\varsigma_0 = \varsigma_i \cdot \alpha_i^\vee(\tau_i)$ as elements of T . Thus, we have

$$[\varsigma_0] = [\varsigma_i] \cdot [\tau_i]_i$$

in $\mathbf{Z}_{\mathbb{A}^1}[T]$. We will fix such a choice of τ_i , for every $i \in \{1, \dots, r\}$, in what follows.

Using these observations, we may rewrite the above change of reduced expression isomorphism (4.21) as well as the morphisms ϕ_i and δ_{ij} (see (4.18) and (4.19)) as follows.

Proposition 4.36. *With the above notation, we have the following, for all units u, v in a finitely generated field extension of k .*

(a) *For every $i \in \{1, \dots, r\}$, we have*

$$\Upsilon_i((u) \otimes [\tilde{w}_i \cdot 1]) = (u) \otimes [\check{w}_i \cdot \varsigma_0] - \check{\phi}_i(u, \tau_i) \cdot [\varsigma_i].$$

(b) *For every $i \in \{1, \dots, r\}$, we have*

$$\Upsilon_i(\check{\phi}_i(u, v)) = \check{\phi}_i(u, v) \cdot [\varsigma_0] - \check{\phi}_i(u, \tau_i) \cdot [\varsigma_i](v)_i + \check{\phi}_i(\eta(u)(v)(\tau_i)) \cdot [\varsigma_i]$$

in $\mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\check{w}_i T]$.

(c) *For $i \neq j$, we have:*

$$\Upsilon_i(\check{\delta}_{ij}(u, v)) = \check{\delta}_{ij}(u, v) \cdot [\varsigma_0] - \check{\phi}_i(u, \tau_i) \cdot [\varsigma_i](v)_j + \check{\phi}_j(v, \tau_i) \cdot [\varsigma_j](u)_i.$$

Proof. We have $\Upsilon_i((u) \otimes [\tilde{w}_i \cdot 1]) = \langle \tau_i \rangle(u) \otimes [\check{w}_i \cdot \varsigma_i]$. Since $\langle \tau_i \rangle = 1 + \eta(\tau_i)$, we get

$$\begin{aligned} \Upsilon_i((u) \otimes [\tilde{w}_i \cdot 1]) &= \langle \tau_i \rangle(u) \otimes [\check{w}_i \cdot \varsigma_i] \\ &= (u) \otimes [\check{w}_i \cdot \varsigma_i] + (\eta(u)(\tau_i)) \otimes [\check{w}_i \cdot \varsigma_i] \\ &= (u) \otimes [\check{w}_i \cdot \varsigma_i] + (u) \otimes [\check{w}_i \cdot \varsigma_i](\tau_i)_i - (u) \otimes [\check{w}_i \cdot \varsigma_i](\tau_i)_i \\ &\quad + (\eta(u)(\tau_i)) \otimes [\check{w}_i \cdot \varsigma_i] \\ &= (u) \otimes [\check{w}_i \cdot \varsigma_i](1 + (\tau_i)_i) - \check{\phi}_i(u, \tau_i) \cdot [\varsigma_i] \\ &= (u) \otimes [\check{w}_i \cdot \varsigma_i][\tau_i]_i - \check{\phi}_i((u, \tau_i) \cdot [\varsigma_i]) \\ &= (u) \otimes [\check{w}_i \cdot \varsigma_0] - \check{\phi}_i(u, \tau_i) \cdot [\varsigma_i], \end{aligned}$$

by Remark 4.35. This proves (a). Parts (b) and (c) follow similarly by using the relation $\langle \tau_i \rangle \cdot [\varsigma_i] = [\varsigma_0]$, for each i and the formulas for $\check{\phi}_i$ and $\check{\delta}_{ij}$ analogous to (4.18) and (4.19). \square

Corollary 4.37. *The automorphism*

$$\bar{\Upsilon} : \left(\bigoplus_i \mathbf{K}_2^{\text{MW}} \right) \oplus \left(\bigoplus_{(i,j); i < j} \mathbf{K}_2^{\text{M}} \right) \rightarrow \left(\bigoplus_i \mathbf{K}_2^{\text{MW}} \right) \oplus \left(\bigoplus_{(i,j); i < j} \mathbf{K}_2^{\text{M}} \right)$$

induced by the change of reduced expression isomorphism on $C_1^{\text{cell}}(G)$:

$$\bigoplus_i \Upsilon_i : \bigoplus_i \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\tilde{w}_i T] \xrightarrow{\cong} \bigoplus_i \mathbf{K}_1^{\text{MW}} \otimes \mathbf{Z}_{\mathbb{A}^1}[\check{w}_i T]$$

through the identification $\left(\bigoplus_i \mathbf{K}_2^{\text{MW}} \right) \oplus \left(\bigoplus_{(i,j); i < j} \mathbf{K}_2^{\text{M}} \right) \cong Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$ given by Lemma 4.10 satisfies the following, for any finitely generated field extension F of k and $u, v \in F^\times$.

(a) For every i , we have

$$\bar{\Upsilon}(\tilde{\phi}_i(u, v)) = \check{\phi}_i(\langle \tau_i \rangle(u)(v)) \in \mathbf{K}_2^{\text{MW}}.$$

(b) For every $i \neq j$, we have

$$\bar{\Upsilon}(\tilde{\delta}_{ij}(u, v)) = \check{\delta}_{ij}(u, v) \in \mathbf{K}_2^{\text{M}}.$$

Proof. This is a direct consequence of the formulas in Proposition 4.36, using the observation that after going modulo T (that is, applying $-\otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$), one has

$$\check{\phi}_i(u, \tau_i)[\varsigma_i](v)_j = 0$$

and

$$\check{\phi}_j(v, \tau_j)[\varsigma_j](u)_i = 0,$$

since $(u)_i = [u]_i - [1]$ and $(v)_j = [v]_j - [1]$ are both in the augmentation ideal; that is, the kernel of the morphism $\mathbf{Z}_{\mathbb{A}^1}[T] \rightarrow \mathbb{Z}$. \square

We are now set to reduce to the case $\lambda_i > \lambda_j$ already treated above using the following lemma and the computations on change of reduced expression for w_0 carried out above.

Lemma 4.38. *Let $[i, j]$ be a pair of indices such that $\lambda_i < \lambda_j$ in the sense of Notation 4.5 with respect to the reduced expression $w_0 = \sigma_\ell \cdots \sigma_1$ given by (4.2). There exists another reduced expression $\sigma'_\ell \cdots \sigma'_1$ of w_0 for which, with obvious notations, one has $\lambda'_i > \lambda'_j$.*

Proof. Let $W_{\{i, j\}} \subset W$ be the subgroup generated by s_i and s_j . Note that $W_{\{i, j\}}$ is also a Coxeter group, being the Weyl group of the subgroup G_{ij} of G of semi-simple rank 2 generated by T , S_{α_i} and S_{α_j} . Let $v_0 \in W_{\{i, j\}}$ be the longest element of $W_{\{i, j\}}$. Then $u_0 = w_0 v_0$ has the property that for any $v \in W_{\{i, j\}}$, $\ell(u_0 v) = \ell(u_0) + \ell(v)$. This follows easily from the properties of the longest word in Coxeter groups (see [3, Proposition 2.20]).

Now, multiplication by u_0 on the left induces a map $W_{\{i, j\}} \rightarrow W$ preserving length and takes v_0 to w_0 . Given a reduced expression of u_0 and a reduced expression of v_0 in $W_{\{i, j\}}$ thus gives a reduced expression of w_0 in W . In this way, we reduce the statement of the lemma to the rank 2 case, as there are only two possible reduced expressions for v_0 in the rank 2 case. One easily verifies that one of these has the required property. \square

We thus obtain the following analogue of Theorem 4.30 in the case $\lambda_i < \lambda_j$, using Corollary 4.37.

Theorem 4.39. *Let G and F be as in Theorem 4.29. Let i and j be distinct indices in $\{1, \dots, r\}$ and assume $\lambda_i < \lambda_j$ for our chosen reduced expression (4.2) of w_0 . For any $u, v \in F^\times$, we denote by $\bar{\partial}_{ij}((u)(v))$ the class of*

$$\partial_2((u) \otimes (v) \otimes [\tilde{w}_{ij} \cdot 1]) \in Z_1(G)$$

in $Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$. Then for any $u, v \in F^\times$, we have (with obvious notations):

(a) If α_i and α_j are not adjacent in the Dynkin diagram of G , then

$$\bar{\partial}_{ij}((u)(v)) = \bar{\delta}_{ji}(v, u) \in (\mathbf{K}_2^{\text{M}})_{ji} \subset Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}.$$

(b) If α_i and α_j are adjacent in the Dynkin diagram of G with $n_{ji} = 2$, then

$$\bar{\partial}_{ij}((u)(v)) = \bar{\delta}_{ji}(v, u) + h \cdot \bar{\phi}_j(v, u) \in (\mathbf{K}_2^{\text{M}})_{ji} \oplus (\mathbf{K}_2^{\text{MW}})_j \subset Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}.$$

- (c) If α_i and α_j are adjacent in the Dynkin diagram of G with n_{ji} odd, then there exists $\tau_j \in k^\times$ such that:

$$\bar{\partial}_{ij}((u)(v)) = \bar{\delta}_{ji}(v, u) + (n_{ji})_\epsilon \langle \tau_j \rangle \cdot \bar{\phi}_j(v, u) \in (\mathbf{K}_2^M)_{ji} \oplus (\mathbf{K}_2^{\text{MW}})_j \subset Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}.$$

Proof. If α_i and α_j are not adjacent in the Dynkin diagram of G , then $s_i s_j = s_j s_i$. But then we have $w_{ij} = w_{ji}$ and we can always assume $\lambda_i > \lambda_j$ to obtain (a) from Theorem 4.30. For (b), we conclude from Theorem 4.30 and Corollary 4.37 that there exists $\tau_j \in k^\times$ such that

$$\bar{\partial}_{ij}((u)(v)) = \bar{\delta}_{ji}(v, u) + h \cdot \langle \tau_j \rangle \cdot \bar{\phi}_j(v, u) \in (\mathbf{K}_2^M)_{ji} \oplus (\mathbf{K}_2^{\text{MW}})_j \subset Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}.$$

But by [23, Lemma 3.9 (ii)], we have $h \cdot \langle \tau_j \rangle = h$ and (b) is proved. Part (c) is proved in exactly the same way and is left to the reader. \square

5. THE MAIN THEOREM, APPLICATIONS AND MISCELLANEOUS REMARKS

5.1. Proof of the main theorem.

Let S be a split, almost simple, simply connected semi-simple k -group of rank 1. Such a group is (non-canonically) isomorphic to \mathbf{SL}_2 and thus, $\pi_1^{\mathbb{A}^1}(S)$ is (non-canonically) isomorphic to \mathbf{K}_2^{MW} . The action of the adjoint group \mathbf{PGL}_2 on \mathbf{SL}_2 induces a nontrivial action on $\pi_1^{\mathbb{A}^1}(\mathbf{SL}_2) = \mathbf{K}_2^{\text{MW}}$. However, the action of the adjoint group \mathbf{PGL}_2 on the quotient

$$\mathbf{K}_2^M = \mathbf{K}_2^{\text{MW}} / \eta$$

is trivial, and in fact, the induced epimorphism $\pi_1^{\mathbb{A}^1}(S) \twoheadrightarrow \mathbf{K}_2^M$ is canonical.

We are now ready to prove the main Theorem. We will freely make use of basic results on root systems from [7, Chapter VI].

Theorem 5.1. *Let G be a split, semisimple, almost simple, simply connected k -group, with a fixed maximal torus $T \subset G$ and a fixed Borel subgroup $B \subset G$ containing T . We denote by Φ the corresponding root system, by Φ^\vee its dual and by $\Delta \subset \Phi$ the set of simple roots corresponding to B . Let $\alpha \in \Delta$ be a long root (so that α^\vee is a small coroot) and let $S_\alpha \subset G$ be the split, semisimple, almost simple, simply connected k -subgroup of rank 1 generated by U_α and $U_{-\alpha}$.*

- (a) *If G is not of symplectic type, then the inclusion $S_\alpha \subset G$ induces a canonical isomorphism*

$$\mathbf{K}_2^M \xrightarrow{\cong} \pi_1^{\mathbb{A}^1}(G)$$

induced by the morphism $\pi_1^{\mathbb{A}^1}(S_\alpha) \rightarrow \pi_1^{\mathbb{A}^1}(G)$ through the canonical epimorphism $\pi_1^{\mathbb{A}^1}(S) \rightarrow \mathbf{K}_2^M$.

- (b) *If G is of symplectic type, then the inclusion $S_\alpha \subset G$ induces a non-canonical isomorphism*

$$\mathbf{K}_2^{\text{MW}} \xrightarrow{\cong} \pi_1^{\mathbb{A}^1}(G).$$

More precisely, an isomorphism $\mathbf{K}_2^{\text{MW}} \cong \pi_1^{\mathbb{A}^1}(G)$ corresponds to the choice of a weak pinning of S_α , or equivalently, an isomorphism $\mathbf{SL}_2 \cong S_\alpha$ up to the adjoint action of \mathbf{PGL}_2 .

Proof. We choose a labelling $(\alpha_1, \dots, \alpha_r)$ of Δ as in [7, §4]. Let M be a strictly \mathbb{A}^1 -invariant sheaf over Sm_k . We prove Theorem 5.1 by explicitly computing the kernel of the morphism

$$(5.1) \quad \mathrm{Hom}_{Ab_{\mathbb{A}^1}(k)}(Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}, M) \rightarrow \mathrm{Hom}_{Ab_{\mathbb{A}^1}(k)}(C_2^{\mathrm{cell}}(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}, M)$$

induced by the morphism $C_2^{\mathrm{cell}}(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \rightarrow Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$ of the exact sequence (4.1). We will denote this kernel by \mathcal{M} . Clearly, we have

$$\mathcal{M} = \mathrm{Hom}_{Ab_{\mathbb{A}^1}(k)}(\mathrm{Coker}(\partial_2 : C_2^{\mathrm{cell}}(G) \rightarrow Z_1(G)), M) = \mathrm{Hom}_{Ab_{\mathbb{A}^1}(k)}(\mathbf{H}_1^{\mathbb{A}^1}(G), M);$$

from (4.1). We will show that

$$\mathcal{M} = \begin{cases} {}_{\eta}M_{-2}(k) = \mathrm{Hom}_{Ab_{\mathbb{A}^1}(k)}(\mathbf{K}_2^{\mathrm{M}}, M), & \text{if } G \text{ is not of symplectic type;} \\ M_{-2}(k) = \mathrm{Hom}_{Ab_{\mathbb{A}^1}(k)}(\mathbf{K}_2^{\mathrm{MW}}, M), & \text{if } G \text{ is of symplectic type.} \end{cases}$$

for every $M \in Ab_{\mathbb{A}^1}(k)$; the theorem then follows from the Yoneda lemma and the fact that $\mathbf{H}_1^{\mathbb{A}^1}(G) = \boldsymbol{\pi}_1^{\mathbb{A}^1}(G)$ as already mentioned in the introduction.

We now choose a weak pinning of G , and a reduced expression of the longest word w_0 of W . We can then orient the cellular \mathbb{A}^1 -chain complex of G as explained in Section 4 and describe the morphism (5.1) using Theorem 4.30 and Theorem 4.39. We first deduce from Lemma 4.10 that

$$\mathrm{Hom}_{Ab_{\mathbb{A}^1}(k)}(Z_1(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}, M) = M_{-2}(k)^r \times \left({}_{\eta}M_{-2}(k) \right)^{\frac{r(r-1)}{2}},$$

since $\mathrm{Hom}_{Ab_{\mathbb{A}^1}(k)}(\mathbf{K}_2^{\mathrm{MW}}, M) = M_{-2}(k)$ for every $M \in Ab_{\mathbb{A}^1}(k)$ by [23, Lemma 2.34, Theorem 3.37]. Let N denote the number of codimension 2 Bruhat cells in G ; then with Notation 3.21, we have

$$\mathrm{Hom}_{Ab_{\mathbb{A}^1}(k)}(C_2^{\mathrm{cell}}(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}, M) = M_{-2}(k)^N.$$

Theorems 4.30 and 4.39 along with the fact that $\delta_{ji}(v, u) = -\delta_{ij}(u, v)$ imply that there exists units $\tau_i \in k^\times$ such that the morphism (5.1) above has the form:

$$M_{-2}(k)^r \times ({}_{\eta}M_{-2}(k))^{\frac{r(r-1)}{2}} \rightarrow M_{-2}(k)^N; \quad ((x_i)_i; (y_{ij})_{i < j}) \mapsto (z_{[i,j]})_{ij}$$

such that for given (i, j) , $i \neq j$ the following holds:

- (1) if α_i and α_j are not adjacent in the Dynkin diagram of G , then

$$z_{[i,j]} = -y_{ij};$$

- (2) if α_i and α_j are adjacent and $n_{ji} = \langle \alpha_j, \alpha_i^\vee \rangle = 2$, then

$$z_{[i,j]} = -y_{ij} + h \cdot x_j;$$

- (3) if α_i and α_j are adjacent and $n = n_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle \in \{1, 3\}$, then

$$z_{[i,j]} = y_{ij} + \langle \tau_j \rangle \cdot n_\epsilon \cdot x_j.$$

We now describe the kernel of (5.1); let $((x_i)_i; (y_{ij})_{i < j}) \in M_{-2}(k)^r \times ({}_{\eta}M_{-2}(k))^{\frac{r(r-1)}{2}}$ be such that $(z_{[i,j]})_{ij} = 0$. If α_i and α_j are not adjacent, then $z_{[i,j]} = 0$ is equivalent to $y_{ij} = 0$ by (1) above.

Let α_i and α_j be adjacent roots with $i < j$ and $n_{ji} = 1$. Then $z_{[i,j]} = 0 = z_{[j,i]}$ is equivalent to $y_{ij} = \langle -\tau_j \rangle x_j = \langle -\tau_i \rangle x_i$ by (3). Equivalently, we have $x_i = \langle -\tau_i \rangle y_{ij}$ and $x_j = \langle -\tau_j \rangle y_{ij}$. However, since $y_{ij} \in {}_\eta M_{-2}(k)$, on which $\text{GW}(k)$ acts trivially, we see that

$$x_i = y_{ij} = x_j.$$

This implies that x_i is η -torsion for every i . Furthermore, if G is such that $n_{ij} = 1$ for every adjacent pair of roots α_i and α_j with $i < j$, then the morphism of abelian groups

$$\mathcal{M} \rightarrow {}_\eta M_{-2}(k); \quad ((x_i)_i; (y_{(ij)})_{i < j}) \mapsto x_1$$

is an isomorphism (note that we may take x_i for any fixed i in the above isomorphism). Note that this happens precisely when all the simple roots have the same length; that is, the cases when G is of the type A , D or E (in other words, G is simply laced).

Now, assume that G is not simply laced. Then there exists a unique index i such that $n_{i+1,i} \in \{2, 3\}$. Assume that $n_{i+1,i} = 3$; in this case, $i = 1$ and G is of the type G_2 with α_2 being the long root. In this case, we have $n_{1,2} = 1$ and $y_{1,2} = -\langle \tau_2 \rangle \cdot 3_\epsilon \cdot x_2 = \langle -\tau_1 \rangle \cdot x_1$. Since the $\text{GW}(k)$ -action on the y_{12} 's is trivial, we have

$$y_{1,2} = -3_\epsilon \cdot x_2 = x_1.$$

Thus, the morphism of abelian groups

$$\mathcal{M} \rightarrow {}_\eta M_{-2}(k); \quad ((x_i)_i; (y_{(ij)})_{i < j}) \mapsto x_2$$

is an isomorphism. Finally, we consider the case where $n_{i+1,i} = 2$; that is, α_{i+1} is a long root. We then have $y_{i,i+1} = -\langle \tau_{i+1} \rangle \cdot h \cdot x_{i+1} = \langle -\tau_i \rangle \cdot x_i$, by (2). Since the $\text{GW}(k)$ -action on the y_{ij} 's is trivial, we obtain

$$y_{i,i+1} = -h \cdot x_{i+1} = x_i.$$

Thus, x_{i+1} determines $((x_i)_i; (y_{(ij)})_{i < j})$. Suppose that G is not of symplectic type. Then α_{i+1} is connected to another long root and it follows that $x_{i+1} \in {}_\eta M_{-2}(k)$ by the above computations. Hence, the morphism of abelian groups

$$\mathcal{M} \rightarrow {}_\eta M_{-2}(k); \quad ((x_i)_i; (y_{(ij)})_{i < j}) \mapsto x_{i+1}$$

is an isomorphism.

It only remains to consider the case where G is of symplectic type. In this case, $i = r$ and it follows proceeding as above that

$$y_{r-1,r} = -h \cdot x_r = x_{r-1}$$

and the morphism of abelian groups

$$\mathcal{M} \rightarrow M_{-2}(k); \quad ((x_i)_i; (y_{(ij)})_{i < j}) \mapsto x_r$$

is an isomorphism. This completes the proof of the theorem. \square

Remark 5.2. The proof of Theorem 5.1 above gives a much more precise statement of part (a). For instance, if β is another root of longest length, then the inclusion $S_\beta \subset G$ induces an isomorphism

$$\mathbf{K}_2^{\mathbb{M}} \cong \pi_1^{\mathbb{A}^1}(G),$$

which is $(-1)^{d(\alpha,\beta)}$ times the isomorphism of the theorem, where $d(\alpha,\beta) \in \mathbb{N}$ is the distance between the roots α and β in the Dynkin diagram of G . The proof also enables one to describe the morphism

$$\pi_1^{\mathbb{A}^1}(S_\alpha) \rightarrow \pi_1^{\mathbb{A}^1}(G)$$

for any root α ; we leave the details to the reader.

5.2. Central extensions of split, semisimple, simply connected groups.

In this section, we determine the set of isomorphism classes of central extensions of a split, semisimple, simply connected algebraic group over a field k by strictly \mathbb{A}^1 -invariant sheaves as a consequence of our main theorem.

Let G be a sheaf of groups on the big Nisnevich site of smooth schemes $(Sm_k)_{\text{Nis}}$ over k . Let M be a sheaf of abelian groups on $(Sm_k)_{\text{Nis}}$. We begin with a brief recollection of Grothendieck's classification of central extensions of G by M given in [17, VII, no. 1] in terms of multiplicative M -torsors on G . Although Grothendieck's construction works over any topos, we will restrict ourselves to $(Sm_k)_{\text{Nis}}$ and Nisnevich locally trivial torsors. Recall that the group of isomorphism classes of M -torsors on G can be identified with the cohomology group $H_{\text{Nis}}^1(G, M)$. A pair (E, ϕ) is called a *multiplicative M -torsor* on G if E is an M -torsor on G and ϕ is an isomorphism

$$\phi : \text{pr}_1^* E + \text{pr}_2^* E \xrightarrow{\cong} \mu^* E$$

of M -torsors on $G \times G$, where $\text{pr}_i : G \times G \rightarrow G$ denotes the i th projection and $+$ denotes the group operation on $H_{\text{Nis}}^1(G, M)$. Let $\text{CExt}(G, M)$ denote the set of isomorphism classes of central extensions of G by M as Nisnevich sheaves of groups on Sm_k , which are exact sequences of sheaves of the form

$$1 \rightarrow M \rightarrow E \rightarrow G \rightarrow 1,$$

where M is a subsheaf of the center of E . This set forms a group under the operation of Baer sum. Such an E is an M -torsor on G equipped with an isomorphism $\text{pr}_1^* E + \text{pr}_2^* E \xrightarrow{\cong} \mu^* E$, since M is contained in the center of E . This gives an isomorphism of groups between $\text{CExt}(G, M)$ and the group of isomorphism classes of multiplicative M -torsors on G .

Now assume that M is a strictly \mathbb{A}^1 -invariant sheaf. By [23, Remark 1.8], the Eilenberg-MacLane space $K(M, n)$ is \mathbb{A}^1 -local, for every $n \in \mathbb{N}$. We have identifications

$$H_{\text{Nis}}^1(G, M) = \text{Hom}_{D_{\mathbb{A}^1}(k)}(C_*^{\mathbb{A}^1}(G), M[1])$$

and

$$H_{\text{Nis}}^1(G \wedge G, M) = \text{Hom}_{D_{\mathbb{A}^1}(k)}(C_*^{\mathbb{A}^1}(G \wedge G), M[1]).$$

Furthermore, if G is a cellular scheme in the sense of Definition 2.11, then one may replace $D_{\mathbb{A}^1}(k)$ with $D(\text{Ab}_{\mathbb{A}^1}(k))$ and $C_*^{\mathbb{A}^1}(G)$ with $C_*^{\text{cell}}(G)$ in the above identifications. We note the following simple observation, which follows from the fact that the morphism $C_*^{\mathbb{A}^1}(G \times G) \rightarrow C_*^{\mathbb{A}^1}(G)$ induced by $\text{pr}_1^* + \text{pr}_2^* - \mu^*$ factors through the morphism $C_*^{\mathbb{A}^1}(G \times G) \rightarrow C_*^{\mathbb{A}^1}(G \wedge G)$ induced by the quotient map $G \times G \rightarrow G \wedge G$.

Lemma 5.3. *For G and M as above, the group of multiplicative M -torsors over G is isomorphic to the kernel of the group homomorphism*

$$H_{\text{Nis}}^1(G, M) \rightarrow H_{\text{Nis}}^1(G \wedge G, M)$$

induced by the morphism $\text{pr}_1^ + \text{pr}_2^* - \mu^* : C_*^{\mathbb{A}^1}(G \wedge G) \rightarrow C_*^{\mathbb{A}^1}(G)$ under the identifications given above.*

Let $F_\bullet G$ denote the *skeletal filtration* on the simplicial classifying space BG of G :

$$* = F_0 BG \subset F_1 BG = \Sigma G \subset \cdots \subset F_{n-1} BG \subset F_n BG \subset \cdots \subset BG,$$

which comes with cofiber sequences

$$F_{n-1} BG \rightarrow F_n BG \rightarrow \Sigma^n G^{\wedge n}$$

for every n , where we set $\Sigma^i \mathcal{X} := S^i \wedge \mathcal{X}$ for any space \mathcal{X} and any $i \in \mathbb{N}$. For the reasons of connectivity, we have an identification

$$\mathrm{Hom}_{\mathcal{H}(k)}(BG, K(M, 2)) = \mathrm{Hom}_{\mathcal{H}(k)}(F_3 BG, K(M, 2)).$$

The long exact sequence of simplicial homotopy classes of maps into $K(M, 2)$ corresponding to the cofiber sequence $F_2 BG \rightarrow F_3 BG \rightarrow \Sigma^3 G^{\wedge 3}$ has the form

$$\begin{aligned} \mathrm{Hom}_{\mathcal{H}_s(k)}(\Sigma^3 G^{\wedge 3}, K(M, 2)) &\rightarrow \mathrm{Hom}_{\mathcal{H}_s(k)}(F_3 BG, K(M, 2)) \\ &\rightarrow \mathrm{Hom}_{\mathcal{H}_s(k)}(F_2 BG, K(M, 2)) \rightarrow \mathrm{Hom}_{\mathcal{H}_s(k)}(\Sigma^2 G^{\wedge 3}, K(M, 2)). \end{aligned}$$

But $\mathrm{Hom}_{\mathcal{H}_s(k)}(\Sigma^3 G^{\wedge 3}, K(M, 2)) = 0$. Since $K(M, 2)$ is \mathbb{A}^1 -local, we obtain the exact sequence

$$0 \rightarrow \mathrm{Hom}_{\mathcal{H}(k)}(F_3 BG, K(M, 2)) \rightarrow \mathrm{Hom}_{\mathcal{H}(k)}(F_2 BG, K(M, 2)) \rightarrow \mathrm{Hom}_{\mathcal{H}(k)}(\Sigma^2 G^{\wedge 3}, K(M, 2))$$

by replacing $\mathcal{H}_s(k)$ in the above exact sequence with $\mathcal{H}(k)$. Now assume that G is a split, semisimple, simply connected algebraic group over k . By [22, Theorem A.2], $\pi_0^{\mathbb{A}^1}(G^{\wedge 3})$ is a quotient of $T^{\wedge 3}$. Therefore, it follows that the group

$$\mathrm{Hom}_{\mathcal{H}(k)}(\Sigma^2 G^{\wedge 3}, K(M, 2)) = H_{\mathrm{Nis}}^0(G^{\wedge 3}, M)$$

is trivial, provided M is such that $M_{-3}(k) = 0$. Hence, we have an identification

$$\mathrm{Hom}_{\mathcal{H}(k)}(F_3 BG, K(M, 2)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{H}(k)}(F_2 BG, K(M, 2)).$$

Now, the long exact sequence corresponding to the cofiber sequence

$$\Sigma G = F_1 BG \rightarrow F_2 BG \rightarrow \Sigma^2 G^{\wedge 2}$$

has the form

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_{\mathcal{H}(k)}(F_2 BG, K(M, 2)) &\rightarrow \mathrm{Hom}_{\mathcal{H}(k)}(F_1 BG, K(M, 2)) = H_{\mathrm{Nis}}^1(G, M) \\ &\rightarrow \mathrm{Hom}_{\mathcal{H}(k)}(\Sigma G^{\wedge 2}, K(M, 2)) = H_{\mathrm{Nis}}^1(G \wedge G, M). \end{aligned}$$

Since G is \mathbb{A}^1 -connected, $G \wedge G$ is \mathbb{A}^1 -1-connected. Consequently, $\pi_1^{\mathbb{A}^1}(G \wedge G)$ is trivial and hence, $H_{\mathrm{Nis}}^1(G \wedge G, M) = 0$. We summarize the above discussion, which along with our main theorem reproves and generalizes [8, Proposition 4.6], [16, Theorem B] as follows.

Theorem 5.4. *Let G be a split, semisimple, simply connected algebraic group over a field k and let M be a strictly \mathbb{A}^1 -invariant sheaf such that $M_{-3}(k) = 0$. Then there are isomorphisms*

$$\mathrm{CExt}(G, M) \cong \mathrm{Hom}_{\mathcal{H}(k)}(BG, K(M, 2)) \cong H_{\mathrm{Nis}}^1(G, M).$$

Furthermore, the group $H_{\mathrm{Nis}}^1(G, M) = H_{\mathrm{Zar}}^1(G, M)$ can be explicitly computed as in Corollary 12 in the introduction.

5.3. Cellular \mathbb{A}^1 -homology of the flag variety G/B in low degrees.

Let G be a split, semisimple, almost simple, simply connected algebraic group over k of rank r . Fix a maximal k -split torus T of G and a Borel subgroup B containing T . In this subsection, we describe the cellular \mathbb{A}^1 -chain complex for the full flag variety G/B , which is much simpler compared to the one for G described above, because of the fact that G/B is cellular in the sense of Definition 2.7. Also note that the canonical quotient morphism $G/T \rightarrow G/B$ is an \mathbb{A}^1 -weak equivalence.

The Bruhat decomposition of G induces a decomposition $G/B = \bigcup_{w \in W} B\dot{w}B/B$, which induces a stratification by open subsets

$$\emptyset = \bar{\Omega}_{-1} \subsetneq \bar{\Omega}_0 \subsetneq \bar{\Omega}_1 \subsetneq \cdots \subsetneq \bar{\Omega}_\ell = G,$$

where ℓ is the length of the longest element w_0 in W and for every i , one has

$$\bar{\Omega}_i - \bar{\Omega}_{i-1} = \coprod_{\ell(w)=\ell-i} B\dot{w}B/B \simeq \coprod_{\ell(w)=\ell-i} \mathbb{A}^{\ell-i}.$$

The cellular \mathbb{A}^1 -chain complex of G/B is oriented by the choice of a weak pinning of G (Definition 3.17) and the choice of a normal reduced expression (Remark 3.19) as a product of simple reflections for every element of the Weyl group W of G . The cellular \mathbb{A}^1 -chain complex of G/B associated with the Bruhat decomposition has the form (see Remark 4.8)

$$\tilde{C}_i^{\text{cell}}(G/B) = \bigoplus_{\ell(w)=\ell-i} \mathbf{Z}_{\mathbb{A}^1}(\mathbb{G}_m^{\wedge i}) \otimes \mathbb{Z}[w] = \bigoplus_{\ell(w)=\ell-i} \mathbf{K}_i^{\text{MW}} \otimes \mathbb{Z}[w].$$

Theorems 4.30 and 4.39 enable us to explicitly describe differentials $\bar{\partial}_i$ in $\tilde{C}_*^{\text{cell}}(G/B)$ in degrees $i \leq 2$. We first observe that for every $i \geq 1$ we have

$$\text{Hom}_{\text{Ab}_{\mathbb{A}^1}(k)}(\mathbf{K}_{i+1}^{\text{MW}}, \mathbf{K}_i^{\text{MW}}) = W(k) \text{ and } \text{Hom}_{\text{Ab}_{\mathbb{A}^1}(k)}(\mathbf{K}_1^{\text{MW}}, \mathbb{Z}) = 0.$$

This shows that $\bar{\partial}_1 = 0$. The words $w \in W$ with length $\ell - 1$ are all of the form $w_0 s_i$, for $1 \leq i \leq n$ and those with length $\ell - 2$ are all of the form $w_0 s_i s_j$, for $1 \leq i, j \leq n$. We further have the relation $w_0 s_i s_j = w_0 s_j s_i$ if and only if the roots α_i and α_j are not adjacent in the Dynkin diagram of G . We describe the differential $\bar{\partial}_2$ in the case of G of rank 2 as a straightforward consequence of Theorems 4.30 and 4.39.

Theorem 5.5. *Let G be a split, semisimple, almost simple, simply connected algebraic group over k of rank r with a fixed maximal k -split torus T and a Borel subgroup B containing T . We have isomorphisms of complexes*

$$C_*^{\text{cell}}(G/B) \cong C_*^{\text{cell}}(G/T) \cong C_*^{\text{cell}}(G) \otimes_{\mathbf{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}.$$

and the following holds.

- (a) For i, j with $i \neq j$, the component of $\bar{\partial}_2$ from $\mathbf{K}_2^{\text{MW}} \otimes \mathbb{Z}[w_0 s_i s_j]$ to $\mathbf{K}_1^{\text{MW}} \otimes \mathbb{Z}[w_0 s_i]$ is always 0.
- (b) If moreover α_i and α_j are adjacent in the Dynkin diagram of G , then the component of $\bar{\partial}_2$ from $\mathbf{K}_2^{\text{MW}} \otimes \mathbb{Z}[w_0 s_i s_j]$ to $\mathbf{K}_1^{\text{MW}} \otimes \mathbb{Z}[w_0 s_j]$ is given by multiplication by $n_\epsilon \cdot \eta \in W(k)$, where $n = n_{ji} = \alpha_j^\vee(\alpha_i)$.

Corollary 5.6. *With the notation of Theorem 5.5, we have $\mathbf{H}_0^{\text{cell}}(G/B) = \mathbf{H}_0^{\text{cell}}(G/T) = \mathbb{Z}$ and*

$$\mathbf{H}_1^{\text{cell}}(G/B) = \mathbf{H}_1^{\text{cell}}(G/T) = \begin{cases} T, & \text{if } G \text{ is not of symplectic type;} \\ \left(\prod_{i=1}^{r-1} \mathbb{G}_m \right) \times \mathbf{K}_1^{\text{MW}}, & \text{if } G \text{ is of symplectic type.} \end{cases}$$

Remark 5.7. The cellular \mathbb{A}^1 -chain complex of the full flag variety G/B of G as in Theorem 5.5 is considerably simpler to handle compared to $C_*^{\text{cell}}(G)$. Using an appropriate generalization of the methods of Section 3 and Section 4, one may compute the entire cellular \mathbb{A}^1 -homology of G/B . The details will appear in a sequel to this paper.

Remark 5.8. Let G be a split, semisimple, simply connected algebraic group over k . There exists an \mathbb{A}^1 -fibration sequence

$$(5.2) \quad G \rightarrow G/T \xrightarrow{\phi} BT.$$

Recall that BT can be given a cellular structure by choosing a basis of the character group $Y = \text{Hom}(T, \mathbb{G}_m)$ of T , which gives an isomorphism $T \cong \prod_{i=1}^r \mathbb{G}_m$ and thereby gives $(B\mathbb{G}_m)^r = (\mathbb{P}^\infty)^r$ as a model of BT . Using the results of Section 2.5 and the Künneth formula, one can compute the complete cellular \mathbb{A}^1 -homology of BT . The morphism $\phi : G/T \rightarrow BT$ induces an epimorphism $\mathbf{H}_1^{\text{cell}}(\phi) : \mathbf{H}_1^{\text{cell}}(G/T) \rightarrow \mathbf{H}_1^{\text{cell}}(BT) = T$, since G is \mathbb{A}^1 -connected. It can be easily verified that $\mathbf{H}_1^{\text{cell}}(\phi)$ is an isomorphism if and only if G is not of symplectic type.

Remark 5.9. Let G be a split, semisimple, simply connected algebraic group over k . There exists an exact sequence of the form

$$(5.3) \quad \mathbf{H}_2^{\text{cell}}(G/T) \xrightarrow{\mathbf{H}_2^{\text{cell}}(\phi)} \mathbf{H}_2^{\text{cell}}(BT) \rightarrow \mathbf{H}_1^{\text{cell}}(G) \rightarrow \mathbf{H}_1^{\text{cell}}(G/T) \xrightarrow{\mathbf{H}_1^{\text{cell}}(\phi)} T \rightarrow 0.$$

A part of the exact sequence (5.3) can be concluded in an elementary way: the long exact sequence of \mathbb{A}^1 -homotopy sheaves associated with the \mathbb{A}^1 -fibration $G \rightarrow G/T \xrightarrow{\phi} BT$ gives the exact sequence

$$\pi_1^{\mathbb{A}^1}(G) \rightarrow \pi_1^{\mathbb{A}^1}(G/T) \rightarrow T \rightarrow 0,$$

which on abelianization gives the exact sequence of last three terms in (5.3). However, this information is not sufficient for the calculation of $\pi_1^{\mathbb{A}^1}(G) \simeq \mathbf{H}_1^{\mathbb{A}^1}(G)$. The exact sequence (5.3) can be obtained as the five-term exact sequence associated with the Serre spectral sequence or the motivic Rothenberg-Steenrod spectral sequence corresponding to the \mathbb{A}^1 -fibration (5.2), computing the cellular \mathbb{A}^1 -homology of BT .

Remark 5.10. Let G be a split, semisimple, almost simple, simply connected algebraic group over k . The morphism $\mathbf{H}_2^{\text{cell}}(BT) \rightarrow \mathbf{H}_1^{\text{cell}}(G)$ factors through $\mathbf{H}_2^{\text{cell}}(BT)_W$, the group of coinvariants for the action of the Weyl group W of G on $\mathbf{H}_2^{\text{cell}}(BT)$. It can be shown that there are isomorphisms

$$\mathbf{H}_1^{\text{cell}}(G) \simeq \mathbf{H}_2^{\text{cell}}(BT)_W \simeq \mathbf{K}_2^{\text{M}},$$

when G is not of symplectic type. However, when G is of symplectic type, one can show that

$$\mathbf{H}_2^{\text{cell}}(BT)_W \simeq \mathbf{K}_2^{\text{M}}/2 \oplus {}_\eta \mathbf{K}_2^{\text{MW}}.$$

These computations are achieved by determining the cokernel of the morphism $\mathbf{H}_2^{\text{cell}}(\phi)$ in (5.3) using, for example, comparison with Suslin homology and results of Demazure [10] on

the multiplicative structure of the Chow ring of G/B . A detailed analysis of coinvariants of the Weyl group action on $\mathbf{H}_*^{\text{cell}}(BT)$ and its relationship with $\mathbf{H}_*^{\text{cell}}(BG)$ will be taken up in the sequel to this paper.

5.4. Formalization of the method and relation to a theorem of E. Cartan.

We begin this section by formalizing the method of our proof of the main theorem (Theorem 5.1), so that the method can be applied under suitable realization functors to obtain results pertaining to other homology theories (such as Suslin homology).

We consider a triangulated \otimes -category \mathcal{D} endowed with a t -structure compatible with the \otimes -structure. As we use the homological point of view, we will denote by $\mathcal{D}_{\geq 0}$ (respectively, $\mathcal{D}_{\leq 0}$) the full sub-category of non-negative object (respectively, of non-positive objects) for the t -structure. If $\Sigma : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$ is the suspension functor for the triangulated structure, we denote, for any $n \in \mathbb{Z}$, by $\mathcal{D}_{\geq n}$ the full subcategory of objects D such that $\Sigma^{\circ-n}(D)$ is non-negative. We write $H_i^t(D)$ for the homology in degree i of an object $D \in \mathcal{D}$. Then for any non-negative object $D \in \mathcal{D}_{\geq 0}$, there is a exact triangle of the form

$$D_{\geq 1} \rightarrow D \rightarrow H_0^t(D) \rightarrow D_{\geq 1}[1]$$

with $D_{\geq 1} \in \mathcal{D}_{\geq 1}$ and $H_0^t(D) \in \mathcal{A}$, the heart of the t -structure, which is an abelian category. We refer the reader to [5] for the precise definition and basic properties of t -structures on triangulated categories.

Definition 5.11. We call a functor

$$C_* : \mathcal{H}(k) \rightarrow \mathcal{D}$$

a *reasonable homological chain functor* for smooth k -varieties if it satisfying the following properties:

- (1) C_* is a symmetric \otimes -functor;
- (2) $C_*(\mathbb{A}^1)$ is the zero object of \mathcal{D} ; and
- (3) if U, V are open subschemes of a smooth k -scheme X covering X , the following obvious Mayer-Vietoris diagram in \mathcal{D} :

$$C_*(U \cap V) \rightarrow C_*(U) \oplus C_*(V) \rightarrow C_*(X) \rightarrow \Sigma(C_*(U \cap V))$$

is an exact triangle.

If $C_* : \mathcal{H}(k) \rightarrow \mathcal{D}$ is a reasonable homological chain functor, then it is easy to see that $C_*(\mathbb{P}^1) \cong \Sigma C_*(\mathbb{G}_m)$ and $C_*(\mathbb{A}^2 - \{0\}) \cong \Sigma C_*(\mathbb{G}_m^{\wedge 2})$ are both objects of $\mathcal{D}_{\geq 1}$. For M and N in \mathcal{A} , we set $M \otimes_0 N := H_0^t(M \otimes N)$. We set

$$\mathbb{Z}^t(1) := H_1^t(\mathbb{P}^1) = \tilde{H}_0^t(\mathbb{G}_m)$$

and

$$\mathbb{Z}^t(2) := H_1^t(\mathbb{A}^2 - \{0\}) = \tilde{H}_0^t((\mathbb{G}_m)^{\wedge 2}) \cong \mathbb{Z}^t(1) \otimes_0 \mathbb{Z}^t(1).$$

Furthermore we denote by $\eta^t : \mathbb{Z}^t(2) \rightarrow \mathbb{Z}^t(1)$ the H_1^t applied to the Hopf map $\mathbb{A}^2 - \{0\} \rightarrow \mathbb{P}^1$.

Observe that any unit u of k defines an element $(u) \in \text{Hom}_{\mathcal{D}}(\mathbb{Z}, \mathbb{Z}^t(1))$, and also defines an automorphism $\langle u \rangle$ of $\mathbb{Z}^t(1)$. We then define

$$\mathbb{Z}^t(n) := \mathbb{Z}^t(1)^{\otimes n},$$

for $n \geq 1$. Then $\eta^t \otimes \mathbb{Z}^t(1)$ induces a morphism $\mathbb{Z}^t(3) \rightarrow \mathbb{Z}^t(2)$. For any pair of units (u, v) in k we have a corresponding symbol $(u)(v) \in \text{Hom}_{\mathcal{D}}(\mathbb{Z}, \mathbb{Z}^t(2))$ and for any units α, u and v in k , the induced action of $\langle \alpha \rangle$ on the symbol $(u)(v)$ is given by (see Remark 8)

$$\langle \alpha \rangle \cdot (u)(v) = (u)(v) + \eta(\alpha)(u)(v).$$

Theorem 5.12. *Let $C_* : \mathcal{H}(k) \rightarrow \mathcal{D}$ be a reasonable homological chain functor for smooth k -varieties as above. Then for any split simply connected semi-simple group G , $C_*(G) \in \mathcal{D}_{\geq 1}$. Moreover, if G is almost simple, then we have the following:*

- (a) *If G is not of symplectic type, then the inclusion $S_\alpha \subset G$ induces a canonical isomorphism*

$$\mathbb{Z}^t(2)/\eta^t \xrightarrow{\cong} H_1^t(C_*(G))$$

induced by the morphism $H_1^t(C_(S_\alpha)) \rightarrow H_1^t(C_*(G))$ through the canonical epimorphism $H_1^t(C_*(S)) \cong \mathbb{Z}^t(2) \rightarrow \mathbb{Z}^t(2)/\eta^t$.*

- (b) *If G is of symplectic type, then the inclusion $S_\alpha \subset G$ induces a non-canonical isomorphism*

$$\mathbb{Z}^t(2) \xrightarrow{\cong} H_1^t(C_*(G)).$$

Proof. The proof Theorem 5.1 can clearly be adapted word-by-word to the more general case of a triangulated \otimes -category admitting reasonable homological chain functor C_* for smooth k -varieties. Observe, for instance, that the use of the \mathbb{A}^1 -homotopy purity theorem used above can always be obtained by an explicit open tubular neighborhood which is isomorphic to a trivial vector bundle (of rank one or two) over the closed subscheme (a Bruhat cell) under consideration (see Lemma 4.18). The relations on symbols that we use are proven in $\mathcal{H}(k)$ so that they are respected on applying the functor $C_* : \mathcal{H}(k) \rightarrow \mathcal{D}$, so that we get exactly the same relations in the symbols computed in \mathcal{D} . We leave the details to the reader. \square

Theorem 5.12 can be used to uniformly explain and reprove some classical results from topology and known results about motives under suitable realization functors, as can be seen below.

Example 5.13.

- (1) The main example of a reasonable homological chain functor for smooth k -varieties is the \mathbb{A}^1 -chain functor $C_*^{\mathbb{A}^1} : \mathcal{H}(k) \rightarrow D_{\mathbb{A}^1}(k)$. In that case Theorem 5.12 is nothing but our main theorem (Theorem 5.1).
- (2) If $\mathcal{D} = DM(k)$ and $C_* : \mathcal{H}(k) \rightarrow DM(k)$ is the functor constructed by Voevodsky associating with every space its motive, we see that $\mathbb{Z}^t(n) = \mathbf{K}_n^M$ and $H_1^t = \mathbf{H}_1^S$, the first Suslin homology sheaf. Thus, in this case, Theorem 5.12 precisely becomes [16, Theorem A], first obtained by S. Gille (Corollary 15).
- (3) If $\mathcal{D} = D(\mathcal{A}b)$ is the derived category of abelian groups, $k = \mathbb{R}$ and $C_* : \mathcal{H}(k) \rightarrow D(\mathcal{A}b)$ is the functor $X \mapsto C_*^{\text{sing}}(X(\mathbb{R}); \mathbb{Z})$, induced by the singular chain complex of the topological space of real points with integral coefficients, then $\mathbb{Z}^t(1) = H_1^{\text{sing}}(\mathbb{P}^1(\mathbb{R}); \mathbb{Z}) \cong \mathbb{Z}$ and η is the multiplication by 2 : $\mathbb{Z} \rightarrow \mathbb{Z}$. Theorem 5.12 in this case proves the following: for any split, semisimple, simply connected algebraic group G over \mathbb{R} , the topological space $G(\mathbb{R})$ is path-connected; if moreover G is almost simple and not of symplectic type, there is a canonical isomorphism

$$\pi_1(G) = \mathbb{Z}/2\mathbb{Z};$$

and if G is of symplectic type, there is a non canonical isomorphism

$$\pi_1(G) = \mathbb{Z}.$$

- (4) If $\mathcal{D} = D(\mathcal{A}b)$ is the derived category of abelian groups, $k = \mathbb{C}$ and $C_* : \mathcal{H}(k) \rightarrow D(\mathcal{A}b)$ is the functor $X \mapsto C_*^{\text{sing}}(X(\mathbb{C}); \mathbb{Z})$, induced by the singular chain complex of the topological space of complex points with integral coefficients, then $\mathbb{Z}^t(1) = H_1^{\text{sing}}(\mathbb{P}^1(\mathbb{C}); \mathbb{Z}) = 0$ and $\eta = 0$. Theorem 5.12 in this case proves that for any split, semisimple, simply connected algebraic group G over \mathbb{C} , the topological space $G(\mathbb{C})$ is path-connected and simply connected.

We now explain how our methods used in the proofs of the main results of the paper (especially, cellular structures) can further be used to give a “motivic” proof of the following theorem of E. Cartan [12, page 496] (see [9] and [6, Theorems B and B’] for the classical proofs using different methods).

Theorem 5.14. *For any split, semisimple, almost simple, simply connected algebraic group G over \mathbb{C} , the topological space $G(\mathbb{C})$ is 2-connected and there is a canonical isomorphism:*

$$\pi_3(G(\mathbb{C})) \cong \mathbb{Z}.$$

Proof. We will use the stratification of $G(\mathbb{C})$ induced by the Bruhat decomposition (3.1) of G . We observe that the inclusion $\Omega_2(\mathbb{C}) \subset G(\mathbb{C})$ is 4-connected; this follows from topological purity by observing that the Bruhat cells in the complement of $\Omega_2(\mathbb{C})$ have complex codimension ≥ 3 and consequently, real codimension ≥ 6 . Hence, $H_i^{\text{sing}}(\Omega_2(\mathbb{C}); \mathbb{Z}) \cong H_i^{\text{sing}}(G(\mathbb{C}); \mathbb{Z})$ for $i \in \{0, \dots, 4\}$. By Example 5.13(4), $G(\mathbb{C})$ is path-connected and simply connected. By the Hurewicz theorem, it suffices to prove that $H_2^{\text{sing}}(\Omega_2(\mathbb{C}); \mathbb{Z}) = 0$ and $H_3^{\text{sing}}(\Omega_2(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}$.

The principal $B(\mathbb{C})$ -bundle $G(\mathbb{C}) \rightarrow G/B(\mathbb{C})$ along with the fact that the natural map $T(\mathbb{C}) \rightarrow B(\mathbb{C})$ is a homotopy equivalence gives the fibration

$$T(\mathbb{C}) \Rightarrow G(\mathbb{C}) \rightarrow G/B(\mathbb{C}).$$

Consider the restriction of this fibration

$$T(\mathbb{C}) \Rightarrow \Omega_2(\mathbb{C}) \rightarrow \overline{\Omega}_2(\mathbb{C}),$$

where $\overline{\Omega}_2(\mathbb{C}) := \Omega_2(\mathbb{C})/B(\mathbb{C})$. We use the homological Serre spectral sequence with integral coefficients for the above fibration.

Now, $\overline{\Omega}_2(\mathbb{C})$ is a cellular space with 2-cells parameterized by the simple roots or the words of length $\ell - 1$ and 4-cells by the words of length $\ell - 2$. Let $W^{(i)} \subset W$ denote the subset of the Weyl group W of G consisting of the words of length $\ell - i$, for each i (recall that ℓ is the length of the longest element of W). For simplicity, we write H_n for singular homology with coefficients in \mathbb{Z} . Then $\overline{\Omega}_2(\mathbb{C})$ is simply connected and its integral singular homology is given by

$$H_0(\overline{\Omega}_2(\mathbb{C})) = \mathbb{Z}; \quad H_2(\overline{\Omega}_2(\mathbb{C})) = \bigoplus_{w \in W^{(1)}} \mathbb{Z}; \quad H_4(\overline{\Omega}_2(\mathbb{C})) = \bigoplus_{w \in W^{(2)}} \mathbb{Z};$$

and $H_n(\overline{\Omega}_2(\mathbb{C})) = 0$, if $n \notin \{0, 2, 4\}$. The integral singular homology of $T(\mathbb{C}) = \prod_{i=1}^r \mathbb{G}_m(\mathbb{C}) \cong \prod_{i=1}^r S^1$ is the exterior algebra $\Lambda^* Y$, where Y is the cocharacter group $\text{Hom}(\mathbb{G}_m, T)$, whose elements are in degree 1. The $E_{p,q}^2$ -term of the Serre spectral sequence with integral coefficients is thus:

$$E_{p,q}^2 = H_p(\overline{\Omega}_2(\mathbb{C}); H_q(T(\mathbb{C}))) = H_p(\overline{\Omega}_2(\mathbb{C})) \otimes_{\mathbb{Z}} H_q(T(\mathbb{C})) = H_p(\overline{\Omega}_2(\mathbb{C})) \otimes_{\mathbb{Z}} \Lambda^q Y$$

and the Serre spectral sequence has the form

$$(5.4) \quad E_{p,q}^2 = H_p(\overline{\Omega}_2(\mathbb{C})) \otimes_{\mathbb{Z}} \Lambda^q Y \Rightarrow H_{p+q}(\Omega_2(\mathbb{C})).$$

It follows that $E_{0,q}^2 = \Lambda^q Y$, $E_{2,q}^2 = Y \otimes_{\mathbb{Z}} \Lambda^q Y$ and $E_{4,q}^2 = Y^{(2)} \otimes_{\mathbb{Z}} \Lambda^q Y$ where we set $Y^{(2)} := \bigoplus_{W^{(2)}} \mathbb{Z}$. We wish to understand the differential $d^2 : E_{*,*}^2 \rightarrow E_{*-2,*+1}^2$ in (5.4).

Consider the cofibrations

$$\Omega_0(\mathbb{C}) \subset \Omega_1(\mathbb{C}) \rightarrow Th(\nu_1)(\mathbb{C})$$

and

$$\Omega_1(\mathbb{C}) \subset \Omega_2(\mathbb{C}) \rightarrow Th(\nu_2)(\mathbb{C})$$

obtained by taking the complex points of the corresponding \mathbb{A}^1 -cofibration sequences $\Omega_0 \subset \Omega_1 \rightarrow Th(\nu_1)$ and $\Omega_1 \subset \Omega_2 \rightarrow Th(\nu_2)$. The filtration $\Omega_0(\mathbb{C}) \subset \Omega_1(\mathbb{C}) \subset \Omega_2(\mathbb{C})$ is exactly the inverse image of the skeletal filtration of $\overline{\Omega}_2(\mathbb{C})$ under the ‘‘fibration’’ $\Omega_2(\mathbb{C}) \rightarrow \overline{\Omega}_2(\mathbb{C})$ up to reindexing (as there are only even-dimensional cells). Now $\Omega_0(\mathbb{C}) \cong T(\mathbb{C})$ and consequently, its homology is $\Lambda_* Y$. Then $Th(\nu_1) \cong \bigvee_{i=1}^r (\mathbb{A}^1 / (\mathbb{A}^1 - \{0\})) \wedge (T_+)$ so that $Th(\nu_1)(\mathbb{C}) \cong \bigvee_{i=1}^r S^2 \wedge (T(\mathbb{C})_+)$ and its reduced singular homology is given by

$$H_*^{\text{sing}}(Th(\nu_1)(\mathbb{C}))[-2] = \bigoplus_{i=1}^r \Lambda^* Y = \Lambda^* Y \otimes_{\mathbb{Z}} Y.$$

This is precisely the column $p = 2$ of the Serre spectral sequence (5.4). In the same way, we obtain $Th(\nu_2)(\mathbb{C}) \cong \bigvee_{W^{(2)}} S^4 \wedge (T(\mathbb{C})_+)$ so that its reduced singular homology is

$$H_*^{\text{sing}}(Th(\nu_2)(\mathbb{C}))[-4] = \bigoplus_{W^{(2)}} \Lambda_*(Y) = \Lambda_*(Y) \otimes_{\mathbb{Z}} Y^{(2)},$$

which is precisely the column $p = 4$ of the Serre spectral sequence (5.4).

The differential $d^2 : E_{*,*}^2 \rightarrow E_{*-2,*+1}^2$ is then analyzed as follows. In the cofibration $\Omega_0 \subset \Omega_1 \rightarrow Th(\nu_1)$, each Bruhat cell $Bw_i B = Y_i$ of codimension 1 admits (using our conventions from Section 4.1) an explicit tubular neighborhood in Ω_1 isomorphic to $\mathbb{A}^1 \times Y_i$, so that we get a homotopy cocartesian square of spaces of the form:

$$\begin{array}{ccc} \prod_{i=1}^r (\mathbb{A}^1 - \{0\}) \times Y_i & \longrightarrow & \prod_{i=1}^r \mathbb{A}^1 \times Y_i \\ \downarrow & & \downarrow \\ \Omega_0 & \longrightarrow & \Omega_1. \end{array}$$

Taking complex points and using the homotopy equivalence between $Y_i(\mathbb{C})$ and $T(\mathbb{C})$, the restriction of the connecting homomorphism

$$H_{p+2}(Th(\nu_1)(\mathbb{C})) \rightarrow H_{p+1}(\Omega_0(\mathbb{C}))$$

to the i th direct summand in the decomposition $H_{p+2}(Th(\nu_1)(\mathbb{C})) \cong \bigoplus_{i=1}^r H_{p+2}(S^2 \wedge (T(\mathbb{C})_+))$ is given by the composition

$$\Lambda^p Y = H_p(Y_i(\mathbb{C})) \subset H_{p+1}((\mathbb{C}^1 - \{0\}) \times Y_i(\mathbb{C})) \rightarrow H_{p+1}(\Omega_0(\mathbb{C})) = H_{p+1}(T(\mathbb{C})) = \Lambda^{p+1} Y.$$

Moreover, since the above morphism and the homotopy equivalence $Y_i(\mathbb{C}) \rightarrow T(\mathbb{C})$ are both left $T(\mathbb{C})$ -invariant, evaluating on the homology turns it into a morphism of $H_*(T(\mathbb{C})) = \Lambda^* Y$ -modules.

The differential $d^2 : E_{2,0}^2 \rightarrow E_{0,1}^2$ is the identity map of Y ; this follows from the fact that for each i , the morphism $\mathbb{A}^1 - \{0\} \rightarrow (\mathbb{A}^1 - \{0\}) \times (Y_i) \rightarrow \Omega_0 \cong T$ is given by the i th coroot α_i^\vee . More generally, it follows that $d^2 : E_{2,*}^2 \rightarrow E_{0,*+1}^2$ is given by the product (in Λ^*Y) map

$$\Lambda^*Y \otimes_{\mathbb{Z}} Y \rightarrow \Lambda^*Y.$$

Since this is a surjective morphism, $H_p(\Omega_2(\mathbb{C})) = 0$ for $p \leq 2$. Analyzing the spectral sequence further, we see that $\pi_3(G(\mathbb{C})) = H_3(G(\mathbb{C})) = H_3(\Omega_2(\mathbb{C}))$ is the homology of the complex

$$(5.5) \quad Y^{(2)} \rightarrow Y \otimes_{\mathbb{Z}} Y \twoheadrightarrow \Lambda_2(Y)$$

at $Y \otimes_{\mathbb{Z}} Y$, where the left morphism is given by the differential

$$d^2 : Y^{(2)} = E_{4,0}^2 \rightarrow E_{2,1}^2 = Y \otimes_{\mathbb{Z}} Y$$

of the Serre spectral sequence (5.4). We claim that the restriction of the differential d^2 to the factor \mathbb{Z} corresponding to $w_{ij} = w_0 s_i s_j$ in $Y^{(2)} = \bigoplus_{W^{(2)}} \mathbb{Z}$ is given by

$$(5.6) \quad \mathbb{Z} \rightarrow Y \otimes_{\mathbb{Z}} Y; \quad 1 \mapsto s_j(\alpha_i^\vee) \otimes \alpha_j^\vee - \alpha_j^\vee \otimes \alpha_i^\vee \in \bigoplus_{W^{(1)}} Y = Y \otimes_{\mathbb{Z}} \left(\bigoplus_{W^{(1)}} \mathbb{Z} \right) = Y \otimes_{\mathbb{Z}} Y.$$

This follows from the proofs of Theorem 4.30 and Theorem 4.39. Recall that in the proof of Theorem 4.30, we used for each w_{ij} an explicit open neighborhood of Z_{ij} in \tilde{X}_{ij} given by the total space of a trivial rank 2-vector bundle embedded as an open subset in \tilde{X}_{ij} and consequently, an explicit model $(\mathbb{A}^2 - \{0\}) \times Z_{ij} \rightarrow \Omega_1$ for the connecting morphism appearing in the differential of the cellular \mathbb{A}^1 -chain complex. Proceeding as above and using our computations, we may identify for each w_{ij} the morphism given by the composite

$$\begin{aligned} \mathbb{Z} &= H_3((\mathbb{C}^2 - \{0\}) \times Z_{ij}(\mathbb{C})) \\ &\cong H_3(S^3 \times T(\mathbb{C})) \rightarrow H_3(\Omega_1(\mathbb{C})) \rightarrow H_3(\bigvee_{i=1}^r S^2 \wedge (T(\mathbb{C})_+)) \cong H_3(\Omega_1/\Omega_0(\mathbb{C})) = Y \otimes_{\mathbb{Z}} Y \end{aligned}$$

with the morphism (5.6) above. Observe that because of the use of complex points and integral singular homology, the issue of change of orientations showing up in the proofs of Theorem 4.30 and Theorem 4.39 disappears, as any complex vector bundle is canonically oriented.

Now to conclude, we just observe that the dual of the diagram (5.5) appears in the spectral sequence of the filtration computing $H^*(G; \mathbf{K}_2^M)$ in degrees ≤ 2 , that is, computing $H^*(\Omega_2; \mathbf{K}_2^M)$ with respect to the filtration $\Omega_0 \subset \Omega_1 \subset \Omega_2$. Since $H^2(G; \mathbf{K}_2^M) = H^2(\Omega_2; \mathbf{K}_2^M) = \mathbb{Z}$ canonically, we conclude that the complex dual to (5.5) has homology in the middle equal to \mathbb{Z} (as it is a diagram of finite type free \mathbb{Z} -modules). This gives us the isomorphism

$$\pi_3(G(\mathbb{C})) = H_3(G(\mathbb{C})) = H_3(\Omega_2(\mathbb{C})) = \mathbb{Z}.$$

□

Remark 5.15. The (5.5) appearing in the proof of Theorem 5.14 is precisely dual to [8, (4.4.3), page 38]. In fact the two situations are closely related for the following reason. There is a canonical natural transformation of functors on smooth k -schemes

$$H^p(X; \mathbb{Z}(q)) \rightarrow H^p(X(\mathbb{C}); \mathbb{Z})$$

where $H^p(X; \mathbb{Z}(q))$ are the integral motivic cohomology groups of Voevodsky [32]. This natural transformation induces a morphism of the cohomological spectral sequences in integral motivic cohomology for Ω_2 (or G) induced by the above flag to the cohomological spectral

sequence in integral cohomology for $\Omega_2(\mathbb{C})$ (or $G(\mathbb{C})$) induced by the complex points of the same flag. The latter is as we observe above the Serre spectral sequence (after conveniently reindexing). Now, for a smooth k -scheme X , one has

$$H^p(X; \mathbb{Z}(q)) \cong H^{p-q}(X; \mathbf{K}_q^M),$$

for $p \geq 2q - 2$ [31, Corollary 2.4]. Hence, the cohomological spectral sequence in integral motivic cohomology for Ω_2 coincides with the same for $H^p(X; \mathbf{K}_q^M)$ in the area we are concerned with (after a suitable reindexing). The above comparison morphism from integral motivic cohomology to integral singular cohomology of the complex points thus identifies our sequence (5.5) to the dual of [8, (4.4.3), page 38]. It follows that in fact we have proved the following motivic variant of Cartan's theorem on π_3 :

Corollary 5.16. *For any split, semisimple, almost simple, simply connected algebraic group G over \mathbb{C} , $G(\mathbb{C})$ is 2-connected and the canonical morphism*

$$\mathbb{Z} = H^1(G; \mathbf{K}_2^M) \cong H^3(G; \mathbb{Z}(2)) \rightarrow H^3(G(\mathbb{C}); \mathbb{Z})$$

is an isomorphism.

Remark 5.17. We can also easily conclude that the morphism $Y^{(2)} = \bigoplus_{W(2)} \mathbb{Z} \rightarrow Y \otimes Y$ in (5.5) is a monomorphism. It is proven in [8, Proof of Proposition 4.6] that the morphism dual to the above morphism is surjective. In our proof of Theorem 5.14 above, we showed that the homology in the middle of (5.5) is free of rank one. As each group in (5.5) is free of finite type, the injectivity of $Y^{(2)} \rightarrow Y \otimes Y$ follows by observing that the rank of $Y^{(2)}$ is exactly one less than the rank of the kernel of $Y \otimes_{\mathbb{Z}} Y \rightarrow \Lambda_2(Y)$, which is the rank of the second symmetric power $S^2(X)$ on $X = Y^*$. This is easy to verify and it implies that $H^2(G; \mathbf{K}_2^M) = 0$. From Corollary 2.29 we know that $H^i(G; \mathbf{K}_2^M) = 0$ for $i > 2$. This gives another proof of [8, Proposition 4.6], which we restate here for the convenience of the reader.

Corollary 5.18. *For any split, semisimple, almost simple, simply connected algebraic group G over k , $H^0(G; \mathbf{K}_2^M) = K_2^M(k)$, $H^1(G; \mathbf{K}_2^M) = \mathbb{Z}$ and $H^i(G; \mathbf{K}_2^M) = 0$ for $i \geq 2$.*

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