Milnor's conjecture on quadratic forms and mod 2 motivic complexes

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Abstract: Let F be field of characteristic different from 2. In this paper we give a new proof of Milnor's conjecture on the graded ring associated to the powers of the fundamental ideal of the Witt ring of quadratic forms over F, first proven by Orlov, Vishik and Voevodsky. Our approach also relies on Voevodsky's affirmation of Milnor's conjecture on the mod 2 Galois cohomology of fields of characteristic different from 2, but, besides this fact, we only use some elementary homological algebra in the abelian category of Zariski sheaves on the category of smooth k-varieties, involving classical results on sheaves of Witt groups, Rost's cycle modules and sheaves of 0-equidimensional cycles.

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1 Introduction

Let F be field of characteristic $\neq 2$. In this paper we give a new proof of Milnor's conjecture identifying the mod 2 Milnor K-theory of F to the graded ring associated to the filtration of the Witt ring $W(F)$ of anisotropic quadratic forms over F by the powers of its fundamental ideal [15]. This result was first obtained by Orlov, Vishik and Voevodsky in [23] where they proved more. This also appears in [11, Remark 3.3]. In both cases however, sophisticated technics and results from Voevodsky's proof of Milnor conjecture on the mod 2 Galois cohomology of fields [32, 33] are used, such as triangles involving the Rost motives, the Milnor operations Q_i.

Our approach uses Voevodsky's affirmation of Milnor's conjecture on the mod 2 Galois cohomology of fields, but, however, it is quite different in spirit from the previous ones. Fix a perfect base field k, let Sm_k denote the category of smooth k-schemes and let $\mathcal{A}b_k$ denote the abelian category of sheaves of abelian groups in the Zariski topology on Sm_k. Besides Voevodsky's result we will only use some elementary homological algebra in the abelian category $\mathcal{A}b_k$ involving sheaves of Witt groups, Rost's cycle modules [24] and sheaves of 0-equidimensional cycles [28]. It is also rather different from our original proof announced in [16]. There we were using the Adams spectral sequence based on mod 2 motivic cohomology and the computation by Voevodsky of the whole corresponding Steenrod algebra. In the approach taken here we don't use these anymore. We will explain the relationship between these two approaches in [19].

Let us denote by $W(F)$ the Witt ring of anisotropic quadratic forms over F [14, 25]. The kernel of the mod 2 rank homomorphism $W(F) \to \mathbb{Z}/2$ is denoted by $I(F)$ and called *the fundamental ideal of* $W(F)$. For each integer

n, the n-th power of $I(F)$ is denoted by $I^n(F)$.

For any unit $u \in F^\times$, the symbol $< u >$ will denote the class in $W(F)$ of the quadratic form of rank one $u.X^2$ and the symbol $<< u >>$ will denote the class of the *Pfister form* $1+ < -u >= 1- < u > \in W(F)$, indeed an element in $I(F)$. The following *Steinberg relation*:

$$<< u >> . << 1-u >>= 0 \tag{1}$$

which holds in $I^2(F)$ for $u \in F^\times - \{1\}$, is a reformulation of the well-known relation $< u > + < 1-u >= 1+ < u.(1-u) >$ (see [25] for instance). One also has the following equality for any pair $(u,v) \in (F^\times)^2$

$$<< u >> + << v >> - << u.v >>=<< u >><< v >> \in I^2(F)$$

which shows that $u \mapsto << u >>$ induces a group homomorphism

$$F^\times/(F^\times)^2 \to I(F)/I^2(F) \tag{2}$$

The Milnor K-theory $K_*^M(F)$ of the field F introduced in [15] is the quotient of the tensor algebra $Tens_{\mathbb{Z}}F^\times$ on the abelian group of units $F^\times$ by the two sided ideal generated by tensors of the form $u \otimes (1-u)$, for $u \in F^\times - \{1\}$. The mod 2 Milnor K-theory of F is the quotient

$$k_*(F) := K_*^M(F)/2$$

As observed in [15] the above computations give a canonical graded ring homomorphism extending (2)

$$s_F : k_*(F) \to \oplus_n I^n(F)/I^{n+1}(F)$$

which we call the *Milnor homomorphism* and which was shown in *loc. cit.*to be surjective in any degree and an isomorphism in degrees ≤ 2. The Milnor conjecture on the Witt ring stated as question 4.3 on page 332 in *loc. cit.*, is the content of the following statement:

Theorem 1.1 *For any field F of characteristic not 2 and any integer n the Milnor homomorphism*

$$s_n(F) : k_n(F) \to I^n(F)/I^{n+1}(F)$$

is an isomorphism.

Remark 1.2 The powers of the fundamental ideal of F form altogether a commutative graded ring denoted by $I^*(F)$, in fact a graded $W(F)$-algebra because $I^0(F) = W(F)$. We have a canonical morphism of graded $W(F)$-algebras $Tens_{W(F)}(I(F)) \to I^*(F)$ where the left hand side denotes the tensor algebra over $W(F)$ of the $W(F)$-module $I(F)$. The relation (1) above, which holds in $I^2(F)$, shows that this morphism factors trough the quotient algebra $K_*^W(F) := Tens_{W(F)}(I(F))/ << u >> \otimes << 1-u >>$ by the two-sided ideal generated by the tensors $<< u >> \otimes << 1-u >> \in I(F) \otimes_{W(F)} I(F)$. It is quite natural to call $K_*^W(F)$ the *Witt K-theory* of F. In [17] we have proven that the induced morphism of $W(F)$-algebras

$$K_*^W(F) \to I^*(F)$$

is an isomorphism. This is in fact a reformulation of the main result of [2], which relies on Voevodsky's results and on Theorem 1.1. Observe conversely that Theorem 1.1 can be recovered from the isomorphism $K_*^W(F) \cong I^*(F)$ by tensorization by $\mathbb{Z}/2$ over $W(F)$.

We already mentioned that the surjectivity of the Milnor morphism holds, so that the main point of Theorem 1.1 is the injectivity. Our proof will consist in constructing inductively a left inverse to $s_n(F)$, the so-called *n-th invariant of quadratic forms*

$$e_n(F) : I^n(F)/I^{n+1}(F) \to k_n(F)$$

The statement in the Theorem corresponding to a fixed integer n will be called in the sequel *the Milnor conjecture on the Witt ring of F in weight n.*

Denote by $H^*(F;\mathbb{Z}/2(n))$ the mod 2 motivic cohomology groups of F in weight n as defined by Suslin-Voevodsky in [27]. These groups $H^*(F;\mathbb{Z}/2(*))$ altogether form a bigraded commutative ring. We have the particular element $\tau := -1 \in \mu_2(F) = H^0(F;\mathbb{Z}/2(1))$.

Our main result is:

Theorem 1.3 *Let k be a perfect field a characteristic $\neq 2$ and let $N > 0$ be an integer. Assume that the following assumptions hold:*

$\mathbf{H}_1$(N) *: Milnor's conjecture on the Witt ring of k holds in weights $\leq N$.*

$\mathbf{H}_2$(N-1) *: For any finite type field extension $F|k$ and for any integer $1 \leq n \leq N-1$, the cup product by τ*

$$H^{n-1}(F;\mathbb{Z}/2(n-1)) \stackrel{\tau\cup}{\to} H^{n-1}(F;\mathbb{Z}/2(n))$$

is onto.

Then Milnor's conjecture on the Witt ring holds for any field extension $F|k$ in weights $\leq N$.

Proof that Theorem 1.3 $\Rightarrow$ Theorem 1.1. The group $H^i(F;\mathbb{Z}/2(n))$ is the group of sections on F of the i-th cohomology sheaf $\mathcal{H}^i(\mathbb{Z}/2(n))$ of an explicit chain complex[1] $\mathbb{Z}/2(n)$ in $\mathcal{A}b_k$, the mod 2 motivic complex in weight n defined in [27]. The ring structure is induced by explicit morphisms of complexes $\mathbb{Z}/2(n) \otimes \mathbb{Z}/2(m) \to \mathbb{Z}/2(n+m)$ by *loc. cit.*. The cup product by τ is thus induced by a morphism of the form

$$\tau\cup : \mathbb{Z}/2(n-1) \to \mathbb{Z}/2(n) \tag{3}$$

Voevodsky's main theorem [32, 33] implies the Beilinson-Lichtenbaum conjecture on mod 2 motivic cohomology, which identifies, for $i \in \{0,\dots,n\}$, the sheaf $\mathcal{H}^i(\mathbb{Z}/2(n))$ to the sheaf associated in the Zariski topology to the presheaf $X \mapsto H^i_{et}(X;\mu_2^{\otimes n})$, see [27, 32]. This identification being compatible to the cup-product and because the cup-product by $\tau \in \mu_2(F) = H^0_{et}(Spec(k);\mu_2)$ in étale cohomology induces isomorphisms $H^i_{et}(X;\mu_2^{\otimes n}) \cong H^i_{et}(X;\mu_2^{\otimes(n+1)})$, this implies that the morphism (3) induces isomorphisms on cohomology sheaves of degrees $\leq n-1$, for any n. This establishes *a fortiori* $\mathbf{H}_2$*(N-1)* for any N.

Now choose for the field k a prime field of characteristic $\neq 2$. The Milnor conjecture on the Witt ring holds for k by [15], which proves $\mathbf{H}_1$*(N)* for any N. Theorem 1.3 now implies Milnor's conjecture on the Witt ring of field extensions $F|k$ in any weight, establishing Theorem 1.1.□

Remark 1.4 It is possible to prove Theorem 1.3 without $\mathbf{H}_1$*(N)* but this would make the exposition more complicated.

[1] for us "chain complex" means that the differential is of degree -1

Remark 1.5 Our method clearly emphasizes that Voevodsky's proof of Milnor's conjecture on mod 2 Galois cohomology in weights $\leq N-1$ implies the Milnor conjecture on the Witt ring in weights $\leq N$.

In the rest of the paper, we will concentrate on the proof of Theorem 1.3. We now fix once for all a perfect field k of characteristic $\neq 2$. Let us describe our strategy. Recall that we denote by $\mathcal{A}b_k$ the abelian category of sheaves of abelian groups in the Zariski topology on the category Sm_k of smooth k-schemes. We denote by

$$\underline{\mathbf{K}}_n^M \in \mathcal{A}b_k$$

the sheaf of *unramified Milnor K-theory* in weight n constructed in [13, 24]. Its fiber on a field $F|k$ is $K_n^M(F)$; see section 2.2 for a recollection. We will denote by

$$\underline{\mathbf{k}}_n := \underline{\mathbf{K}}_n^M/2$$

the cokernel in $\mathcal{A}b_k$ of the multiplication by 2 on $\underline{\mathbf{K}}_n^M$. We will denote by

$$\underline{\mathbf{W}} \in \mathcal{A}b_k$$

the associated sheaf to the presheaf of Witt groups on Sm_k: $X \mapsto W(X)$ constructed in [14] for instance.

We will then show how the filtration of the Witt ring $W(F) = \underline{\mathbf{W}}(F)$ of each field extension $F|k$ by the powers of their fundamental ideal naturally arises from a decreasing filtration by sub-sheaves

$$\cdots \subset \underline{\mathbf{I}}^n \subset \cdots \subset \underline{\mathbf{W}}$$

For each n, we set $\underline{\mathbf{i}}_n := \underline{\mathbf{I}}^n/\underline{\mathbf{I}}^{n+1} \in \mathcal{A}b_k$. The Milnor homomorphism for fields then arises for each n from an epimorphism of sheaves:

$$\underline{\mathbf{s}}_n : \underline{\mathbf{k}}_n \to \underline{\mathbf{i}}_n$$

called the *Milnor morphism* in weight n whose kernel is denoted by $\underline{\mathbf{j}}_n$.

Our strategy to prove Theorem 1.3 can now be decomposed as follows.

First by induction we may assume the Milnor conjecture on the Witt ring is proven for all fields $F|k$ in weights $\leq N-1$, so that $\underline{\mathbf{j}}_n = 0$ for $0 \leq n \leq N-1$. Then:

(1) Using $\mathbf{H}_1$*(N)* and $\mathbf{H}_2$*(N-1)* prove the vanishing of the groups $Ext^i_{\mathcal{A}b_k}(\underline{\mathbf{k}}_n, \underline{\mathbf{j}}_N)$ for $0 \leq n \leq N-1$ and $i \in \{0,1,2\}$.

(2) Using elementary homological algebra in $\mathcal{A}b_k$, deduce the Milnor conjecture on the Witt ring in weight N for all field extensions $F|k$ from **(1)**.

Proof of step (1) will be given in section 3.3 below in a more general form; see Theorem 3.10. The idea is to use two types of information about the Suslin-Voevodsky motivic complexes $\mathbb{Z}/2(n)$. The first one is the vanishing for $n > 0$ of the groups of morphisms in the derived category of $\mathcal{A}b_k$ of the form

$$Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}/2(n), M[*])$$

for some type of sheaves M, like j_N, which are birational invariant. This is done by adapting a simple geometric argument due to Voevodsky [31]. The second type of information concerns the cohomology sheaves $\mathcal{H}^i(\mathbb{Z}/2(n))$. These vanish for $i > n$, for $i = n$ we have the Suslin-Voevodsky isomorphism[2]: $\underline{\mathbf{k}}_n \cong \mathcal{H}^n(\mathbb{Z}/2(n))$ and hypothesis $\mathbf{H}_2$*(N-1)* provides an epimorphism $\underline{\mathbf{k}}_{n-1} \to \mathcal{H}^{n-1}(\mathbb{Z}/2(n))$ for $n < N$. The vanishing in step **(1)** is then deduced from these facts combined to the universal coefficient spectral sequence (see Lemma 3.9):

$$E_2^{p,q} = Ext^p_{\mathcal{A}b_k}(\mathcal{H}^q(\mathbb{Z}/2(n)); M) \Rightarrow Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}/2(n), M[p-q]) \qquad \square$$

Proof of step (2) will be given in section 3.4. Here is a sketch. Assume the Milnor conjecture on the Witt ring in weights $\leq N-1$ for fields $F|k$ and the vanishing in **(1)** are both established.

For each integer n, set $\underline{\mathbf{W}}_n := \underline{\mathbf{W}}/\underline{\mathbf{I}}^{n+1}$. The sheaf $\underline{\mathbf{W}}_{N-1}$ thus admits a filtration with associated subquotients of the form $\underline{\mathbf{I}}^n/\underline{\mathbf{I}}^{n+1} = \underline{\mathbf{i}}_n$, for some $0 \leq n \leq N-1$. The Milnor conjecture on the Witt ring in weights $\leq N-1$ gives isomorphisms $\underline{\mathbf{k}}_n \cong \underline{\mathbf{i}}_n$, and one deduces from the long exact sequence of Ext groups and **(1)** the vanishing

$$Ext^i_{\mathcal{A}b_k}(\underline{\mathbf{W}}_{N-1}, \underline{\mathbf{j}}_N) = 0$$

for $i \in \{0,1,2\}$. Using the short exact sequence

$$0 \to \underline{\mathbf{j}}_N \to \underline{\mathbf{k}}_N \to \underline{\mathbf{i}}_N \to 0 \tag{4}$$

[2] see [33, 7] and section A.3

one deduces that the morphism $Ext^1_{\mathcal{A}b_k}(\underline{\mathbf{W}}_{N-1}, \underline{\mathbf{k}}_N) \to Ext^1_{\mathcal{A}b_k}(\underline{\mathbf{W}}_{N-1}, \underline{\mathbf{i}}_N)$ is an isomorphism. There exists thus a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \underline{\mathbf{k}}_N & \to & \underline{\mathbf{\Gamma}}_N & \to & \underline{\mathbf{W}}_{N-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \underline{\mathbf{i}}_N & \to & \underline{\mathbf{W}}_N & \to & \underline{\mathbf{W}}_{N-1} & \to & 0 \end{array} \tag{5}$$

with exact horizontal rows where the bottom horizontal row is the obvious one and where the left vertical map is the Milnor epimorphism. Next we show that the canonical epimorphism

$$\underline{\mathbf{W}} \to \underline{\mathbf{W}}_N$$

canonically lifts trough the epimorphism $\underline{\mathbf{\Gamma}}_N \to \underline{\mathbf{W}}_N$ to a morphism

$$\underline{\mathbf{W}} \to \underline{\mathbf{\Gamma}}_N$$

This is one of the main argument: it uses the fact that the presentation of the Witt ring (or rather sheaf) uses units as generators and “open subschemes of product of $\mathbb{G}_m$ as relations”; taking into account the inductive assumption that $\underline{\mathbf{j}}_{N-1} = 0$ which implies $\underline{\mathbf{j}}_N$ is a birational invariant sheaf, the existence of the lifting follows easily (see section 3.4).

To finish the proof one observes that the composition $\underline{\mathbf{I}}^N \subset \underline{\mathbf{W}} \to \underline{\mathbf{\Gamma}}_N \to \underline{\mathbf{W}}_{N-1}$ is trivial by construction, and that the induced morphism

$$\underline{\mathbf{I}}^N \to \underline{\mathbf{k}}_N = (Ker : \underline{\mathbf{\Gamma}}_N \to \underline{\mathbf{W}}_{N-1})$$

induces a left inverse to the Milnor morphism in weight N

$$\underline{\mathbf{e}}_N : \underline{\mathbf{I}}^N / \underline{\mathbf{I}}^{N+1} \to \underline{\mathbf{k}}_N \qquad \square$$

Remark 1.6 Let $\mathcal{F}_k$ be the category of finite type field extensions $F|k$. It is tempting to work in the abelian category of functors from $\mathcal{F}_k$ to the category of abelian groups implicitly considered by Serre in [9], instead of the more elaborated category $\mathcal{A}b_k$. If one could prove an analogue of the step **(1)** (or Theorem 3.10) there, then our strategy could be simplified further.

Notations. For a scheme X and an integer $i \in \mathbb{N}$, we will denote by $X^{(i)}$ the set of points of codimension i of the scheme X.

Given $x \in X \in Sm_k$, we will denote by $\mathcal{O}_{X,x}$ the local ring of X at x. For a given presheaf of sets M on Sm_k we will denote by $M(\mathcal{O}_{X,x})$, or simply by M_x, the fiber of M at $x \in X$, in the Zariski topology. Important examples for us will be that of a finite type field extension $F|k \in \mathcal{F}_k$, considered as the local ring of the generic points of its models in Sm_k or that of a *geometric discrete valuation* ring. Such a discrete valuation ring $\mathcal{O}_v$ is one with field of fractions F of finite type over k and which is isomorphic to the local ring of some $X \in Sm_k$ at some point x of codimension 1. The associated valuation v on F will be called a *geometric discrete valuation* on F; its residue field will be denoted by $\kappa(v)$.

2 The Milnor morphism as a morphism of sheaves

2.1 Unramified Witt groups and related sub-sheaves

Definition 2.1 *A sheaf of sets M on Sm_k in the Zariski topology is said to be* 0-pure *if for any irreducible $X \in Sm_k$ with field of functions F the canonical map*

$$M(X) \to M(F)$$

is injective and induces a bijection

$$M(X) = \cap_{y \in X^{(1)}} M(\mathcal{O}_{X,y}) \subset M(F)$$

Let us denote by $W : Sm_k \to \mathcal{A}b, X \mapsto W(X)$ the presheaf of Witt groups on Sm_k; see [14] for instance, or [22] for a quick recollection. We will denote by $\underline{\mathbf{W}}$ the associated sheaf in the Zariski topology. The following result is a reformulation of some results of [22]:

Theorem 2.2 *(Ojanguren-Panin) The sheaf $\underline{\mathbf{W}}$ is 0-pure.*

Proof. Fix an irreducible $X \in Sm_k$ with field of functions F. Clearly, because $\underline{\mathbf{W}}$ is a sheaf in the Zariski topology, the morphism

$$\underline{\mathbf{W}}(X) \to \Pi_{x \in X} \underline{\mathbf{W}}(\mathcal{O}_{X,x})$$

is injective. Now by *loc. cit.* for each $x \in X$ the morphism

$$\underline{\mathbf{W}}(\mathcal{O}_{X,x}) = W(\mathcal{O}_{X,x}) \to W(F) = \underline{\mathbf{W}}(F)$$

is injective. This implies that the canonical morphism

$$\underline{\mathbf{W}}(X) \to \underline{\mathbf{W}}(F)$$

is injective. Now again because $\underline{\mathbf{W}}$ is a sheaf in the Zariski topology, the previous injection identifies $\underline{\mathbf{W}}(X)$ with $\cap_{x \in X} \underline{\mathbf{W}}(\mathcal{O}_{X,x}) = \cap_{x \in X} W(\mathcal{O}_{X,x}) \subset W(F)$. By Theorem A of *loc. cit.*, for each point $x \in X$ one has $W(\mathcal{O}_{X,x}) = \cap_y W(\mathcal{O}_{X,y})$ where y runs over the set of all prime ideal in $\mathcal{O}_{X,x}$ of height one, that is to say points $y \in X^{(1)}$ whose closure contains x. It follows then that

$$\underline{\mathbf{W}}(X) = \cap_{x \in X} W(\mathcal{O}_{X,x}) = \cap_{y \in X^{(1)}} W(\mathcal{O}_{X,y}) \subset W(F)$$

which establishes the Theorem.$\square$

Let $n \in \mathbb{N}$ be an integer. For any irreducible $X \in Sm_k$ with function field F we set

$$\underline{I}^n(X) = I^n(F) \cap \underline{\mathbf{W}}(X) \subset W(F)$$

We extend the definition to any $X \in Sm_k$ by setting

$$\underline{I}^n(X) := \oplus_{x \in X^{(0)}} \underline{\mathbf{I}}^n(X_x) \subset \oplus_{x \in X^{(0)}} \underline{\mathbf{W}}(X_x) = \underline{\mathbf{W}}(X)$$

where $X_x \subset X$ denotes the irreducible component containing $x \in X^{(0)}$.

Theorem 2.3 *Given any morphism $f : X \to Y$ in Sm_k and any $n \in \mathbb{N}$ the morphism*

$$\underline{\mathbf{W}}(f) : \underline{\mathbf{W}}(Y) \to \underline{\mathbf{W}}(X)$$

maps the subgroup $\underline{I}^n(Y) \subset \underline{\mathbf{W}}(Y)$ into $\underline{I}^n(X) \subset \underline{\mathbf{W}}(X)$. Thus the correspondence $X \mapsto \underline{I}^n(X)$ admits a unique structure of presheaf of abelian groups, denoted by $\underline{\mathbf{I}}^n$, such that the inclusions $\underline{I}^n(X) \subset \underline{\mathbf{W}}(X)$ define a monomorphism of presheaves $\underline{\mathbf{I}}^n \subset \underline{\mathbf{W}}$. Moreover, $\underline{\mathbf{I}}^n$ is a 0-pure sheaf.

Proof. By Lemma A.1 of the Appendix below, it is sufficient to prove that for any geometric discrete valuation v on a finite type field extension $F|k$, the morphism

$$\rho_v : W(\mathcal{O}_v) = \underline{\mathbf{W}}(\mathcal{O}_v) \to \underline{\mathbf{W}}(\kappa(v)) = W(\kappa(v)$$

maps $\underline{I}^n(\mathcal{O}_v) = I^n(F) \cap W(\mathcal{O}_v)$ into $I^n(\kappa(v))$. Let $\mathcal{O}_v^h$ denote the henselization of $\mathcal{O}_v$, F_v^h its fraction field. By naturality we have the following commutative diagram of Witt rings

$$\begin{array}{ccc} W(\mathcal{O}_v) & \to & W(\mathcal{O}_v^h) \\ \downarrow & & \downarrow \\ W(\kappa(v) & = & W(\kappa(v)) \end{array}$$

and $\underline{I}^n(\mathcal{O}_v)$ clearly maps to $\underline{I}^n(\mathcal{O}_v^h) := I^n(F_v^h) \cap W(\mathcal{O}_v^h) \subset W(F_v^h)$. Lemma 2.4 (4) below implies the result.□

Lemma 2.4 *Let v be a discrete valuation on a field F, with residue field $\kappa(v)$ of characteristic not 2, and let π be a uniformizing element for v. Then*

1) The ring homomorphism $W(\mathcal{O}_v) \to W(F)$ is injective and the diagram $0 \to W(\mathcal{O}_v) \to W(F) \stackrel{\partial_v^\pi}{\to} W(\kappa(v)) \to 0$ is a short exact sequence, where ∂_v^π is the residue morphism of [15, 25].

2) For each $n > 0$ the residue morphism $\partial_v^\pi : W(F) \to W(\kappa(v))$ maps $I^n(F)$ onto $I^{n-1}(\kappa(v))$.

3) Let $\mathcal{O}_v^h$ denote the henselization of $\mathcal{O}_v$, F_v^h its fraction field. Then the ring homomorphism $\rho : W(\mathcal{O}_v^h) \to W(\kappa(v))$ is an isomorphism and the canonical $W(\mathcal{O}_v^h)$-algebra homomorphism

$$W(\mathcal{O}_v^h)[T]/(T^2 - 1) \to W(F_v^h) \ , \ T \mapsto < \pi >$$

an isomorphism (Springer).

4) For each $n > 0$ the intersection $I^n(F_v^h) \cap W(\mathcal{O}_v^h)$ is equal to the n-th power $I^n(\mathcal{O}_v^h)$ of $I(\mathcal{O}_v^h)$ and the exact sequence of 1) induces an exact sequence

$$0 \to I^n(\mathcal{O}_v^h) \to I^n(F_v^h) \stackrel{\partial_v^\pi}{\to} I^{n-1}(\kappa(v)) \to 0$$

Proof. Statement 1) is proven in [25, Theorems 2.1, 2.2]. It follows from the proof of [15] Corollary 5.2 that the residue morphism maps $I^n(F)$ into $I^{n-1}(\kappa(v))$. The surjectivity is easy.

The first isomorphism in statement 3) is *loc. cit.* Theorem 2.4 and the second one is Corollary 2.6, due to Springer.

The last statement is proven in *loc. cit.* §5 in the case of complete discrete valuation rings but using 1), 2) and 3) the proof carries over to our case: one proceeds as in *loc. cit.*, proof of Corollary 5.2, establishing inductively that $I^n(F_v^h) = \rho^{-1}(I^n(\kappa(v))) \oplus << \pi >> .\rho^{-1}(I^{n-1}(\kappa(v)))$.$\square$

Remark 2.5 Let A be a regular local ring A in which 2 is invertible with fraction field F. Assume that $W(A) \to W(F)$ is injective (for instance if it contains a field of characteristic $\neq 2$ by [22]). We don't know in general whether or not for any $n \in \mathbb{N}$ the group

$$\underline{\mathbf{I}}^n(A) := I^n(F) \cap W(A) \subset W(A)$$

is always the n-th power of the ideal $I(A) := I(F) \cap W(A)$, though the previous Lemma establishes it for an henselian discrete valuation ring.

2.2 Unramified Milnor K-theory and Rost's cycle modules

Let n be an integer. For any field F and any n-tuple $(u_1, \dots, u_n) \in (F^\times)^n$ of units we will denote by $\{u_1\} \dots \{u_n\} \in K_n^M(F)$ its image through the obvious map $(F^\times)^n \to K_n^M(F)$. Recall from [15] that for any discrete valuation v on the field F, with residue field $\kappa(v)$ there exists one and only one homomorphism:

$$\partial_v : K_n^M(F) \to K_{n-1}^M(\kappa(v)) \tag{6}$$

satisfying

$$\partial_v(\{\pi\}.\{u_2\} \dots \{u_n\}) = \{\overline{u_2}\} \dots \{\overline{u_n}\}$$

if $v(\pi) = 1$ and $v(u_i) = 0$ for each $i \geq 2$. For $u \in \mathcal{O}_v^\times$ the notation $\overline{u} \in \kappa(v)$ means the image of u in $\kappa(v)^\times$. This homomorphism is called the *residue homomorphism* associated to v.

For any smooth k-variety X, we let $\underline{\mathbf{K}}_n^M(X)$ denote the kernel, introduced in [13], of all the residue homomorphisms associated to points in X of codimension 1:

$$\underline{\mathbf{K}}_n^M(X) := Ker\big(\oplus_{x \in X^{(0)}} K_n^M(\kappa(x)) \stackrel{\sum \partial_x}{\to} \oplus_{y \in X^{(1)}} K_{n-1}^M(\kappa(y)) \big)$$

The correspondence $X \mapsto \underline{\mathbf{K}}_n^M(X)$ is turned into a Zariski sheaf on Sm_k by Rost in [24]. This sheaf will be denoted by $\underline{\mathbf{K}}_n^M$ and called the sheaf of *unramified Milnor K-theory in weight n*.

More generally, let us recall briefly the notion of *cycle module* over k from [24]. We just remind that a cycle module M_* is a triple $(M_*, \phi^*, \partial_v)$ consisting of a functor

$$M_* : \mathcal{F}_k \to \mathcal{A}b_*$$

with $\mathcal{A}b_*$ being the category of graded abelian groups, a transfer morphism $\varphi^* : M_*(F) \to M_*(E)$ of degree 0 for each *finite* extension $E \subset F$ in $\mathcal{F}_k$ and a residue morphism $\partial_v : M_*(F) \to M_*(\kappa(v))$ of degree -1 for any *geometric discrete valuation* v on $F|k \in \mathcal{F}_k$. These data satisfy some axioms which we will not recall here; see *loc. cit.* p. 329 and p. 337.

The Milnor K-theory groups K_*^M, endowed with the transfers morphisms of Kato [12], and the above residue morphisms, form the fundamental example of cycle module, as it follows from [15, 5, 12, 24]. The mod m Milnor K-theory groups K_*^M/m also form a cycle module for any integer m. In fact, the category of cycle modules, with the obvious notion of morphisms, is abelian: the kernel and cokernel of a morphism are performed termwise on each $F|k \in \mathcal{F}_k$, and the axioms of [24, p. 329 and p. 337] follow formally. In the sequel, we will simply denote by k_* the cycle module of mod 2 Milnor K-theory.

Given a cycle module M_* and $X \in Sm_k$ we will denote by $A^0(X; M_*)$ the group of unramified sections of M_0 on X, by which we mean[3] the kernel of the sum of residue morphism at point of codimension 1:

$$\oplus_{x \in X^{(0)}} M_0(\kappa(x)) \stackrel{\sum \partial_x}{\to} \oplus_{y \in X^{(1)}} M_{-1}(\kappa(y))$$

In [24, §12] these groups are turned canonically into a Zariski sheaf on Sm_k which we denote by $\underline{\mathbf{M}}_0$. A sheaf of abelian groups in the Zariski topology on Sm_k will be said to come from a cycle module, or to have a cycle module structure, if it is isomorphic to a sheaf of the form $\underline{\mathbf{M}}_0$. It is easy to check that such sheaves are 0-pure in the sense of Definition 2.1.

[3]Here we slightly differ from [24] where Rost considers rather the direct sum $\oplus_{n \in \mathbb{Z}} A^0(X; M_*(n))$, see below.

Remark 2.6 By Déglise [7] the sheaf $\underline{\mathbf{M}}_0$ admits a canonical structure of homotopy invariant sheaf with transfers in the sense of Voevodsky [28], and these two notions of homotopy invariant sheaves with transfers and of cycle modules are essentially the same.

Lemma 2.7 *For any (termwise) short exact sequence* $0 \to M'_* \to M_* \to M"_* \to 0$ *of cycle modules, the diagram*

$$0 \to \underline{\mathbf{M}'}_0 \to \underline{\mathbf{M}}_0 \to \underline{\mathbf{M}"}_0 \to 0$$

is a short exact sequence of sheaves in the Zariski topology.

Proof. We will freely use the notations from [24]. For any $Y \in Sm_k$, or any localization Y of a smooth k-scheme, and any cycle module N_* is defined a cochain complex $C^*(Y; N_*)$, see *loc. cit.*p. 355 and p. 359. A short exact sequence of cycle modules $0 \to M'_* \to M_* \to M''_* \to 0$ then induces a short exact sequence of cochain complexes

$$0 \to C^*(Y; M'_*) \to C^*(Y; M_*) \to C^*(Y; M''_*) \to 0$$

and a corresponding long exact sequence of associated cohomology groups. But Theorem (6.1) of *loc. cit.*establishes that for any $x \in X \in Sm_k$ and any cycle module N_*, the cochain complex $C^*(Spec(\mathcal{O}_{X,x}); N_*)$ has trivial cohomology in > 0 degrees. The short exact sequence

$$0 \to C^*(Spec(\mathcal{O}_{X,x}); M'_*) \to C^*(Spec(\mathcal{O}_{X,x}); M_*) \to C^*(Spec(\mathcal{O}_{X,x}); M''_*) \to 0$$

thus produces a short exact sequence

$$0 \to A^0(Spec(\mathcal{O}_{X,x}); M'_*) \to A^0(Spec(\mathcal{O}_{X,x}); M_*) \to A^0(Spec(\mathcal{O}_{X,x}); M''_*) \to 0$$

in other words, of the form $0 \to \underline{\mathbf{M}'}_0(\mathcal{O}_{X,x}) \to \underline{\mathbf{M}}_0(\mathcal{O}_{X,x}) \to \underline{\mathbf{M}''}_0(\mathcal{O}_{X,x}) \to 0$, which gives the result.$\square$

For any cycle module M_* and any integer $n \in \mathbb{Z}$, we denote by $M_*(n)$ the cycle module obtained in the obvious way by setting $(M_*(n))_m = M_{m+n}$ and endowed with the corresponding shifted data. For instance, for each $n \in \mathbb{N}$ we have $\underline{\mathbf{K}}_n^M = \underline{\mathbf{K}_*^{\mathbf{M}}(\mathbf{n})}_0$ and we define the sheaf of *unramified mod* 2 *Milnor* K*-theory in weight* n as

$$\underline{\mathbf{k}}_n := \underline{\mathbf{k}_*(\mathbf{n})}_0 = \underline{\mathbf{K}}_n^M/2$$

More generally for a cycle module M_* we will simply set $\underline{\mathbf{M}}_n := \underline{\mathbf{M}_*(\mathbf{n})}_0$.

Definition 2.8 *Let $n \in \mathbb{Z}$ be an integer. We will say that a cycle module M_* is of weights $\leq n$ if and only if for any $F|k \in \mathcal{F}_k$ the group $M_i(F)$ vanishes for $i < -n$.*

Observe that if M_* is in weights $\leq n$, $M_*(m)$ is in weights $\leq n+m$. Also M_* is in weights $\leq n$ if and only if for any $F|k \in \mathcal{F}_k$, $M_{-n-1}(F) = 0$. In that case the sheaves $\underline{\mathbf{M}}_{-m}$ all vanish for $m \geq n+1$. The cycle modules K_*^M and k_* are in weights ≤ 0.

Lemma 2.9 *Let $X \in Sm_k$ and let $Z \subset X$ be closed subscheme everywhere of codimension $\geq n$, then for any sheaf M coming from a cycle module of weights $\leq n-1$ the groups*

$$H^*(X, (X-Z); M)$$

vanish for any $*$.

Proof. This follows from the results of [24]: for any cycle module N_* one can compute $H^*(X, X-Z; \underline{N_0})$ as the cohomology of the complex $C^*(X, N_*)/C^*(X-Z; N_*)$ which is of the form

$$\oplus_{x\in X^{(0)}/\overline{x}\subset Z} N_0(\kappa(x)) \to \cdots \to \oplus_{x\in X^{(i)}/\overline{x}\subset Z} N_{-i}(\kappa(x)) \to \dots$$

Now because the codimension of Z is $\geq n$, that complex is trivial up to codimension n. But if N_* is of weights $\leq n-1$, it is trivial from there as well, so it vanishes.$\square$

We will also need the following lemma whose first part is proven in [24, 12] and whose second part is due to Bass and Tate [5, Prop. 4.5 b)].

Lemma 2.10 *Let v be a geometric discrete valuation on a finite type field $F|k$. Then:*

1) The diagram $0 \to \underline{\mathbf{K}}_n^M(\mathcal{O}_v) \to \underline{\mathbf{K}}_n^M(F) \xrightarrow{\partial_v} \underline{\mathbf{K}}_{n-1}^M(\kappa(v)) \to 0$ is a short exact sequence. Moreover, given a uniformizing element $\pi \in \mathcal{O}_v$ the following diagram is commutative

$$\begin{array}{ccc}
\underline{\mathbf{K}}_n^M(\mathcal{O}_v) & \subset & \underline{\mathbf{K}}_n^M(F) \\
\rho_v \downarrow & & \downarrow \partial_v(\{\pi\} \cup (-)) \\
\underline{\mathbf{K}}_n^M(\kappa(v)) & = & \underline{\mathbf{K}}_n^M(\kappa(v))
\end{array}$$

2) For any unit $u \in \mathcal{O}_v^\times$ the symbol $\{u\} \in \underline{\mathbf{K}}_1^M(F)$ lies in $\underline{\mathbf{K}}_1^M(\mathcal{O}_v)$ and moreover the group $\underline{\mathbf{K}}_n^M(\mathcal{O}_v)$ is generated by symbols of the form

$$\{u_1\}\dots\{u_n\}$$

with the u_i's in $\mathcal{O}_v^\times$.

2.3 Sheafifying the Milnor homomorphism

For each integer $n \in \mathbb{N}$ and any field $F|k \in \mathcal{F}_k$ we set

$$i_n(F) := I^n(F)/I^{n+1}(F)$$

We denote by $i_*(F)$ the corresponding graded abelian group.

Theorem 2.11 *There exists one and only one structure of cycle module on the correspondence*

$$\mathcal{F}_k \to \mathcal{A}b_*; F \mapsto i_*(F)$$

such that the Milnor homomorphisms

$$s_*(F) : k_*(F) \to i_*(F)$$

altogether define a morphism of cycle modules, an epimorphism indeed.

Proof. We will show below that the transfers morphisms

$$\varphi^* : k_*(F) \to k_*(E)$$

for finite extensions $E \subset F$ in $\mathcal{F}_k$ and the residue morphisms

$$\partial_v : k_*(F) \to k_{*-1}(\kappa(v))$$

for geometric discrete valuations v on $F|k \in \mathcal{F}_k$, descend to morphisms on i_*. Endowed with these induced morphism, i_* becomes a cycle module because the axioms are automatic consequences of the corresponding ones for mod 2-Milnor K-theory. Moreover, by construction, the Milnor homomorphism preserves the cycle module structures, defining an epimorphism of cycle modules of the form $k_* \to i_*$. The uniqueness of the structure is clear.

To prove our claim for the residue morphisms, let v be any geometric discrete valuation on $F|_k \in \mathcal{F}_k$, and π be a uniformizing element for v. Let us denote by

$$\partial_v^\pi : i_n(F) \to i_{n-1}(\kappa(v))$$

the homomorphism induced by the one on $I^n(F)$ given in Lemma 2.4 2) above. Clearly from the explicit description of residues in Milnor K-theory and in Witt theory [15, 25] the diagram

$$\begin{array}{ccc} k_n(F) & \stackrel{\partial_v}{\to} & k_{n-1}(\kappa) \\ \downarrow & & \downarrow \\ i_n(F) & \stackrel{\partial_v^\pi}{\to} & i_{n-1}(\kappa) \end{array}$$

is commutative. This also shows that ∂_v^π doesn't depend on π.

Let $E \subset F$ be any finite extension of fields and n be an integer. Assume first this extension is purely unseparable. Then in that case, because the characteristic p is odd, one has $< x^p > = < x >$ in $W(F)$ and $\{x^p\} = \{x\} \in k_1(F)$; for any $x \in F$, some power x^{p^n} is an E. Thus the extension of scalars $W(E) \to W(F)$ and $k_*(E) \to k_*(F)$ are both epimorphisms. But the same formula shows that the Frobenius induces the identity morphism $W(E) = W(E)$; as some iterated of the Frobenius on F maps to $E \subset F$ the factorization of the identity as $W(E) \to W(F) \to W(E)$ this shows the extension of scalars is in fact an isomorphism. The degree $[F : E]$, being a power of p, is odd so that the standard property of the transfer morphism implies it is a monomorphism on mod 2 Milnor K-theory, showing that $k_*(E) \to k_*(F)$ is an isomorphism as well. This clearly imply our claim on the transfers in that case.

Assume now the extension $E \subset F$ is separable. Let $t : F \to E$ denote the trace morphism and $t_* : W(F) \to W(E)$, $q \mapsto t \circ q$ the corresponding Scharlau transfer [25, §2.5]. By Arason [1, Satz 3.3] t_* maps $I^n(F)$ into $I^n(E)$ and thus induces a natural morphism $t_* : i_n(F) \to i_n(E)$.

To check that the diagram

$$\begin{array}{ccc} k_n(F) & \stackrel{\varphi^*}{\to} & k_n(E) \\ \downarrow & & \downarrow \\ i_n(F) & \stackrel{t_*}{\to} & i_n(E) \end{array} \tag{7}$$

is commutative one proceeds as follows. Let $E \subset E'$ be a separable algebraic extension of odd degree, and $F' := E' \otimes_E F$, a finite separable algebra over E'. Clearly the square above maps by extension of scalars to the corresponding one involving E' and F'. But $E \to E'$ and $F \to F'$ being of odd degree, the extension morphisms $i_n(E) \to i_n(E')$ is a monomorphism: this is proven after the proof of [1, Satz 3.3]. Thus to prove the commutativity of our square (7) it suffices to prove it for the one obtained by extending the scalars to E'. If we choose for $E \subset E'$ separable algebraic extension such that the absolute Galois group of E' is a 2-Sylow of that of E, we may assume further that $E' \to F'$ admits a finite increasing filtration $E' \subset E'_1 \subset \cdots \subset E'_r = F'$ by quadratic extensions. Thus we reduced our claim to the commutativity of (7) in case $E \subset F$ is a quadratic extension. By [5, Corollary 5.3 p. 29], $k_*(F)$ is generated as a module over $k_*(E)$ by $k_{\leq 1}(F)$. So it suffices (by the projection formula both for φ^* and t_*) to prove it for $n = 1$, which is an easy computation (use for instance [25, Lemma 5.8]).□

For each $n \in \mathbb{N}$, $\underline{\mathbf{i}}_n$ denotes the associated sheaf to the cycle module $i_*(n)$. The Milnor morphism of cycle modules

$$s_* : k_* \to i_*$$

being an epimorphism it defines for each $n \in \mathbb{N}$ an epimorphism of sheaves in the Zariski topology

$$\underline{\mathbf{s}}_n : \underline{\mathbf{k}}_n \to \underline{\mathbf{i}}_n$$

by Lemma 2.7. This morphism is called the *Milnor morphism* in weight n.

2.4 The isomorphism $I^n/I^{n+1} = i_n$

The following result justifies *a posteriori* our quick definition of $\underline{\mathbf{i}}_n$ in the introduction as $\underline{\mathbf{I}}^n/\underline{\mathbf{I}}^{n+1}$.

Theorem 2.12 *Let $n \geq 0$ be an integer.*

1) The canonical transformation between functors on $\mathcal{F}_k$: $I^n(F) \to i_n(F)$ arises from a unique morphism of sheaves $\underline{\mathbf{I}}^n \to \underline{\mathbf{i}}_n$.

2) The kernel of this morphism is the subsheaf $\underline{\mathbf{I}}^{n+1} \subset \underline{\mathbf{I}}^n$.

3) This morphism is an epimorphism (in the Zariski topology) and thus induces a canonical isomorphism $\underline{\mathbf{I}}^n/\underline{\mathbf{I}}^{n+1} \cong \underline{\mathbf{i}}_n$.

Proof. 1) We use Lemma A.2. The first property follows easily from the fact that the morphisms

$$I^n(F) \to i_n(F)$$

commute to residue morphisms (see the proof of Theorem 2.11). The second property means that for any geometric discrete valuation v on $F|k \in \mathcal{F}_k$ the following diagram commutes

$$\begin{array}{ccc} \underline{\mathbf{I}}^n(\mathcal{O}_v) & \to & \underline{\mathbf{i}}_n(\mathcal{O}_v) \\ \downarrow & & \downarrow \\ \underline{\mathbf{I}}^n(\kappa(v)) & \to & i_n(\kappa(v)) \end{array}$$

This follows again from Lemma 2.4 by mapping $\mathcal{O}_v$ to $\mathcal{O}_v^h$. This defines the morphism $\underline{\mathbf{I}}^n \to \underline{\mathbf{i}}_n$ for each n.

2) It is sufficient to prove that for each $x \in X \in Sm_k$ the diagram

$$0 \to \underline{\mathbf{I}}^{n+1}(\mathcal{O}_{X,x}) \to \underline{\mathbf{I}}^n(\mathcal{O}_{X,x}) \to \underline{\mathbf{i}}_n(\mathcal{O}_{X,x}) \tag{8}$$

is an exact sequence.

Choose for each point y of codimension 1 in $Spec(\mathcal{O}_{X,x})$ a uniformizing element π_y of the associated discrete valuation. From the fact that $\underline{\mathbf{W}}(\mathcal{O}_{X,x}) = W(\mathcal{O}_{X,x})$ is the kernel of all the residue morphisms $\partial_y^{\pi_y}$, from Theorem 2.2 and from Lemma 2.4 one constructs the commutative diagram in which F is the fraction field of $\mathcal{O}_{X,x}$ and the right horizontal maps are residues:

$$\begin{array}{ccccccc} 0 & \to & \underline{\mathbf{I}}^{n+1}(\mathcal{O}_{X,x}) & \to & I^{n+1}(F) & \to & \oplus_{y \in Spec(\mathcal{O}_{X,x})^{(1)}} I^n(\kappa(y)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \underline{\mathbf{I}}^n(\mathcal{O}_{X,x}) & \to & I^n(F) & \to & \oplus_{y \in Spec(\mathcal{O}_{X,x})^{(1)}} I^{n-1}(\kappa(y)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \underline{\mathbf{i}}_n(\mathcal{O}_{X,x}) & \to & i_n(F) & \to & \oplus_{y \in Spec(\mathcal{O}_{X,x})^{(1)}} i_{n-1}(\kappa(y)) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The horizontal rows and the last two vertical rows being each exact sequences, the claim follows.

3) We want now to establish the surjectivity of the right homomorphism in (8). Assume first that k is infinite. By [8, Proposition 4.3], the group $\underline{\mathbf{K}}_n^M(\mathcal{O}_{X,x})$ is generated by symbols $\{u_1\}\dots\{u_n\}$ with the u_i's units in $\mathcal{O}_{X,x}$. Thus so is the (quotient) group $\underline{\mathbf{i}}_n(\mathcal{O}_{X,x})$. Now we conclude because these symbols $\{u_1\}\dots\{u_n\}$ in $\underline{\mathbf{i}}_n(\mathcal{O}_{X,x})$ are clearly in the image of $\underline{\mathbf{I}}^n(\mathcal{O}_{X,x}) \to \underline{\mathbf{i}}_n(\mathcal{O}_{X,x})$ which is thus onto.

Assume now that k is finite. Let $k \subset K$ be an algebraic extension with K infinite, perfect and of odd degree. By Lemma 2.14 below we see that the cokernel of $\underline{\mathbf{I}}^n(\mathcal{O}_{X,x}) \to \underline{\mathbf{i}}_n(\mathcal{O}_{X,x})$ injects into the cokernel of $\underline{\mathbf{I}}^n(\mathcal{O}_{X,x}|K) \to \underline{\mathbf{i}}_n(\mathcal{O}_{X,x}|K)$, which is zero by what we have seen above. The theorem is now proven in any case.$\square$

Remark 2.13 It is possible to give a more elementary and more natural proof of the previous Theorem which doesn't use the result of [8] quoted above. It relies on the work [26] and on [24]: for any irreducible $X \in Sm_k$ with function field F and any $n \in \mathbb{N}$ one constructs as in [26] an explicit and canonical complex $C^*(X;\underline{\mathbf{I}}^n)$ of the form (see also [4]):

$$I^n(F) \to \oplus_{y\in Spec(\mathcal{O}_{X,x})^{(1)}} I^{n-1}(\kappa(y);\tau_y) \to \oplus_{z\in Spec(\mathcal{O}_{X,x})^{(2)}} I^{n-2}(\kappa(z);\Lambda^2(\tau_z)) \to \dots$$

in which τ_y means the tangent vector space a point $y \in X$, the dual of the $\kappa(y)$-vector space $m_y/(m_y)^2$. The technic from Rost [24] shows that this complex gives for a smooth local ring $\mathcal{O}_{X,x}$ a resolution of $\underline{\mathbf{I}}^n(\mathcal{O}_{X,x})$. This complex can be shown to extends on the right the horizontal lines of the diagram above used in the proof of part 2); from this it is quite easy to prove the surjectivity of $\underline{\mathbf{I}}^n(\mathcal{O}_{X,x}) \to \underline{\mathbf{i}}_n(\mathcal{O}_{X,x})$.

The following Lemma is directly inspired by [3, 22].

Lemma 2.14 *Let $k \subset L$ be a finite extension and $x \in X \in Sm_k$. Let F be the fraction field of $\mathcal{O}_{X,x}$.*

1) For each $n \in \mathbb{N}$ the Scharlau transfer $s_* : W(F \otimes_k L) \to W(F)$ with respect to a non-zero k-linear map $s : L \to k$ [25, §2.5] maps $W(\mathcal{O}_{X,x}\otimes_k L) \subset W(F \otimes_k L)$ to $W(\mathcal{O}_{X,x}) \subset W(F)$ and maps $\underline{\mathbf{I}}^n(F \otimes_k L)$ to $\underline{\mathbf{I}}^n(F)$.

Thus it induces a natural morphism

$$\begin{array}{ccc} \underline{\mathbf{I}}^n(\mathcal{O}_{X,x}\otimes_k L) & \to & \underline{\mathbf{I}}^n(\mathcal{O}_{X,x}) \\ || & & || \\ W(\mathcal{O}_{X,x}\otimes_k L)\cap \underline{\mathbf{I}}^n(F\otimes_k L) & \to & W(\mathcal{O}_{X,x})\cap \underline{\mathbf{I}}^n(F) \end{array}$$

2) For any non-trivial s the natural morphism of 1) is compatible to the morphism

$$\underline{\mathbf{i}}_n(\mathcal{O}_{X,x}\otimes_k L)\to \underline{\mathbf{i}}_n(\mathcal{O}_{X,x})$$

induced by the transfer $\underline{\mathbf{i}}_n(F\otimes_k L)\to \underline{\mathbf{i}}_n(F)$ of the cycle module structure on i_* (see 2.11).

3) If $[L:k]$ is odd, the morphism $\underline{\mathbf{I}}^n(\mathcal{O}_{X,x})\to \underline{\mathbf{i}}_n(\mathcal{O}_{X,x})$ is a direct summand of the morphism $\underline{\mathbf{I}}^n(\mathcal{O}_{X,x})\to \underline{\mathbf{i}}_n(\mathcal{O}_{X,x})$.

Proof. The fact that the Scharlau transfer maps $W(\mathcal{O}_{X,x}\otimes_k L)\subset W(F\otimes_k L)$ to $W(\mathcal{O}_{X,x})\subset W(F)$ follows from [22, §2 & §3]. The fact that the Scharlau transfer (for any choice of s) maps $I^n(F)$ into $I^n(E)$ is [1, Satz 3.3]. This proves 1).

By the previous result of Arason, the induced transfer $t_*: i_n(F)\to i_n(E)$ doesn't depend on the choice of s. Choosing for s the trace morphism we thus get the transfer for the cycle module structure, see the proof of Theorem 2.11 above. This proves 2).

If $[L:k]$ is odd, choose for s the morphism of [25, Lemma 5.8 p. 49]. By *loc. cit.* the composition $\underline{\mathbf{I}}^n(\mathcal{O}_{X,x})\to \underline{\mathbf{I}}^n(\mathcal{O}_{X,x}\otimes_k L)\to \underline{\mathbf{I}}^n(\mathcal{O}_{X,x})$ is multiplication by the class $s_*(1)=1$. The same holds for $\underline{\mathbf{i}}_n$ by [1]. The Lemma is proven.□

3 Proof of the main Theorem

3.1 Some $\mathbb{A}^1$-homological algebra

For any chain complex C_* in $\mathcal{A}b_k$, any sheaf $M\in \mathcal{A}b_k$, and any integer $n\in\mathbb{Z}$ we denote by

$$Hom_{D(\mathcal{A}b_k)}(C_*, M[n])$$

the group of morphisms in the derived category $D(\mathcal{A}b_k)$ of the abelian category $\mathcal{A}b_k$ from C_* to the n-th shift $M[n]$ of M. That group can be computed as follows: choose an injective resolution $M \to I_*$ of M in $\mathcal{A}b_k$ of the form $M \to I_0 \to I_{-1} \to I_{-2} \to I_{-3} \to \dots$. Then $Hom_{D(\mathcal{A}b_k)}(C_*, M[n])$ is the group of morphisms of chain complexes $C_* \to I_*[n]$ modulo the subgroup of those which are homotopic to zero.

For technical purpose, we slightly extend the notion of smooth k-scheme. Let us denote by Sm'_k the full subcategory of that of all k-schemes consisting of k-schemes which are possibly infinite disjoint union of smooth k-schemes. Any such $X \in Sm'_k$ can be written as $\amalg_\alpha X_\alpha$ with each X_α irreducible and in Sm_k; the X_α's are called the irreducible components of X. The associated free sheaf of abelian groups on X is the sheaf $\mathbb{Z}(X) = \oplus_\alpha \mathbb{Z}(X_\alpha) \in \mathcal{A}b_k$.

A morphism of sheaves of abelian groups of the form

$$f : \mathbb{Z}(X) \to \mathbb{Z}(Y)$$

with X and Y in Sm'_k, is said to be *elementary* if for any irreducible component X_α of X the restriction of that morphism to the summand $\mathbb{Z}(X_\alpha)$ can be written as a finite sum

$$\Sigma_i n_{\alpha,i} \cdot f_{\alpha,i}$$

with $n_{\alpha,i} \in \mathbb{Z}$ and with each $f_{\alpha,i}$ corresponding to a morphism of k-schemes $X_\alpha \to Y$. An elementary resolution of a sheaf M is a resolution $R_* \to M$ whose terms R_n, $n \in \mathbb{N}$ are free sheaves on smooth k-schemes (possibly in Sm'_k) and whose boundaries are elementary morphisms.

Lemma 3.1 *For any $M \in \mathcal{A}b_k$, there exists an elementary resolution of the form*

$$\mathbb{Z}(X_*) := \cdots \to \mathbb{Z}(X_n) \to \cdots \to \mathbb{Z}(X_0) \to M \to 0$$

Proof. Let $\mathcal{P}_k$ be the abelian category of presheaves of abelian groups on Sm_k. Then for any $X \in Sm_k$, $\mathbb{Z}(X)$ is just the associated sheaf to the presheaf $Z(X) \in \mathcal{P}_k$ which maps Y to the free abelian group on the set $Hom_k(Y, X)$. These $Z(X)$ are projective generators in $\mathcal{P}_k$. One can thus find a resolution $R'_* \to M$ in $\mathcal{P}_k$ with $R'_n = 0$ for $n < 0$, and such that R'_n a direct sum of sheaves of the form $Z(X)$ with $X \in Sm_k$. Then the sheafification of that resolution gives a resolution in $\mathcal{A}b_k$ with the required properties.$\square$

Definition 3.2 *A sheaf $M \in \mathcal{A}b_k$ is said to be* Zariski strictly $\mathbb{A}^1$-invariant *if for any $X \in Sm_k$ the natural homomorphism*

$$H^*(X; M) \to H^*(X \times \mathbb{A}^1; M)$$

is an isomorphism.

By [24, §9], any sheaf arising from a cycle module is Zariski strictly $\mathbb{A}^1$-invariant. Any homotopy invariant sheaf with transfers in the sense of [28] is Zariski strictly $\mathbb{A}^1$-invariant by [29].

Recall the construction $\underline{C}_*$ from [27, 28] (in [28] it is denoted by $C^\bullet$). It associates to a sheaf $N \in \mathcal{A}b_k$ the complex of sheaves $\underline{C}_*(N)$ with n-th term the sheaf $X \mapsto N(\Delta^n \times X)$, with Δ^n the n-th algebraic simplex[4], and with differential in degree n, $\Sigma_{i=0}^n (-1)^i \partial_i$, with ∂_i induced by the i-th coface $\Delta^{n-1} \to \Delta^n$.

Lemma 3.3 *Let M be a Zariski strictly $\mathbb{A}^1$-invariant sheaf.*

1) For any non-negatively graded chain complex C_* in $\mathcal{A}b_k$ the morphism

$$Hom_{D(\mathcal{A}b_k)}(C_*; M) \to Hom_{D(\mathcal{A}b_k)}(C_* \otimes \mathbb{Z}(\mathbb{A}^1); M)$$

induced by the projection $C_* \otimes \mathbb{Z}(\mathbb{A}^1) \to C_*$, is an isomorphism.

2) For any sheaf $N \in \mathcal{A}b_k$ the morphism $N \to \underline{C}_*(N)$ induces an isomorphism

$$Hom_{D(\mathcal{A}b_k)}(\underline{C}_*(N), M[*]) \cong Hom_{D(\mathcal{A}b_k)}(N, M[*]) = Ext^*_{\mathcal{A}b_k}(N, M)$$

Proof. 1) By definition, for M to be Zariski strictly $\mathbb{A}^1$-invariant exactly means that for any $X \in Sm_k$ the homomorphism

$$Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(X); M[*]) = H^*(X; M) \to$$

$$H^*(X \times \mathbb{A}^1; M) = Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(X) \otimes \mathbb{Z}(\mathbb{A}^1); M)$$

is an isomorphism. The part 1) of the Lemma then follows easily from standard homological and the fact that the sheaves $\mathbb{Z}(X)$ are generators of $\mathcal{A}b_k$.

[4]i.e. $\Delta^n = Spec(k[T_0, \ldots, T_n]/(\Sigma_i T_i - 1))$

The part 2) of the Lemma follows from 1) exactly in the same way as in [20, Corollary 3.8 p.89].□

An $\mathbb{A}^1$-*homotopy* between morphisms $f, g : C_* \to D_*$ of chain complexes in $\mathcal{A}b_k$ is a morphism $h : C_* \otimes \mathbb{Z}(\mathbb{A}^1) \to D_*$ which induces f (resp. g) through the 0 (resp. 1) section $Spec(k) \to \mathbb{A}^1$.

As a consequence of 1) of the Lemma, any two $\mathbb{A}^1$-homotopic morphisms $f, g : C_* \to D_*$, with C_* non-negatively graded induce the same morphism $Hom_{D(\mathcal{A}b_k)}(D_*; M) \to Hom_{D(\mathcal{A}b_k)}(C_*; M)$, for M Zariski strictly $\mathbb{A}^1$-invariant.

3.2 Vanishing of some groups $Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}/2(n), M[*])$

For $X \in Sm_k$, we let $\mathbb{Z}_{tr}(X)$ denote the sheaf which maps $Y \in Sm_k$ to the group of finite correspondences from Y to X, that is to say the free abelian group $c(Y, X)$ on the set of irreducible closed subschemes $Z \subset Y \times X$ which are finite on Y and which dominate an irreducible component of Y [28, 27].

For X_1 and X_2 pointed smooth k-schemes we let

$$\mathbb{Z}_{tr}(X_1 \wedge X_2) \in \mathcal{A}b_k$$

denote the cokernel of the obvious morphism given by the base points $\mathbb{Z}_{tr}(X_1) \oplus \mathbb{Z}_{tr}(X_2) \to \mathbb{Z}_{tr}(X_1 \times X_2)$. Iterating this construction we get for a family $(X_1, \dots, X_n)$ of pointed smooth k-schemes the sheaf [27]:

$$\mathbb{Z}_{tr}(X_1 \wedge \cdots \wedge X_n)$$

For any integer n, the *motivic chain complex* in weight n of Suslin-Voevodsky [27, 28] is the chain complex $\mathbb{Z}(n) := \underline{C}_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))[-n]$.

The following result is a variation on a idea from [31]:

Theorem 3.4 *Let $n \geq 1$ and let M be sheaf which comes from a cycle module of weights $\leq n-1$. Then one has*

$$Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(n), M[m]) = 0 \tag{9}$$

for any integer $m \in \mathbb{Z}$.

The proof will be given below, after a couple of preliminary Lemmas. Of course the case $m < 0$ is trivial.

For $X \in Sm_k$, we let $z_{eq}(X)$ denote the sheaf which maps $Y \in Sm_k$ to the free abelian group $z_{eq}(X)(Y)$ on the set of irreducible closed subschemes $Z \subset Y \times X$ which are quasi-finite on Y and which dominates an irreducible component of Y [28]. We have the following geometric lemma:

Lemma 3.5 *(Voevodsky [30]) There exist explicit quasi-isomorphisms of chain complexes in $\mathcal{A}b_k$ of the form*

$$\mathbb{Z}(n)[2n] = \underline{C}_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))[+n] \leftarrow \underline{C}_*(\mathbb{Z}_{tr}(\mathbb{P}^n)/\mathbb{Z}_{tr}(\mathbb{P}^{n-1})) \rightarrow \underline{C}_*(z_{eq}(\mathbb{A}^n))$$

The following two Lemmas and their Corollary below are inspired by the proof of [31, Proposition 3.3].

Lemma 3.6 *Let $X \in Sm_k$ and let $z \in z_{eq}(\mathbb{A}^n)(X)$. Let us denote by $\Omega(z) \subset X \times \mathbb{A}^n$ the open complement of the support $|z| \subset X \times \mathbb{A}^n$ of z. Let $z|_{\Omega(z)}$ be the pull back of z through the morphism $\Omega(z) \subset X \times \mathbb{A}^n \rightarrow X$. Then there exists a canonical cycle*

$$h(z) \in z_{eq}(\mathbb{A}^n)(\Omega(z) \times \mathbb{A}^1)$$

such that $\partial_1 h(z) = z|_{\Omega(z)}$ and $\partial_0 h(z) = 0$. This cycle is functorial in X.

Moreover, given any (finite) decomposition $z = \Sigma_j n_j z_j$, with $n_j \in \mathbb{Z}$ and $z_j \in z_{eq}(\mathbb{A}^n)(X)$, then first $|z| \subset \cup_j |z_j|$, so that $\cap_j \Omega(z_j) \subset \Omega(z)$ and one has the following equality in $z_{eq}(\mathbb{A}^n)(\cap_j \Omega(z_j) \times \mathbb{A}^1)$

$$h(z) = \Sigma_j n_j h(z_j)$$

The assertion involving $\mathbb{A}^1$-homotopies follows from the explicit construction given in *loc. cit.*. Beware that *a priori* it is not clear whether or not that given a morphism $\mathbb{Z}(X) \xrightarrow{f} \mathbb{Z}(Y)$ and an element $y \in z_{eq}(\mathbb{A}^n)(Y)$, one has $f(\mathbb{Z}(\Omega(f^*(y)))) \subset \mathbb{Z}(\Omega(y))$, although this is true for $f : X \rightarrow Y$ a morphism of schemes. For instance we will be in a situation where $f^*(y) = 0$. This is why will need the next Lemma.

Lemma 3.7 *Given an elementary morphism*

$$f : \mathbb{Z}(X) \rightarrow \mathbb{Z}(Y)$$

an element $y \in z_{eq}(\mathbb{A}^n)(Y)$, considered as a morphism $\mathbb{Z}(Y) \to z_{eq}(\mathbb{A}^n)$, and an open subscheme $\Omega_Y \subset Y \times \mathbb{A}^n$, whose complement Z_Y is quasi-finite equidimensional over Y and contains $|y|$ (in other words $\Omega_Y \subset \Omega(y)$), then there exits an open subscheme

$$\Omega_X \subset X \times \mathbb{A}^n$$

whose complement Z_X is quasi-finite equidimensional over X and contains $|x|$ (in other words $\Omega_X \subset \Omega(x)$), with $x = y \circ f = f^* y$, and such that f maps $\mathbb{Z}(\Omega_X)$ into $\mathbb{Z}(\Omega_Y)$. Moreover, the restriction to $\mathbb{Z}(\Omega_X)$ of the $\mathbb{A}^1$-homotopy $h(x)$ of Lemma 3.6 is compatible with the restriction to $\mathbb{Z}(\Omega_Y)$ of the $\mathbb{A}^1$-homotopy $h(y)$.

Proof. We may assume X is irreducible. Write $f = \Sigma_j n_j f_j$, a finite sum. Define $Z \subset X \times \mathbb{A}^n$ to be the union over the finite set of indices j of the support of the cycles $f_j^*(Z_Y) \in z_{eq}(X \times \mathbb{A}^n)$.

By construction one has

$$|x| = |\Sigma_j n_j f_j^*(y)| \subset \Sigma_j |f_j^*(y)| \subset \Sigma_j |f_j^*(Z_Y)| = Z$$

Clearly $\Omega_X = X \times \mathbb{A}^n - Z$ satisfies the conclusions.

To check that f maps $\mathbb{Z}(\Omega_X)$ into $\mathbb{Z}(\Omega_Y)$ it suffices to observe that $\mathbb{Z}(\Omega_X) = \cap_j \mathbb{Z}(\Omega(X \times \mathbb{A}^n \backslash f_j^*(Z_Y)))$ so that each morphism f_j separately maps $\mathbb{Z}(\Omega_X)$ into $\mathbb{Z}(\Omega_Y)$.

Following this, one proves easily the assertion on the $\mathbb{A}^1$-homotopies using Lemma 3.6 above.$\square$

Corollary 3.8 *Let $\rho_* : \mathbb{Z}(X_*) \stackrel{\simeq}{\to} z_{eq}(\mathbb{A}^n)$ be an elementary resolution (given by Lemma 3.1). Then there exists a subcomplex*

$$\mathbb{Z}(\Omega_*) \subset \mathbb{Z}(X_*) \otimes \mathbb{Z}(\mathbb{A}^n) = \mathbb{Z}(X_* \times \mathbb{A}^n)$$

such that for each $q \geq 0$, Ω_q is an open subscheme in $X_q \times \mathbb{A}^n$ whose complement is a closed subscheme Z_q quasi-finite equidimensional over X_q, such that the composition

$$\mathbb{Z}(\Omega_*) \subset \mathbb{Z}(X_*) \otimes \mathbb{Z}(\mathbb{A}^n) = \mathbb{Z}(X_* \times \mathbb{A}^n) \to \mathbb{Z}(X_*) \stackrel{\rho_*}{\to} z_{eq}(\mathbb{A}^n)$$

is homotopic to zero.

Proof. We construct the subcomplex

$$\mathbb{Z}(\Omega_*) \subset \mathbb{Z}(X_*) \otimes \mathbb{Z}(\mathbb{A}^n)$$

by an induction on the degree $q \geq 0$ using Lemmas 3.6 and 3.7 above, by imposing that for each q, Ω_q is an open subset of $\Omega(z_q)$, where $z_q = 0$ for $q > 0$ and $z_0 = \rho_0 : X_0 \to z_{eq}(\mathbb{A}^n)$, and that the homotopy is the restriction in each degree q of the homotopy on $\mathbb{Z}(\Omega(z_q) \times \mathbb{A}^1)$ given by Lemma 3.6.

To start with, apply Lemma 3.6 to the morphism $\rho_0 : \mathbb{Z}(X_0) \to \mathbb{Z}(\mathbb{A}^n)$ which we see as an element $z_0 \in z_{eq}(X_0 \times \mathbb{A}^n)$. We get an open subset $\Omega_0 = \Omega(z_0) \subset X_0 \times \mathbb{A}^n$ whose complement is quasi-finite equidimensional over X_0 such that the composition

$$\mathbb{Z}(\Omega_0) \subset \mathbb{Z}(X_0) \otimes \mathbb{Z}(\mathbb{A}^n) = \mathbb{Z}(X_0 \times \mathbb{A}^n) \to \mathbb{Z}(X_0) \stackrel{\rho_0}{\to} z_{eq}(\mathbb{A}^n)$$

is homotopic to zero, through the explicit homotopy of the Lemma 3.6.

Now Lemma 3.7 applied to the boundary, an elementary morphism by assumption, $\delta_1 : \mathbb{Z}(X_1) \to \mathbb{Z}(X_0)$, and to z_0 allows one to define Ω_1 (observe that even if $\delta_1^*(z_0) = Z_1 = 0$), Lemma 3.7 works and is non trivial!) . The process continues thanks to Lemma 3.6 and the functorial property of the homotopy.□

Proof of Theorem 3.4. Set $M := \underline{\mathbf{M}}_0$. To prove the vanishing of the Theorem, it is clearly sufficient, by Lemmas 3.5 and 3.3, to prove for each $m \in \mathbb{N}$ the vanishing

$$Ext^m_{\mathcal{A}b_k}(z_{eq}(\mathbb{A}^n); M) = Hom_{D(\mathcal{A}b_k)}(z_{eq}(\mathbb{A}^n); M[m]) = 0$$

We proceed inspired by Voevodsky's proof of [31, Prop. 3.6]. Choose an elementary resolution $\mathbb{Z}(X_*) \to z_{eq}(\mathbb{A}^n)$. As M is Zariski strictly $\mathbb{A}^1$-invariant, Lemma 3.3 implies that the morphisms

$$\mathbb{Z}(X_*) \otimes \mathbb{Z}(\mathbb{A}^n) \to \mathbb{Z}(X_*) \to z_{eq}(\mathbb{A}^n)$$

induce isomorphisms

$$Hom_{D(\mathcal{A}b_k)}(z_{eq}(\mathbb{A}^n); M[m]) \cong Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(X_*); M[m])$$

$$\cong Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(X_*) \otimes \mathbb{Z}(\mathbb{A}^n); M[m])$$

By Corollary 3.8 and Lemma 3.3 again, the restriction morphism

$$Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(X_*) \otimes \mathbb{Z}(\mathbb{A}^n); M[m]) \to Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(\Omega_*); M[m])$$

is 0. The group $Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(X_*) \otimes \mathbb{Z}(\mathbb{A}^n); M[m])$ is thus a quotient of the group $Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(X_*) \otimes \mathbb{Z}(\mathbb{A}^n)/\mathbb{Z}(\Omega_*); M[m])$. By construction the complex $\mathbb{Z}(X_*) \otimes \mathbb{Z}(\mathbb{A}^n)/\mathbb{Z}(\Omega_*)$ is degreewise a direct sum of sheaves of the form $\mathbb{Z}(X)/\mathbb{Z}(X-Z)$ with Z everywhere of codimension $\geq n$; the groups $Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}(X)/\mathbb{Z}(X-Z), M[m]) = H^m(X, X-Z; M)$ thus vanish by Lemma 2.9. The Theorem follows.□

3.3 Vanishing of some extension groups $Ext^i_{\mathcal{A}b_k}(k_n, M)$

Let C_* be a non-negatively graded chain complex in $\mathcal{A}b_k$ and $n \in \mathbb{Z}$ an integer. We denote by $\mathcal{H}_n C_*$ its n-th homology sheaf or in other words its $(-n)$-th cohomology sheaf $\mathcal{H}^{-n}C_*$. We have the following well-known construction, which can be derived from standard homological algebra [10]:

Lemma 3.9 *(Universal coefficient spectral sequence) Let C_* be a non-negatively graded chain complex in $\mathcal{A}b_k$ and let M be a sheaf of abelian groups on Sm_k. Then there exists a natural, strongly convergent spectral sequence of cohomological type of the form*

$$E_2^{p,q} = Ext^p_{\mathbb{A}b_k}(\mathcal{H}_q C_*; M) \Rightarrow Hom_{D(\mathcal{A}b_k)}(C_*, M[p+q])$$

We are now in position to prove our main result on the vanishing of Ext groups:

Theorem 3.10 *Let $n > 0$ be an integer and M_* be a cycle module. Then:*

1) If $M_{-n}(k) = 0$ then $Hom_{\mathcal{A}b_k}(\underline{\mathbf{k}}_n, \underline{\mathbf{M}}_0) = 0$;

2) If M_* is of weights $\leq n-1$ then $Ext^1_{\mathcal{A}b_k}(\underline{\mathbf{k}}_n, \underline{\mathbf{M}}_0) = 0$;

3) If M_* is of weights $\leq n-1$ and $M_{-n+1}(k) = 0$ and if $\mathbf{H}_2$(N-1) holds and $1 \leq n \leq N-1$, then:

$$Ext^2_{\mathcal{A}b_k}(\underline{\mathbf{k}}_n, \underline{\mathbf{M}}_0) = 0$$

Proof. Set $M := \underline{\mathbf{M}}_0$. Let $n > 0$ be an integer. Let's first prove 1). An easy computation as in [27, Lemma 3.3 p. 23] shows that

$$M((\mathbb{G}_m)^n) = \oplus_{m=0}^n M_{-m}(k)^{\binom{n}{m}}$$

As a consequence the intersection of the kernels of the morphisms $M(\iota_i) : M((\mathbb{G}_m)^n) \to M((\mathbb{G}_m)^{n-1})$, with $\iota_i : (\mathbb{G}_m)^{n-1} \to (\mathbb{G}_m)^n$ the closed subscheme defined by i-th coordinate $= 1$, is exactly $M_{-n}(k)$. Let $\phi : \underline{\mathbf{k}}_n \to M$ be a morphism and let $(\mathbb{G}_m)^n \to \underline{\mathbf{k}}_n$ be the morphism of sheaves of sets corresponding to the obvious symbol in $\underline{\mathbf{k}}_n((\mathbb{G}_m)^n)$. From the previous observation we see that the composition $(\mathbb{G}_m)^n \to \underline{\mathbf{k}}_n \xrightarrow{\phi} M$ is zero because $M_{-n}(k) = 0$ and its composition with the ι_i's is 0 because a symbol of length n containing 1 is trivial. Thus ϕ is the zero morphism on sections over fields, and is thus zero because $M(X) \subset \oplus_{x \in X^{(0)}} M_0(\kappa(x))$ for any $X \in Sm_k$.

Now let's prove 2). Assume M_* is of weights $\leq n-1$. By Theorem 3.4 the groups $Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}/2(n), M[*])$ vanish. The universal coefficient spectral sequence of Lemma 3.9 thus converges to 0. By definition of the complex $\mathbb{Z}/2(n)$ [27], see also section 3.2, the homology sheaves $\mathcal{H}_i(\mathbb{Z}/2(n))$ vanish for $i < -n$ and by Theorem A.7 we have a canonical isomorphism:

$$\underline{\mathbf{k}}_n \cong \mathcal{H}_{-n}(\mathbb{Z}/2(n))$$

Looking at the E_2-term gives the vanishing (already known by 1)):

$$Hom_{\mathcal{A}b_k}(\underline{\mathbf{k}}_n, M) = Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}/2(n), M[-n]) = 0$$

and the vanishing:

$$Ext^1_{\mathcal{A}b_k}(\underline{\mathbf{k}}_n, M) \subset Hom_{D(\mathcal{A}b_k)}(\mathbb{Z}/2(n), M[-n+1]) = 0$$

To prove 3) we go further into the study of the E_2-term which also gives a canonical isomorphism

$$Hom_{\mathcal{A}b_k}(\mathcal{H}_{-n+1}(\mathbb{Z}/2(n)), M) \cong Ext^2_{\mathcal{A}b_k}(\underline{\mathbf{k}}_n, M) \tag{10}$$

By $\mathbf{H}_2$*(N-1)* , for any integer $1 \leq n \leq N-1$ the morphism of sheaves

$$\underline{\mathbf{k}}_{n-1} = \mathcal{H}_{-n+1}(\mathbb{Z}/2(n-1)) \to \mathcal{H}_{-n+1}(\mathbb{Z}/2(n)) \tag{11}$$

induced by the cup-product by τ, induces an epimorphism on sections over any fields $F|k \in \mathcal{F}_k$. Because M is 0-pure, this easily implies that the induced morphism $Hom_{\mathcal{A}b_k}(\mathcal{H}_{-n+1}(\mathbb{Z}/2(n)), M) \to Hom_{\mathcal{A}b_k}(\underline{\mathbf{k}}_{n-1}, M)$ is injective. Together with the isomorphism (10) we get an injection

$$Ext^2_{\mathcal{A}b_k}(\underline{\mathbf{k}}_n, M) \subset Hom_{\mathcal{A}b_k}(\underline{\mathbf{k}}_{n-1}, M)$$

but by the case 1) already proven, the group on the right vanishes because $M_{-n+1}(k) = 0$ by assumption.$\square$

3.4 Construction of e_N

We can now complete the proof of our main result Theorem 1.3, following the lines of the introduction.

Let $N > 0$ be a fixed integer. We assume hypothesis $\mathbf{H}_1$*(N)* and $\mathbf{H}_2$*(N-1)* hold. Proceeding by increasing induction we may assume the Milnor conjecture on the Witt ring in weights $\leq N-1$ for fields $F|k$ is proven.

Let us denote by j_* the kernel in the category of cycle modules of the Milnor epimorphism $k_* \to i_*$ constructed in Theorem 2.11. By our inductive assumption, $j_n = 0$ for any $n \leq N-1$. By $\mathbf{H}_2$*(N-1)* and Theorem 3.10, for any integer $1 \leq n \leq N-1$ one has the vanishing

$$Ext^i_{\mathcal{A}b_k}(\underline{\mathbf{k}}_n, \underline{\mathbf{j}}_N) = 0$$

for $i \in \{0, 1, 2\}$. This vanishing also holds for $n = 0$: use the short exact sequence of sheaves $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \underline{\mathbf{k}}_0 \to 0$ and $\mathbf{H}_1$*(N)* which gives $\underline{\mathbf{j}}_N(k) = 0$.

For each integer $n \in \mathbb{N}$, set $\underline{\mathbf{W}}_n := \underline{\mathbf{W}}/\underline{\mathbf{I}}^{n+1}$. Using the short exact sequences

$$0 \to \underline{\mathbf{i}}_n \to \underline{\mathbf{W}}_n \to \underline{\mathbf{W}}_{n-1} \to 0 \qquad (12)$$

and the inductive assumption that $\underline{\mathbf{k}}_n \cong \underline{\mathbf{i}}_n$ for $0 \leq n \leq N-1$, we conclude from the above vanishing of Ext groups that for any $i \in \{0, 1, 2\}$, the group

$$Ext^i_{\mathcal{A}b_k}(\underline{\mathbf{W}}_{N-1}, \underline{\mathbf{j}}_N)$$

vanishes as well. This implies, by the short exact sequence of Lemma 2.7

$$0 \to \underline{\mathbf{j}}_N \to \underline{\mathbf{k}}_N \to \underline{\mathbf{i}}_N \to 0 \qquad (13)$$

that the homomorphism

$$Ext^i_{\mathcal{A}b_k}(\underline{\mathbf{W}}_{N-1}, \underline{\mathbf{k}}_N) \to Ext^i_{\mathcal{A}b_k}(\underline{\mathbf{W}}_{N-1}, \underline{\mathbf{i}}_N)$$

is an isomorphism for $i \in \{0,1\}$ and a monomorphism for $i = 2$. For $i = 1$, this implies the existence of a sheaf of abelian groups $\underline{\mathbf{\Gamma}}_N$ which fits into a commutative square in $\mathcal{A}b_k$ of the form:

$$\begin{array}{ccccccccc} 0 & \to & \underline{\mathbf{k}}_N & \to & \underline{\mathbf{\Gamma}}_N & \to & \underline{\mathbf{W}}_{N-1} & \to & 0 \\ & & \downarrow & & \downarrow & & || & & \\ 0 & \to & \underline{\mathbf{i}}_N & \to & \underline{\mathbf{W}}_N & \to & \underline{\mathbf{W}}_{N-1} & \to & 0 \end{array} \tag{14}$$

in which the horizontal rows are exact, the bottom one being of the form (12), and which induces the Milnor morphism on the left.

Lemma 3.11 *Let $X \in Sm_k$ be such that $\underline{\mathbf{j}}_N(X) = 0$. Then the morphism*

$$\underline{\mathbf{\Gamma}}_N(X) \to \underline{\mathbf{W}}_N(X)$$

is an isomorphism, and thus so is $\underline{\mathbf{k}}_N(X) \to \underline{\mathbf{i}}_N(X)$ by (14).

Proof. The epimorphism $\underline{\mathbf{\Gamma}}_N \to \underline{\mathbf{W}}_N$ has kernel $\underline{\mathbf{j}}_N$. We get for any $X \in Sm_k$ an exact sequence $0 \to \underline{\mathbf{j}}_N(X) \to \underline{\mathbf{k}}_N(X) \to \underline{\mathbf{i}}_N(X) \to H^1(X; \underline{\mathbf{j}}_N)$. By definition for $Y \in Sm_k$ one has

$$\underline{\mathbf{j}}_N(Y) := Ker\big(\oplus_{y \in Y^{(0)}} j_N(\kappa(y)) \to \oplus_{z \in Y^{(1)}} j_{N-1}(\kappa(z)) \big)$$

(see 2.2) and by our assumptions which implies that, for any field $F|k$, $j_{N-1}(F) = 0$), we see that any open immersion $U \subset Y$ induces an isomorphism $\underline{\mathbf{j}}_N(Y) \cong \underline{\mathbf{j}}_N(U)$; this sheaf is thus flasque and $H^1(X; \underline{\mathbf{j}}_N) = 0$ for any X by [10]. This implies the Lemma.□

This is for instance the case for $X = \mathbb{G}_m$ because $(\underline{\mathbf{j}}_N)(\mathbb{G}_m) = \underline{\mathbf{j}}_N(k) \oplus \underline{\mathbf{j}}_{N-1}(k) = 0$ (by $\mathbf{H}_1$*(N)*). The same observation holds for a product $(\mathbb{G}_m)^n$ by the formula used in the proof above. This also holds for any open subscheme $X \subset (\mathbb{G}_m)^n$ because $\underline{\mathbf{j}}_N(X) \subset \underline{\mathbf{j}}_N((\mathbb{G}_m)^n) = 0$.

As a consequence there exists a unique lift

$$\mathbb{G}_m \to \underline{\mathbf{\Gamma}}_N$$

through $\underline{\mathbf{\Gamma}}_N \to \underline{\mathbf{W}}_N$ of the obvious "symbol"

$$\mathbb{G}_m \to \underline{\mathbf{W}}_N \ , \ \ u \mapsto < u > \in \underline{\mathbf{W}}_N$$

We denote by $\theta : \mathbb{Z}(\mathbb{G}_m) \to \underline{\mathbf{\Gamma}}_N$ the morphism of abelian sheaves induced by this lift.

We denote by $(-) : \mathbb{G}_m \to \mathbb{Z}(\mathbb{G}_m)$, $x \mapsto (x)$ the morphism of sheaves of sets given by the "inclusion of the base" into the free sheaf of abelian groups $\mathbb{Z}(\mathbb{G}_m)$ on it. We let $\Phi_0 : X_0 = \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{Z}(\mathbb{G}_m)$ denote the morphism

$$(U, V) \mapsto (U) - (U \,.\, V^2)$$

where we denote by $U : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ and $V : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ the projections to the first and second factor. We will also need the morphism $\Phi_1 : X_1 = \mathbb{G}_m \to \mathbb{Z}(\mathbb{G}_m)$ defined by

$$(U) \mapsto (U) + (-U)$$

Let $X_2 \subset \mathbb{G}_m \times \mathbb{G}_m$ denote the open complement to the closed subscheme of $\mathbb{G}_m \times \mathbb{G}_m$ defined by the equation $U+V = 0$. Finally we let $\Phi_2 : X_2 \to \mathbb{Z}(\mathbb{G}_m)$ denote the morphism

$$(U, V) \mapsto (U) + (V) - (U + V) - ((U + V).U.V)$$

Lemma 3.12 *For $i \in \{0, 1, 2, \}$, the composition*

$$X_i \to \mathbb{Z}(\mathbb{G}_m) \xrightarrow{\theta} \underline{\mathbf{\Gamma}}_N$$

is trivial, i.e. constant with value the 0 *section of* $\underline{\mathbf{\Gamma}}_N$.

Proof. By the above observation, the morphisms

$$\underline{\mathbf{\Gamma}}_N(X_i) \to \underline{\mathbf{W}}_N(X_i)$$

are isomorphisms. The Lemma now follows from the fact that the corresponding statements hold for the compositions $X_i \to \underline{\mathbf{W}}_N$. These are indeed classical relations which hold in the Witt ring $W(F)$ of any field $F|k$ (see [25, Corollary 9.4 p. 66]) and we conclude by the remark below, which implies that the $\underline{\mathbf{W}}_N$ are 0-pure.□

Remark 3.13 If $0 \to M' \to M \to M" \to 0$ is a short exact sequence of sheaves in the Zariski topology, and that both M' and $M"$ are strictly $\mathbb{A}^1$-invariant in the sense of Appendix A.2, it is clear that M itself is also strictly $\mathbb{A}^1$-invariant. This implies easily by induction, because the $\underline{\mathbf{i}}_N$ are each strictly $\mathbb{A}^1$-invariant, that the $\underline{\mathbf{W}}_N$ are strictly $\mathbb{A}^1$-invariant as well. By Corollary A.5 these are also 0-pure.

Corollary 3.14 *For each finite type field extension $F|k$, the morphism*

$$\theta(F) : \mathbb{Z}(F^\times) \to \underline{\mathbf{\Gamma}}_N(F)$$

factors through the epimorphism $\mathbb{Z}(F^\times) \to W(F)$ and thus defines a natural transformation on $\mathcal{F}_k$:

$$\Theta(F) : W(F) \to \underline{\mathbf{\Gamma}}_N(F)$$

Proof. This is clear from the previous Lemma and the fact that for each field F of char $\neq 2$ the relations $< u > = < u.v^2 >$, $< u > + < -u >$ and $< u > + < v > = < u+v > + < (u+v)uv >$, with $u \in F^\times$ and $v \in F^\times$ ($u + v \neq 0$ in the last case), generate the kernel of the epimorphism

$$\mathbb{Z}(F^\times) \to W(F)$$

by *loc. cit.*.□

Lemma 3.15 *The natural transformation on $\mathcal{F}_k$:*

$$\Theta : W(-) \to \underline{\mathbf{\Gamma}}_N|_{\mathcal{F}_k}$$

obtained above arises from a unique morphism of sheaves of abelian groups

$$\mathbf{\Theta} : \underline{\mathbf{W}} \to \underline{\mathbf{\Gamma}}_N$$

Proof. We observe by remark 3.13 that the sheaves $\underline{\mathbf{i}}_N$ and $\underline{\mathbf{W}}_{N-1}$ being strictly $\mathbb{A}^1$-invariant, the sheaf Γ_N, being an extension between two strictly $\mathbb{A}^1$-invariant sheaves is also strictly $\mathbb{A}^1$-invariant, thus 0-pure. Both sheaves in the statement of Lemma 3.15 being 0-pure, we reduce by Lemma A.2 to checking that for any geometric discrete valuation v on $F|k \in \mathcal{F}_k$:

(1) $\Theta(F)$ maps $W(\mathcal{O}_v) \subset W(F)$ into $\underline{\mathbf{\Gamma}}_N(\mathcal{O}_v) \subset \underline{\mathbf{\Gamma}}_N(F)$.

(2) the following induced diagram is commutative

$$\begin{array}{ccc} W(\mathcal{O}_v) & \to & \underline{\mathbf{\Gamma}}_N(\mathcal{O}_v) \\ \rho_v \downarrow & & \downarrow \rho_v \\ W(\kappa(v)) & \stackrel{\Theta(\kappa(v))}{\to} & \underline{\mathbf{\Gamma}}_N(\kappa(v)) \end{array}$$

It is well known that $W(\mathcal{O}_v) \subset W(F)$ is the subgroup generated by symbols $< u >$ with $u \in \mathcal{O}_v^\times$, see [25]. This shows that the morphism of sheaves

$$\mathbb{Z}(\mathbb{G}_m) \to \underline{\mathbf{W}}$$

is onto on geometric discrete valuation rings $\mathcal{O}_v$. Using the morphism of sheaves $\theta : \mathbb{Z}(\mathbb{G}_m) \to \underline{\mathbf{\Gamma}}_N$ this clearly implies (1) and (2).□

We let $((-)) : \mathbb{G}_m \to \mathbb{Z}(\mathbb{G}_m)$ denote the morphism of sheaves of sets

$$u \mapsto ((U)) := 1 - (U)$$

and more generally for each $n > 0$ we let

$$(\mathbb{G}_m)^n \to \mathbb{Z}(\mathbb{G}_m)$$

denote the morphism of sheaves of sets

$$(U_1, \dots, U_n) \mapsto ((U_1)) \cup \dots \cup ((U_n))$$

(whose definition uses the obvious structure of sheaf of commutative rings on $\mathbb{Z}(\mathbb{G}_m)$ with product denoted by $\cup$).

Lemma 3.16 *For each $n \in \mathbb{N}$ the composition*

$$\epsilon_n : (\mathbb{G}_m)^n \to \mathbb{Z}(\mathbb{G}_m) \xrightarrow{\theta} \underline{\mathbf{\Gamma}}_N$$

is trivial if $n > N$ and its image is contained in the subsheaf $\underline{\mathbf{k}}_N \subset \underline{\mathbf{\Gamma}}_N$ for $n = N$. In that case the induced morphism

$$(\mathbb{G}_m)^N \to \underline{\mathbf{k}}_N$$

is the obvious symbol: $(U_1, \dots, U_n) \mapsto \{U_1\} \dots \{U_N\}$.

Proof. By our above observation, the morphisms

$$\underline{\mathbf{\Gamma}}_N((\mathbb{G}_m)^n) \to \underline{\mathbf{W}}_N((\mathbb{G}_m)^n) \quad \text{and} \quad \underline{\mathbf{k}}_N((\mathbb{G}_m)^n) \to \underline{\mathbf{i}}_N((\mathbb{G}_m)^n)$$

are isomorphisms. Thus it suffices to check each statements on $\underline{\mathbf{W}}_N$ and $\underline{\mathbf{i}}_N$ respectfully, which is clear, because the composition

$$(\mathbb{G}_m)^n \stackrel{\epsilon_n}{\to} \underline{\mathbf{\Gamma}}_N \to \underline{\mathbf{W}}_N$$

is the n-fold Pfister symbol.□

We can now easily combine the above results to prove the Milnor conjecture in weight N, finishing the proof of Theorem 1.3:

Corollary 3.17 *The composition*

$$\underline{\mathbf{I}}^n \subset \underline{\mathbf{W}} \stackrel{\mathbf{\Theta}}{\to} \underline{\mathbf{\Gamma}}_N$$

is zero for $n > N$ and for $n = N$ induces a morphism

$$e_N : \underline{\mathbf{i}}_N \cong \underline{\mathbf{I}}^N/\underline{\mathbf{I}}^{N+1} \to \underline{\mathbf{k}}_N = ker(\underline{\mathbf{\Gamma}}_N \to \underline{\mathbf{W}}_{N-1})$$

This is a left inverse to the Milnor morphism in weight N which is thus an isomorphism.

Proof. Lemma 3.16 implies that the morphism $\underline{\mathbf{I}}^n \subset \underline{\mathbf{W}} \to \underline{\mathbf{\Gamma}}_N$ is zero on fields $F|k \in \mathcal{F}_k$ for $n > N$ and is thus zero. The induced morphism $\underline{\mathbf{I}}^N/\underline{\mathbf{I}}^{N+1} \to \underline{\mathbf{\Gamma}}_N$ composed with $\underline{\mathbf{\Gamma}}_N \to \underline{\mathbf{W}}_{N-1}$ is zero on fields by Lemma 3.16 again and is thus zero. Thus we get

$$e_N : \underline{\mathbf{i}}_N \cong \underline{\mathbf{I}}^N/\underline{\mathbf{I}}^{N+1} \to \underline{\mathbf{k}}_N = ker(\underline{\mathbf{\Gamma}}_N \to \underline{\mathbf{W}}_{N-1})$$

Lemma 3.16 implies that the composition

$$(\mathbb{G}_m)^N \to \underline{\mathbf{I}}^N \to \underline{\mathbf{I}}^N/\underline{\mathbf{I}}^{N+1} \to \underline{\mathbf{k}}_N$$

is the N-symbol; thus the composition

$$(\mathbb{G}_m)^N \to \underline{\mathbf{k}}_N \stackrel{\mathbf{s}_N}{\to} \underline{\mathbf{i}}_N \stackrel{e_N}{\to} \underline{\mathbf{k}}_N$$

is also the obvious one, so that the composition $\underline{\mathbf{k}}_N \stackrel{\mathbf{s}_N}{\to} \underline{\mathbf{i}}_N \stackrel{e_N}{\to} \underline{\mathbf{k}}_N$ is the identity on fields, thus is the identity.□

Remark 3.18 In fact to prove the Milnor conjecture in weight N it is not necessary to construct $\mathbf{\Theta}$ as a morphism of sheaves. The natural transformation Θ on fields suffices, and one can adapt the end of the proof to that situation.

A Complement on sheaves

A.1 Elementary properties of 0-pure sheaves of sets

We will not repeat here the definition of 0-pure sheaves of sets given in 2.1.

Let M be a 0-pure sheaf and denote by

$$M|_{\mathcal{F}_k} : \mathcal{F}_k \to \mathcal{A}b$$

its restriction to finite type field extensions of k. For any geometric discrete valuation v on $F|k \in \mathcal{F}_k$, one has an associated subset $M(\mathcal{O}_v) \subset M(F)$ and a restriction map

$$\rho_v : M(\mathcal{O}_v) \to M(\kappa(v))$$

We will not try to describe explicitly the properties satisfied by these data which characterizes exactly the one coming from a 0-pure sheaf[5]. We will only use the following Lemma:

Lemma A.1 *Let M be a 0-pure sheaf of sets and $N \subset M|_{\mathcal{F}_k}$ be a subfunctor. For any irreducible $X \in Sm_k$ with function field F set*

$$N(X) := N(F) \cap M(X) \subset M(F)$$

and for any $X \in Sm_k$ set $N(X) := \Pi_{\alpha \in X^{(0)}} N(X_\alpha) \subset M(X)$.

Assume that for any geometric discrete valuation v on a finite type field extension $F|k$, the map $\rho_v : M(\mathcal{O}_v) \to M(\kappa(v))$ sends $M(\mathcal{O}_v) \cap N(F)$ into $N(\kappa(v)) \subset M(\kappa(v))$.

Then for any morphism $f : Y \to X$ in Sm_k, the map $M(f) : M(X) \to M(Y)$ maps $N(X)$ into $N(Y)$ and the correspondence $X \mapsto N(X)$ is a 0-pure sheaf of sets.

Proof. We assume X and Y are irreducible with field of fractions E and F. Also it suffices to prove the claim separately for f a smooth morphism and for f a closed immersion.

Assume first $f : Y \to X$ is smooth. The map $M(f) : M(X) \to M(Y)$ extends to a map $M(E) \to M(F)$, and thus maps $N(X) = M(X) \cap N(E)$ into $N(Y) = M(Y) \cap N(F)$. Assume now $f : Y \to X$ is a closed immersion. Let $y \in Y \subset X$ be the generic point of Y and $\mathcal{O}_{X,y}$ its local ring

[5] Though this can be done

in X; a regular local ring of dimension $codim_X(Y) = d$. Choose a regular sequence $(x_1, \ldots, x_d)$ generating the maximal ideal of $\mathcal{O}_{X,y}$. Because k is perfect, we get the existence of an open subscheme $U \subset X$ containing y and a flag $Y \cap U = Y_1 \subset \cdots \subset Y_{d+1} = U$ of integral closed subschemes, smooth over k, such that Y_i is the principal divisor in Y_{i+1} defined by the function x_i. We observe that $N(X) = N(U) \cap M(X)$ and $N(Y) = N(Y_1) \cap M(Y)$. It is thus sufficient to check that $M(U) \to M(Y_1)$ maps $N(U)$ into $N(Y_1)$ and using the above flag we reduce to the case f is a closed immersion with Y a principal divisor in X defined by a function. Denoting by v the discrete valuation on E associated to Y we see that $M(f) : M(X) \to M(Y)$ extends to $\rho_v : M(\mathcal{O}_v) \to M(\kappa(v)) = M(F)$; but then by construction and assumption $N(X) = M(X) \cap N(E) = M(X) \cap N(\mathcal{O}_v)$ maps to $M(Y) \cap N(F) = N(Y)$.

The fact that $X \mapsto N(X)$ is a 0-pure sheaf is proven as follows. We may assume X irreducible. Let $\{U_i\}$ be a finite open covering of X. To check that the obvious diagram

$$N(X) \subset \Pi_i N(U_i) \rightrightarrows \Pi_{i,j} N(U_i \cap U_j)$$

is left exact follows easily from the fact that it imbeds into the corresponding diagram for M. The rest is easy.$\square$

Now we can describe morphisms between 0-pure sheaves in analogous terms. Let M and N be 0-pure sheaves of sets and let $\phi : N \to M$ be a morphism of sheaves. By restriction to $\mathcal{F}_k$ this defines a natural transformation

$$\phi|_{\mathcal{F}_k} : N|\mathcal{F}_k \to M|\mathcal{F}_k$$

between functors $\mathcal{F}_k \to Sets$. Moreover $\phi|_{\mathcal{F}_k}$ has the following two properties, for any geometric discrete valuation v on $F|k \in \mathcal{F}_k$:

(1) $\phi(F)$ maps $N(\mathcal{O}_v)$ into $M(\mathcal{O}_v) \subset N(F)$.

(2) the following induced diagram is commutative

$$\begin{array}{ccc} N(\mathcal{O}_v) & \to & M(\mathcal{O}_v) \\ \rho_v \downarrow & & \downarrow \rho_v \\ N(\kappa(v)) & \stackrel{\phi(\kappa(v))}{\to} & M(\kappa(v)) \end{array}$$

Conversely:

Lemma A.2 *Given 0-pure sheaves of sets M and N, the above correspondence defines a bijection from the set $Hom_k(N,M)$ of morphisms of sheaves of sets on Sm_k from N to M to the set $\mathcal{C}(N,M)$ consisting of natural transformations*

$$\phi : N|\mathcal{F}_k \to M|\mathcal{F}_k$$

satisfying, for any geometric discrete valuation v, properties (1) and (2) above.

Proof. By the 0-purity property, the injectivity of the map $Hom_k(N,M) \to \mathcal{C}(N,M)$is clear. Now let $\phi : N|\mathcal{F}_k \to M|\mathcal{F}_k$ be a natural transformation in $\mathcal{C}(N,M)$. Then for each irreducible $X \in Sm_k$, with function field F, ϕ induces by property (1) and Definition 2.1 a morphism

$$\phi(X) : N(X) = \cap_{y\in X^{(1)}} N(\mathcal{O}_{X,y}) \to \cap_{y\in X^{(1)}} M(\mathcal{O}_{X,y}) = M(X)$$

It only remains to show that the $\phi(X)$ altogether define a morphism of sheaves, that is to say a natural transformation on functors on Sm_k. To do this one proceeds using the same argument as in the proof of Lemma A.1 above: to check the property for pull-back along smooth morphisms one considers everything embedded in sections over the corresponding function fields and to check the property for pull-back along closed immersions one reduces to the case of a principal divisor using property (2).□

A.2 Strictly $\mathbb{A}^1$-invariant sheaves and 0-purity

Definition A.3 *A sheaf $M \in \mathcal{A}b_k$ is said to be* strictly $\mathbb{A}^1$-invariant *if it is Zariski strictly $\mathbb{A}^1$-invariant and if for any smooth k-variety X the comparison homomorphism*

$$H^*(X;M) \to H^*_{Nis}(X;M)$$

from Zariski cohomology to Nisnevich [21] cohomology is an isomorphism.

Any homotopy invariant sheaf with transfers [28, Definition 3.1.10] is strictly $\mathbb{A}^1$-invariant by [*loc. cit.*, Theorem 3.1.12].

Lemma A.4 *For any cycle module M_* the sheaf $\underline{\mathbf{M}}_0$ is strictly $\mathbb{A}^1$-invariant sheaf.*

Proof. We know that it is Zariski strictly $\mathbb{A}^1$-invariant by [24, §9]. The fact that it is a sheaf in the Nisnevich topology and that the comparison homomorphism

$$H^*_{Zar}(X; \underline{\mathbf{M}}_0) \to H^*_{Nis}(X; \underline{\mathbf{M}}_0)$$

is an isomorphism follows from [6, Theorem 8.3.1 and §7.3 Ex 5)].□

Lemma 5.5.4 of [18] gives:

Lemma A.5 *A strictly $\mathbb{A}^1$-invariant sheaf is 0-pure.*

Corollary A.6 *A morphism $M \to N$ of strictly $\mathbb{A}^1$-invariant which induces an isomorphism on fields $F|k$ is an isomorphism.*

Proof. By Lemma A.2 both M and N are 0-pure. Thus such a morphism $M \to N$ is a monomorphism of sheaves. Let C be its cokernel. It is clearly a strictly $\mathbb{A}^1$-invariant sheaf thus a 0-pure sheaf again by Lemma A.2. Moreover it vanishes on each field $F|k$. It is thus 0.□

A.3 Motivic complexes and unramified Milnor K-theory

The cohomology sheaves $\mathcal{H}^i(\mathbb{Z}(n))$ of the Suslin-Voevodsky motivic complex $\mathbb{Z}(n)$ in weight n vanish by construction for $i > n$. A standard result of Suslin-Voevodsky [27] gives the computation of $\mathcal{H}^n(\mathbb{Z}(n))(F)$ for each field $F|k$. More precisely, we know from [27] that there is a canonical quasi-isomorphism $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$. This gives in particular for each field $F|k$ a canonical isomorphism

$$F^\times = H^1(Spec(F); \mathbb{Z}(1))$$

It is shown in [27, Theorem 3.4] that it induces, using the product

$$\mathbb{Z}(n) \otimes \mathbb{Z}(m) \to \mathbb{Z}(n+m) \tag{15}$$

a canonical isomorphism of graded rings

$$\Phi_*(F) : K^M_*(F) \cong \oplus_n H^n(Spec(F); \mathbb{Z}(n)) = \oplus_n \mathcal{H}^n(\mathbb{Z}(n))(F)$$

Altogether these isomorphisms define an isomorphism of functors on $\mathcal{F}_k$

$$\Phi_n : K^M_n \cong \mathcal{H}^n(\mathbb{Z}(n))|_{\mathcal{F}_k}$$

The following result ([33, Corollary 2.4 (and proof)] and [7]) extends naturally the previous isomorphism to an isomorphism of sheaves:

Theorem A.7 *Let $n \in \mathbb{N}$ be an integer.*

1) There exists a unique isomorphism of sheaves

$$\underline{\Phi}_n : \underline{\mathbf{K}}_n^M \cong \mathcal{H}^n(\mathbb{Z}(n))$$

which induces the natural isomorphism of Suslin-Voevodsky on the field extensions of k.

2) For each integer $m > 0$, the above isomorphism induces an isomorphism (in $\mathcal{A}b_k$):

$$\underline{\mathbf{K}}_n^M/m \cong \mathcal{H}^n(\mathbb{Z}/m(n))$$

We include the proof for the comfort of the reader:

Proof. 1) In the proof we simply denote by $\mathcal{H}^n$ the sheaf of $\mathcal{H}^n(\mathbb{Z}(n))$. We first construct $\underline{\Phi}_n$ using Lemma A.2. We thus have to check Properties (1) and (2) of that Lemma. Uniqueness is clear.

Fix a discrete valuation v on $F|k \in \mathcal{F}_k$ of geometric type. Each element of $\mathcal{H}^n(F)$ of the form

$$\Phi_n(\{u_1\}\dots\{u_n\})$$

with the u_i's in $\mathcal{O}_v^\times$, can be expressed, using the morphism $\mathcal{H}^1\otimes\dots\otimes\mathcal{H}^1 \to \mathcal{H}^n$ induced by (15) as the cup-product $\Phi_1(\{u_1\}) \cup \dots \cup \Phi_1(\{u_n\})$ so that it lies in $\mathcal{H}^n(\mathcal{O}_v) \subset \mathcal{H}^n(F)$, because each symbol $\Phi_1(\{u_i\})$ lies in $\mathcal{H}^1(\mathcal{O}_v) = \mathcal{O}_v^\times$. Moreover we clearly have the formula

$$\rho_v(\Phi_n(\{u_1\}\dots\{u_n\})) = \rho_v(\Phi_1(\{u_1\}) \cup \dots \cup \Phi_1(\{u_n\}))$$

$$= \Phi_1(\rho_v(\{u_1\})) \cup \dots \cup \Phi_1(\rho_v(\{u_n\})) = \Phi_n(\rho_v(\{u_1\}\dots\{u_n\}))$$

Properties (1) and (2) of Lemma A.2 follow immediately from that observation and from the result of Bass-Tate [5, Prop. 4.5 (b) p. 22], see also 2.10, that any $x \in \underline{\mathbf{K}}_n^M(\mathcal{O}_v)$ is a sum of symbols of the previous form. This defines

$$\underline{\Phi}_n : \underline{\mathbf{K}}_n^M \to \mathcal{H}^n$$

The n-th cohomology sheaf $\mathcal{H}^n$ being a homotopy invariant sheaf with transfers by [28, Definition 3.1.9], it is strictly $\mathbb{A}^1$-invariant by [*loc. cit.*, Theorem

3.1.12]; so is $\underline{\mathbf{K}}_n^M$ by Lemma 2.7. We conclude that $\underline{\Phi}_n$ is an isomorphism by Corollary A.6.

2) is an easy consequence of 1).□

Remark A.8 As a consequence, the sheaves $\underline{K}_n^M$ and $\underline{\mathbf{k}}_n$ have canonical structures of homotopy invariant sheaf with transfers, given by the above isomorphisms. By [7] any sheaf arising from a cycle module has a canonical structure of homotopy invariant sheaf with transfers; it is also proven in *loc. cit.* that the above isomorphisms of sheaves are compatible with these additional structures.

References

[1] J. Arason, Cohomologische Invarianten Quadratischer Formen. J. Algebra 36 (1975), no. 3, 448–491.

[2] J. K. Arason et R. Elman, Powers of the fundamental ideal in the Witt ring, Journal of Algebra **239**, 150-160 (2001).

[3] P. Balmer, S. Gille, I. Panin and C. Walter), The Gersten conjecture on Witt groups in the equicharacteristic case, Documenta Mathematica 7 (2002), pp. 203-217.

[4] J. Barge et F. Morel, Cohomologie des groupes linéaires, K-théorie de Milnor et groupes de Witt. (French) [Cohomology of linear groups, Milnor K-theory and Witt groups] C. R. Acad. Sci. Paris Série I Math. 328 (1999), no. 3, 191–196.

[5] H. Bass and J. Tate, The Milnor ring of a global field. Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972), pp. 349–446. Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973.

[6] J.-L. Colliot-Thélène, R. T. Hoobler, B. Kahn, The Bloch-Ogus-Gabber theorem. Algebraic K-theory (Toronto, ON, 1996), 31–94, Fields Inst. Commun., 16, Amer. Math. Soc., Providence, RI, 1997.

[7] F. Déglise, Modules homotopiques avec transferts et motifs gnriques. Thèse de l'université Paris VII, disponible à:

http://www-math.univ-paris13.fr/ deglise/these.html

[8] P. Elbaz-Vincent and S. Müller-Stach, Milnor K-theory of rings, higher Chow groups and applications, Inventionnes Math., **148**, p. 177-206 (2002).

[9] S. Garibaldi, A. Merkurjev, J.-P. Serre, *Cohomological Invariants in Galois Cohomology*, University Lecture series, volume **28**, AMS.

[10] A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku. (French) Tohoku Math. J. (2) 9 1957 119–221.

[11] B. Kahn and R. Sujatha, Motivic cohomology and unramified cohomology of quadrics. J. Eur. Math. Soc. (JEMS) 2 (2000), no. 2, 145–177.

[12] K. Kato, A generalization of local class field theory by using K-groups. II. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 3, 603–683.

[13] K. Kato, Milnor K-theory and the Chow group of zero cycles. Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), 241–253, Contemp. Math., 55, Amer. Math. Soc., Providence, RI, 1986.

[14] M.-A. Knus, *Quadratic and Hermitian forms over rings.* With a foreword by I. Bertuccioni. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 294. Springer-Verlag, Berlin, 1991.

[15] J. Milnor, Algebraic K-theory and Quadratic Forms, Inventionnes math. **9**, 318-344 (1970).

[16] F. Morel, Suite spectrale d'Adams et invariants cohomologiques des formes quadratiques, C.R. Acad. Sci. Paris, t. 328, Série I, p. 963-968, 1999.

[17] F. Morel, Sur les puissances de l'idéal fondamental de l'anneau de Witt, CMH.

[18] F. Morel, The stable $\mathbb{A}^1$-connectivity theorems, to appear in K-theory Journal.

[19] F. Morel, Milnor's conjecture on quadratic forms and the operation Sq^2, in preparation.

[20] F. Morel and V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes. Inst. Hautes Études Sci. Publ. Math. No. 90 (1999), 45–143.

[21] Y. Nisnevich, The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), 241–342, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 279, Kluwer Acad. Publ., Dordrecht, 1989.

[22] M. Ojanguren, I. Panin, A purity theorem for the Witt group. Ann. Sci. École Norm. Sup. (4) 32 (1999), no. 1, 71–86.

[23] D. Orlov, A. Vishik and V. Voevodsky, An exact sequence for Milnor's K-theory with applications to quadratic forms, http://www.math.uiuc.edu/K-theory/0454/ .

[24] M. Rost, Chow groups with coefficients. Doc. Math. 1 (1996), No. 16, 319–393 (electronic).

[25] W. Scharlau, *Quadratic and Hermitian forms.* Grundlehren der Mathematischen Wissenschaften, **270**. Springer-Verlag, Berlin, 1985.

[26] M. Schmidt, Wittringhomologie, Dissertation, Universität Regensburg, 1998.

[27] A. Suslin and V. Voevodsky, Bloch-Kato conjecture and motivic cohomology with finite coefficients. The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

[28] V. Voevodsky, Triangulated categories of motives over a field. Cycles, transfers, and motivic homology theories, 188–238, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.

[29] V. Voevodsky, Cohomological theory of presheaves with transfers. Cycles, transfers, and motivic homology theories, 87–137, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.

[30] V. Voevodsky, Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic, IMRN 2002:7 (2002) 351-355.

[31] V. Voevodsky, Reduced power operations in motivic cohomology, Inst. Hautes Études Sci. Publ. Math. No. 98 (2003), 1–57.

[32] V. Voevodsky, Motivic cohomology with Z/2-coefficients, Inst. Hautes Études Sci. Publ. Math. No. 98 (2003), 59–104.

[33] V. Voevodsky, The Milnor conjecture, preprint, 1996, available at http://www.math.uiuc.edu/K-theory/0170/

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