Detecting coupling direction: attractor dimensions and clustering in recurrence networks

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Declaration of Authorship

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Contents

Chapter 1

Introduction

This is an informal introduction, precise definitions will be given from Section 1.4 on.

The motivation for this thesis originates from the article "Geometric detection of coupling directions by means of inter-system recurrence networks" by [Feldhoff et al.]. The authors consider two **dynamical systems** $\mathcal{X}$ and $\mathcal{Y}$ in the **phase space** $\mathbb{R}^n$ for some $n \in \mathbb{N}$, whose trajectories are observable.[1] It is assumed that both systems are governed by their own, intrinsic, dynamics, and are coupled with each other by means of **diffusive coupling**[2]: the dynamics of the type

$$\begin{cases} \dot{x} = f(x) + k_{\mathcal{Y}\mathcal{X}}(y - x) \\ \dot{y} = g(y) + k_{\mathcal{X}\mathcal{Y}}(x - y) \end{cases} \tag{1.0.1}$$

[1]Without giving precise definition, we will see a dynamical system as a triple consisting of a phase space, time and a time-evolution law ([Katok and Hasselblatt, Ch.0.1]). Time can be continuous or discrete and the time-evolution law, which in general can itself depend on time and have infinite memory, is simplified to allow us to find out all future states if we know the state at any particular moment. Thus, the law for continuous time can be described by a system of differential equations and the law for discrete time — by a map $x_{t+1} = F(x_t)$. The resulting "sequence" of states is the **trajectory** one can observe.

There are at least three different situations in which one considers a dynamical system. On the theoretical level, having the equations governing the system, it is indeed possible to observe the whole trajectory. While doing numerical simulations, it is possible to observe all states of a discrete time system, but only some approximations of states of a continuous time system. The third situation is working with real-world data, assuming that this data represents a sequence of states of some underlying dynamical system. If we think that the time in the real world is continuous, we always get partial information from the data. More than that, the vector x_t which represents one state does not describe the whole underlying system, but only some features of it, that is why it is more appropriate to call this vector an observable and not a state. In general, observables have to be converted into state vectors, and an appropriate phase space, i.e., a space that is equivalent to the original state space, has to be chosen. This problem of **phase space reconstruction** is not always trivial, see [Kantz and Schreiber, Ch.3 and 9] for established methods. In this thesis, we do not discuss these difficulties and always consider some $\mathbb{R}^n$ as the original state space. We speak of states and trajectories to simplify the language.

Finally, we want to stress that the work described in this thesis is motivated by application to real-world data. Although some theoretical dynamical systems can hardly be applied to real-world processes, we see in each of these systems, e.g., temperature, percipitation and wind direction measured every day at 2 pm in Chennai.

[2]The choice of diffusive coupling is not motivated in the article. Clearly, it is a simple form of interaction between two systems. The term "diffusive" naturally arises as one considers chemical processes and trajectories of the concentrations of some quantities in connected chambers (see, e.g., [Bar-Eli]). In the same sense it is reasonable to apply this form of coupling to atmospheric processes.

for functions f and g very similar to each other and coupling strengths $k_{\mathcal{YX}}, k_{\mathcal{XY}} \geqslant 0$ is considered. However, it is not known whether one (or both) coupling strength is zero, i.e., in which directions the systems are coupled, and the question is to determine these directions on the basis of trajectories as well as functions f and g.

1.1 Recurrence network approach

[Feldhoff et al.] propose a method to answer this question, which is based on the **recurrence network analysis (RNA)**. RNA, whose systematic study was started by [Donner et al., 2010], combines **recurrence plot analysis** (see [Marwan et al.] for comprehensive discussion) with the **complex network** approach (see, e.g., [Boccaletti et al.]). One starts with a time series, i.e., a sequence $(x_t)_{t=1,\dots,N}$ with all $x_t \in \mathbb{R}^n$ for some $n, N \in \mathbb{N}$, which represents states of a dynamical system, chooses some norm $||\cdot||$ on $\mathbb{R}^n$, a number $\epsilon > 0$ and defines the **adjacency matrix** $A(\epsilon)$ as

$$A_{ij}(\epsilon) = 1_{\{||x_i - x_j|| \leqslant \epsilon\}} - 1_{\{i=j\}}, \tag{1.1.1}$$

where 1_B is the indicator function of the set B. This adjacency matrix automatically defines a **recurrence network**, i.e., a graph where each time index t is a vertex — irrespectively of possible equality of the two corresponding states — and two different vertices i and j are connected with an edge if and only if $||x_i - x_j|| \leqslant \epsilon$, i.e., the corresponding states are close enough to each other with respect to the threshold ϵ.[3] Now one can study the recurrence network in order to reveal some properties of the underlying dynamical system. “Many network-theoretic measures yield sophisticated quantitative charactericstics corresponding to certain phase space properties of a dynamical system.”[4]

In order to study two coupled systems, the authors introduce the concept of the **inter-system recurrence network (IRN)**. Let $A^{\mathcal{X}}(\epsilon_{\mathcal{X}})$ and $B^{\mathcal{Y}}(\epsilon_{\mathcal{Y}})$ be the adjacency matrices for the systems $\mathcal{X}$ and $\mathcal{Y}$ and some thresholds $\epsilon_{\mathcal{X}}$, $\epsilon_{\mathcal{Y}}$. For the inter-system threshold $\epsilon_{\mathcal{XY}} > 0$, define the cross-recurrence adjacency matrix $A^{\mathcal{XY}}(\epsilon_{\mathcal{XY}})$ as

$$A^{\mathcal{XY}}_{ij}(\epsilon_{\mathcal{XY}}) = 1_{\{||x_i - y_j|| \leqslant \epsilon_{\mathcal{XY}}\}}, \tag{1.1.2}$$

where $(x_i)_{i=1,\dots,N_1}$ and $(y_j)_{j=1,\dots,N_2}$ are time series for the systems $\mathcal{X}$ and $\mathcal{Y}$ respectively. The matrix

$$A(\epsilon) = \begin{pmatrix} A^{\mathcal{X}}(\epsilon_{\mathcal{X}}) & A^{\mathcal{XY}}(\epsilon_{\mathcal{XY}}) \\ \left(A^{\mathcal{XY}}(\epsilon_{\mathcal{XY}})\right)^T & A^{\mathcal{Y}}(\epsilon_{\mathcal{Y}}) \end{pmatrix} \tag{1.1.3}$$

with $\epsilon = \left(\epsilon^{\mathcal{X}}, \epsilon^{\mathcal{Y}}, \epsilon^{\mathcal{XY}}\right)^T$ is the **inter-system adjacency matrix** - this way, one gets a recurrence network for both time series together. One may choose different thresholds due to a possibly different geometry of the time-series. It is worth noting that both series do not need to have the same sampling, as the network uses time information only for determining the list of all vertices.

[3] This means that the state x_i (as well as x_j) is “recurrent”, since the trajectory comes at some other point in time close to it again. As one can see, the temporal relation of states plays no role in recurrence networks. It is important to choose an appropriate threshold. For discussion of the threshold for recurrence plots, see [Marwan et al., Sec.3.2.2].

[4] [Donner et al., 2010, Sec.1], see Section 3 for the measures. A list of all relevant measures can be found in [Donges et al., Tab.II]. Chapter 3 of the review [Newman] offers comprehensive discussion.

Further, several measures for the IRN are introduced.[5] One of them is the **local cross-clustering coefficient**. Fix ϵ and define $A = A(\epsilon)$. Denote by $V_{\mathcal{X}}$ and $V_{\mathcal{Y}}$ the subsets of the network vertices corresponding to the time series for $\mathcal{X}$ and $\mathcal{Y}$ respectively and by $v \in V_{\mathcal{X}}$ some arbitrary vertex from $\mathcal{X}$. Then the local cross-clustering coefficient for the vertex v is defined by

$$\hat{\mathcal{C}}_v^{\mathcal{X}\mathcal{Y}} = \frac{\sum_{p,q \in V_{\mathcal{Y}}} A_{vp} A_{vq} A_{pq}}{\left(\sum_{p \in V_{\mathcal{Y}}} A_{vp}\right)\left(\sum_{p \in V_{\mathcal{Y}}} A_{vp} - 1\right)} \tag{1.1.4}$$

and can be interpreted as the probability that two different randomly drawn neighbours of v from the subset $V_{\mathcal{Y}}$ are linked[6]. The **global cross-clustering coefficient**

$$\hat{\mathcal{C}}^{\mathcal{X}\mathcal{Y}} = \frac{1}{|V_{\mathcal{X}}|} \sum_{v \in V_{\mathcal{X}}} \hat{\mathcal{C}}_v^{\mathcal{X}\mathcal{Y}} \tag{1.1.5}$$

shows the probability that for a randomly chosen vertex in $V_{\mathcal{X}}$, two of its different randomly drawn neighbours from $V_{\mathcal{Y}}$ are linked.

Another similar global measure is introduced: the **cross-transitivity**

$$\hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}} = \frac{\sum_{v \in V_{\mathcal{X}},\ p,q \in V_{\mathcal{Y}}} A_{vp} A_{vq} A_{pq}}{\sum_{v \in V_{\mathcal{X}},\ p,q \in V_{\mathcal{Y}},\ p \neq q} A_{vp} A_{vq}} \tag{1.1.6}$$

shows the probability that a randomly chosen "cross-triple", i.e., a vertex $v \in V_{\mathcal{X}}$ and two vertices $p, q \in V_{\mathcal{Y}}$ with both $||x_v - x_p||$ and $||x_v - x_q||$ less or equal ϵ, is in fact a non-degenerate "cross-triangle", i.e., $||x_p - x_q|| \leqslant \epsilon$ and $p \neq q$.[7]

Analogously, one defines $\hat{\mathcal{C}}^{\mathcal{Y}\mathcal{X}}$ and $\hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}}$. A priori, there is no reason to assume that $\hat{\mathcal{C}}^{\mathcal{X}\mathcal{Y}} = \hat{\mathcal{C}}^{\mathcal{Y}\mathcal{X}}$ or $\hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}} = \hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}}$, and this fact is the basis for the proposed method. The authors argue that in case of unidirectional coupling, i.e., one of $k_{\mathcal{X}\mathcal{Y}}$, $k_{\mathcal{Y}\mathcal{X}}$ is zero and the other is not, one cross-transitivity tends to be larger than the other.[8] "Let x_i and x_j be two recurrent states in $\mathcal{X}$. If the coupling direction is $\mathcal{X} \to \mathcal{Y}$[9] and the coupling is large enough, we are likely to also find a state y_k^* in $\mathcal{Y}$, which is (cross-)recurrent to both x_i and x_j, due to the coupling's diffusive nature and thus the tendency to "drag" the trajectory of $\mathcal{Y}$ towards $\mathcal{X}$. The resulting "cross-triangle" adds to the value of both $\hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}}$ and $\hat{\mathcal{C}}^{\mathcal{Y}\mathcal{X}}$ according to their definition. On the other hand, "cross-triangles" constituted by two recurrent states in $\mathcal{Y}$ and one cross-recurrent state in $\mathcal{X}$ are merely coincidental due to the driver-responce-like coupling. We would thus expect to see $\hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}} > \hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}}$ in case of unidirectional coupling $\mathcal{X} \to \mathcal{Y}$ and vice versa for the opposite coupling direction."[10]

The article proceeds with a single numerical example of two coupled Rössler oscillators (equation (4.1.4)), for which the hypothesis is corroborated: for coupling strengths in a certain interval, $\hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}} > \hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}}$ clearly holds for all realisations from an ensemble of 200.

[5] All of them are analogues of the measures for one-system recurrence networks. See previous footnote.

[6] In case $\sum_{p \in V_{\mathcal{Y}}} A_{vp} \leqslant 1$, i.e., v has less than 2 neighbours among $V_{\mathcal{Y}}$, we set $\hat{\mathcal{C}}_v^{\mathcal{X}\mathcal{Y}} = 0$.

[7] In the article, $p \neq q$ in the denomenator's sum is not present in the definition, but is assumed implicitly. The same applies to transitivity, which will be defined later.

[8] Since global clustering and transitivity are similar measures, we will from now on discuss only transitivity, which is also the focus of the thesis. Intuition suggests that all results should at least qualitatively also hold for the global clustering coefficient, though this may be misleading.

[9] This means, $k_{\mathcal{Y}\mathcal{X}} = 0$.

[10] [Feldhoff et al., Sec.2.4]

1.2 Numerics vs. analytical theory. Attractor of the system

Of course, one numerical example does not give enough evidence for the theory. We considered examples of other systems and present the results in Chapter 4. For numerous systems - described by both differential equations or maps - the hypothesis is corroborated again. However, there are other systems, e.g., the so-called Thomas operator or the Rényi map, for which $\hat{\mathcal{T}}^{\mathcal{XY}}$ and $\hat{\mathcal{T}}^{\mathcal{YX}}$ stay in exactly opposite relation to each other. In general, as the numerical estimations show, $\hat{\mathcal{T}}^{\mathcal{XY}}$ and $\hat{\mathcal{T}}^{\mathcal{YX}}$ for two diffusively coupled systems can exhibit complex behaviour.

These findings reveal the need of a better theoretical understanding. Reading thoroughly the argument for the hypothesis given above, one can notice that only "cross-triangles", and not the "cross-triples", are discussed. However, the change of the spatial distribution of system $\mathcal{Y}$ due to its coupling with system $\mathcal{X}$ should affect both triangles and triples. Since the transitivity is the ratio of the number of triangles to the number of triples, the resulting effect is in general unclear. In Section 4.2 we give two simple examples to show that both relations between $\hat{\mathcal{T}}^{\mathcal{XY}}$ and $\hat{\mathcal{T}}^{\mathcal{YX}}$ are possible. The question of what is (in any sense) the typical relation, remains unanswered.

An analytical theory would imply the understanding of attractors of the dynamical systems. For some systems, a nonempty set of initial conditions will after some **transient** time result in a set of trajectories lying close to a certain subset of the phase space. This subset is invariant under the time-evolution law and is called the **attractor** of the system. The corresponding set of initial conditions is called the **basin of attraction**. Systems that have attractors are called **dissipative**[11].

When we talked about trajectories before, we meant the part of trajectories that is close to the attractor. We are interested in a typical behaviour of the system and this is the one after the transient time. An attractor represents the system in its dynamical equilibrium and the properties of the attractor are often everything we want to know about the system.[12] A system can have several attactors; in this case each of them can be studied separately.

Since an attractor is a set of points (in a phase space), it is in the first place characterized by its geometry.[13] The simplest attractors are stable fixed points and limit cycles. If the dimension of the phase space is higher than two, systems can have attractors of very complex geometry.[14] [Ruelle and Takens] coined the word **strange attractor** to describe attractors

[11]Dissipative systems are defined as systems that loose energy in the course of their time evolution. On average a phase space volume of an ensemble of initial conditions decreases during the time evolution of the system. Existence of attractors is the characterizing property of a dissipative system. See [Handbook of Physics, 6.1.3]

[12]For decades researchers focus on the dynamical equilibrium and not the transient time. This has partly historical reasons. Without computers one could not characterize the system in the transient time well enough: some time after Newton, it was clear that it is impossible to give explicit solutions for the dynamics of even only three bodies. The theory of dynamical systems started to develop fast after the breakthrough of [Poincaré], who suggested not to ask questions about individual states but about the trajectory after sufficiently many iterations. Developing stability theory for fixed points and periodic orbits, he laid the foundation of studying attractors. See [Holmes] and [Strogatz, Sec.1.1] for the history of the theory of dynamical systems.

It is not clear whether the focus on the attractor is justified by the motivation to study real-world processes. Any real-world system is not isolated: there are changing forces coming from outside the system and not constant energy exchange with the outer parts, so it is rather always the transient evolution that we observe. The concept of attractor is rationalized by the dissipative processes such as friction or heat transfer which run in all real-world systems.

[13]Here we will describe only those approaches to study attractors which will be more thoroughly investigated in the thesis. For a comprehensive introduction to different approaches see, e.g., [Hilborn, Ch.9].

[14]It was long believed that stable fixed points and limit cycles are the only possible attractors. The

with fractal geometry[15]. Many **chaotic systems**, i.e., systems sensitive to the initial conditions in the sense that a slight change of the starting point drastically changes the subsequent trajectory, have strange attractors. Strange attractors deserve their name and there is no comprehensive theory describing them. To have a better understanding of an attractor one needs to study the corresponding invariant measure.

1.3 Invariant measure, dimension of an attractor

An attractor is a set of points, so we are interested in its geometry. At the same time, this set evolves from the trajectories of the dynamical system, so the distribution of points in the set is also relevant. Normally, the trajectory "visits" different points not equally often and the distribution of points gives additional information about the structure of the attractor. This distribution is described by the **invariant measure**.

Suppose that a dynamical system in discrete time has only one attractor, whose basin is the whole phase space. Define some finite measure (let it be normalized to 1, i.e., a probability density) on the phase space and consider an ensemble of initial conditions distributed according to this measure. After the transient time, all points from this ensemble will be close to the attractor and one could compute a new measure, according to which these points are distributed in the phase space.[16] Normally, this measure will be approximately invariant under the dynamical system, which means that the measure, according to which the points will be distributed after the next iteration, will be approximately the same.

The invariant measure provides enough information about the attractor - clearly, it contains information about the geometry, since it is equal to zero only for points of the phase space which are not on the attractor. However, the question of existence and uniqueness of the invariant measure, not to speak of the analytical expression for it, is in general very difficult.[17] One thus has to revert to less informative and again more geometric properties of the attractor. Different kinds of **dimensions** constitute another class of attractors' charactericstics.

The dimension of a set is intuitively understood as the minimum number of coordinates needed to identify a point in this set. E.g., a square is two-dimensional and a unit circle is one-dimensional - though the circle is normally embedded in a (two-dimensional) plane, it is enough to use one parameter - the angle - to identify each of its points.

There are many different (and not always equivalent) ways to formalize the concept of

Poincaré-Bendixson theorem states that this is indeed true for continuous systems in one- or two-dimensional phase space. In 1927, [van der Pol and van der Mark] described a system, noting the coexistence of two periodic orbits of different period, which, according to [Birkhoff] implied an unstable invariant set of complex geometry. It was not until 1960 that a stable invariant set with fractal geometry arising from a dynamical system described by a map was found (the set is now called the **Smale horseshoe**, see [Smale]). Soon, in 1963, the probably most famous attractor - the **Lorenz attractor** - was discovered ([Lorenz]). The Lorenz attractor arises from a 3-dimensional system in continuous time. See [Holmes] for the history.

[15] Fractals are sets that have repeating patterns at every scale. They are often self-similar sets, i.e., some of their proper subsets are similar to the whole set. We will not give mathematical definitions of fractals or self-similarity, since they involve more theory and are not exactly the topic of this thesis. For the classical work on fractals, see [Mandelbrot]. For the mathematical foundations, see [Falconer].

[16] One can compute the new measure after each iteration. Clearly, a finite ensemble will give only an approximation of the measure. For a good description of this process, see [Lasota and Mackey, Sec.1.2].

[17] In Chapter 3, we discuss a sophisticated theory of the invariant measure developed by Prof. Lasota and colleagues, which we apply to the driven Rényi transformation.

dimension. A very intuitive approach is that of the **box-counting dimension**[18]. Imagine that the phase space $\mathbb{R}^n$ is covered by a grid of n-dimensional cubes of edge length ϵ. Let $N(\epsilon)$ be the number of cubes that have non-empty intersection with the set. Thinking about the simple examples of a point, an interval, a square or a cube, we expect $N(\epsilon)$ to be proportional to ϵ^d, where d is the dimension of the set. The box-counting dimension is defined as (e.g., [Ott, Sec.3.1])

$$D_0 = \lim_{\epsilon \to 0} -\frac{\log N(\epsilon)}{\log \epsilon}, \tag{1.3.1}$$

in case the limit exists. For geometrically simple sets, this definition coincides with the intuitive idea of a dimension. For sets of complex structure, this definition can yield non-integer values. Though surprising at first sight, this fact is what makes the definition so valuable: it gives a measure of the space-filling capacity of the set. E.g., the Koch curve — one of the first examples of a fractal set (see [von Koch]) — is defined by an iterative process and can be depicted only approximately after a finite number of iterations. Any approximation is a one-dimensional curve, but the Koch curve itself fills some space on the plane and has box-counting dimension higher than 1.

The box-counting dimension has a disadvantage in that it is quite demanding to compute it for the given data. The phase space should be divided into boxes, the trajectory should be located and the number of computations increases exponentially with the dimension of the phase space. Other notions of dimension allow easier numerical estimations. The **correlation dimension** was the first concept of dimension created to gain computational advantage (see [Grassberger and Procaccia, Sec.1 and 2]). Another example is the **transitivity dimension**, which arised from the recurrence network approach. We will discuss these notions in Chapter 2.

1.4 Thesis goals and structure

The initial plan for this thesis was to develop an analytical theory for the detection of diffusive coupling in two systems from the data sets of their trajectories, based on the recurrence network analysis. Specifically, we wished to circumstantiate the ideas of [Feldhoff et al.] and develop them, possibly through the analysis by means of the transitivity dimension. This turned out to be a very complex task to accomplish in the available time, so only several steps towards understanding the subject have been made.

After introducing basic definitions and notation in Section 1.4, we proceed in Chapter 2 with new results on transitivity[19] and the transitivity dimension. Having sketched the already known facts, I give a proof that the transitivity dimension of the attractor is an integer equal to the phase space dimension n in case the corresponding invariant density is absolutely continuous w.r.t. Lebesgue, bounded and continuous in at least one point of the phase space where it is nonzero. It probably follows that if the invariant measure has the same properties but w.r.t. a smooth submanifold of dimension $m < n$, the transitivity dimension will be equal to m. This result supports the idea that the transitivity dimension is defined meaningfully, and it does not contradict the well-known fact about the non-integer dimension of fractal sets. Indeed, fractal attractors do not normally have invariant densities which are absolutely

[18]The box-counting dimension is also called the Minkowski-Bouligand dimension.

[19]Transitivity is the analogue of cross-transitivity for one system. The definition will be given in Chapter 2.

continuous w.r.t. Lebesgue and continuous at least in one point.[20] On the other hand, this result is surprising, since it shows that the continuity in one point (local property) is enough for the dimension to have integer value (global property).

The result is in line with the fact that the whole class of Rényi entropy dimensions (the correlation dimension being one of them) has also integer values of the phase space dimension under the same conditions on the invariant measure. Though this last fact seems to be widely accepted among the researchers, I could not find the proof in the literature and give my own.

Further, I study the rate of convergence of the transitivity computed from a sample data towards the theoretical value (i.e., the one computed using the invariant measure) in dependence on the attractor dimension. Third, in Section 2.3, I present an approach to study weak coupling identifying the influence of the driving system with stochastic noise and show on simple examples how the presence of the noise changes the dimension of the attractor of the driven system.

In Chapter 3, I present the theory of the invariant measure of the Rényi transformation developed by Prof. Lasota and colleagues. This is not an original piece of work, my contribution is solely the application of the theory to the driven Rényi transformation as well as the compilation of different articles into one (hopefully, better readible) sequence of lemmas and theorems. However, I found the work on this chapter very useful for the understanding of the difficulties involved in the analytical theory of the invariant measure for a given system. Rényi transformations were chosen for analysis quite early during the work on the thesis, since they demonstrated interesting behaviour by numerical computations of the cross-transitivities and seemed to be not difficult to study analytically. However, own attempts to find the invariant measures did not succeed and the found existing theory turned out to be quite complicated.

Finally, Chapter 4 is devoted to heuristic study of the questions following [Feldhoff et al.]. Different dynamical systems are studied with the same methodology as in [Feldhoff et al.] and the results of the numerical computations are presented. Further, I give simple examples to show that in general any relation between $\hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}}$ and $\hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}}$ (see Section 1.1) is possible.

We finish with a short conclusion.

1.5 Basic definitions, theorems and notation

This section can be skipped and used as a reference while reading the thesis, especially Section 2.3 and Chapter 3.

1.5.1 Norms on $\mathbb{R}^n$, scalar product

In all applications, we consider dynamical systems in the phase space $\mathbb{R}^n$ for some $n \in \mathbb{N}$. On $\mathbb{R}^n$ we use the Euclidean and, most of the time, the supremum norm. The choice is due to the simplicity of computations and is not of high significance, since all norms on $\mathbb{R}^n$ are equivalent.

Definition 1.5.1. *Let $x = (x_1, \dots, x_n)^T$ be a point in $\mathbb{R}^n$. Then*

$$||x||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{1.5.1}$$

[20] In Section 2.1, we give an example and further discuss this issue.

is the **Euclidean norm** *of x and*

$$||x||_\infty = \max_{1 \leqslant j \leqslant n} |x_j| \tag{1.5.2}$$

is the **supremum norm** *of x.*

The **(open) ball** *of radius $r \in \mathbb{R}^n$ with center in $x \in \mathbb{R}^n$ is defined by*

$$B_r(x) = \{y \in \mathbb{R}^n : ||x - y|| < r\}, \tag{1.5.3}$$

where $|| \cdot ||$ can be any norm and should be specified beforehand.

Definition 1.5.2. *Two norms $|| \cdot ||_\alpha$ and $|| \cdot ||_\beta$ on $\mathbb{R}^n$ are called* **equivalent** *if there exist finite positive constants c and C with*

$$c||x||_\alpha \leqslant ||x||_\beta \leqslant C||x||_\alpha \tag{1.5.4}$$

for all $x \in \mathbb{R}^n$.

Theorem 1.5.3. *On $\mathbb{R}^n$, any two norms are equivalent.*

The elementary proof of an even stronger statement can be found in [MacCluer, Th.4.2].

Definition 1.5.4. *For two points $x = (x_1, \ldots, x_n)^T, y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, the* **scalar product** *of x and y is defined as*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i. \tag{1.5.5}$$

1.5.2 Measures

For the theory in Chapter 3, we consider a general phase space X.

Definition 1.5.5. *A family of sets $\mathcal{A} \subset X$ is called a* **σ-algebra** *if*

(i) $X \in \mathcal{A}$,

(ii) for all $A \in \mathcal{A}$, $X^c = X \backslash A \in \mathcal{A}$ and

(iii) for any countably many $A_1, A_2, \cdots \in \mathcal{A}$, $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$.

Definition 1.5.6. *Let $\mathcal{A} \subset X$ be a σ-algebra. A* **measure** *is a function $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\varnothing) = 0$ and such that for any countably many mutually disjoint sets $A_1, A_2, \cdots \in \mathcal{A}$, it holds $\mu\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i)$.*

A measure μ is called **finite**, *if $\mu(X) < \infty$.*

A measure μ is called **σ-finite**, *if there are at most countably many sets $X_1, X_2, \cdots \in \mathcal{A}$ such that $X = \bigcup_{i=1}^\infty X_i$ and $\mu(X_i) < \infty$ for all $i \in \mathbb{N}$.*

Definition 1.5.7. *For a space X, a σ-algebra $\mathcal{A}$ defined on X and a σ-finite measure μ defined on $\mathcal{A}$,*

the pair $(X, \mathcal{A})$ is called the **measurable space** *and*

the triple $(X, \mathcal{A}, \mu)$ is called the **σ-finite measure space**.

For $\mathbb{R}^n$, the canonical σ-finite measure space is $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n)$:

Definition 1.5.8. *$\mathcal{B}(\mathbb{R}^n)$, the* **Borel σ-algebra on** *$\mathbb{R}^n$, is the smallest σ-algebra, which contains all open sets in $\mathbb{R}^n$, i.e., all possible unions of open balls in $\mathbb{R}^n$.*

The existence of this smallest σ-algebra is guaranteed by a theorem, see, e.g., [Klenke, Th.1.16].

Definition 1.5.9. *The* **Lebesgue measure** *λ^n is the unique measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that*

$$\lambda^n((a,b]) = \prod_{i=1}^{n}(b_i - a_i) \tag{1.5.6}$$

for all $a = (a_1, \ldots, a_n)^T, b = (b_1, \ldots, b_n)^T \in \mathbb{R}^n$.

The existence of this unique measure is guaranteed by a theorem, see, e.g., [Klenke, Th.1.55].

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. For a function $f : X \to \mathbb{R}$, we denote the integral of f over a set $A \in \mathcal{A}$ with respect to (w.r.t.) the measure μ by

$$\int_A f d\mu. \tag{1.5.7}$$

For the Lebesgue measure on $\mathbb{R}^n$, we write dx instead of $d\lambda^n$.

Definition 1.5.10. *Let μ and ν be two measures on $(X, \mathcal{A})$.*

The measure μ is called **absolutely continuous** *w.r.t. the measure ν, if $\nu(A) = 0$ implies $\mu(A) = 0$ for every $A \in \mathcal{A}$.*

The measures μ and ν are called **mutually singular***, if there is an $A \in \mathcal{A}$ with $\mu(A) = 0$ and $\nu(X \backslash A) = 0$.*

Theorem 1.5.11 (The Radon-Nikodym theorem)**.** *Let μ and ν be two σ-finite measures on $(X, \mathcal{A})$. Then μ is absolutely continuous w.r.t. ν if and only if there exists a nonnegative measurable function $f : X \to \mathbb{R}$ such that for all $A \in \mathcal{A}$*

$$\mu(A) = \int_A f d\nu. \tag{1.5.8}$$

A proof of this theorem can be found, e.g., in [Teschl, topics, Th.9.2].

Definition 1.5.12. *The function $f : X \to \mathbb{R}$ from the Radon-Nikodym theorem is called the* **Radon-Nikodym derivative***.*

Theorem 1.5.13 (Lebesgue decomposition theorem)**.** *Let μ and ν be two σ-finite measures on $(X, \mathcal{A})$. Then μ can be uniquely decomposed as $\mu = \mu_{ac} + \mu_s$, where μ_{ac} is absoultely continuous w.r.t. ν and μ_s and ν are mutually singular.*

A proof of this theorem uses the Radon-Nikodym theorem, c.f. [Teschl, topics, Th.9.3].[21]

[21] One can proceed the other way around and make the Radon-Nikodym theorem a corollary to the Lebesgue decomposition theorem - see [Klenke, Th.7.33].

There are several ways in which we can express the time-evolution law. Generally, we can speak of transformations of the phase space.[22]

Definition 1.5.14. *A transformation (or a function) $S : X \to X$ is* **measurable***, if $S^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$.*

A measurable transformation is **nonsingular** *if $\mu(A) = 0$ implies $\mu(S^{-1}(A)) = 0$.*

Definition 1.5.15. *Let $S : X \to X$ be a nonsingular transformation. The measure μ is said to be* **invariant** *under S if for every $A \in \mathcal{A}$*

$$\mu(S^{-1}(A)) = \mu(A). \tag{1.5.9}$$

1.5.3 L^p-spaces and Fubini theorem

Definition 1.5.16. *For a set $A \in X$, the* **indicator function** *of A is defined as*

$$1_A(x) = \begin{cases} 1, & x \in A \\ 0, & \textit{otherwise}. \end{cases} \tag{1.5.10}$$

Definition 1.5.17. *The* **support** *of a function $f : X \to \mathbb{R}$ is the set*

$$\textit{supp}(f) = \{x \in X : f(x) \neq 0\}. \tag{1.5.11}$$

Definition 1.5.18. *For $p > 1$, the space of functions $f : X \to \mathbb{R}$ such that*

$$||f||_p = \left(\int_X |f|^p d\mu \right)^{1/p} < \infty \tag{1.5.12}$$

is called an L^p **space** *over X, denoted $L^p(X)$.*

The space of functions $f : X \to \mathbb{R}$ such that

$$||f||_\infty = \sup_{x \in X} |f(x)| < \infty \tag{1.5.13}$$

is called an L^∞ **space** *over X, denoted $L^\infty(X)$.*

Functions $f \in L^1$ are called **integrable***.*

Definition 1.5.19. *Denote*

$$D = D(X, \mathcal{A}, \mu) = \{f \in L^1(X) : ||f|| = 1\}. \tag{1.5.14}$$

Any function $f \in D$ is called a **density***.*

It can be easily shown that L^p spaces are indeed vector spaces for all $1 \leqslant p \leqslant \infty$. Moreover, they are Banach spaces, i.e., complete.[23] If μ is finite, then we have the following inclusions for all $p \in \mathbb{N}$ (see, e.g., [Villani, Th.2]):

$$L^\infty(X) \subset \cdots \subset L^p(X) \subset L^{p-1}(X) \subset \cdots \subset L^2(X) \subset L^1(X). \tag{1.5.15}$$

[22] For discrete time systems, the transformation is just the map $S : X \to X$. For systems in continuous time, we need to define a family of transformations. Following [Lasota and Mackey, Def.7.2.1], we can define a dynamical system $\{S_t\}_{t \in \mathbb{R}}$ on X as a family of transformations $S_t : X \to X$, $t \in \mathbb{R}$, satisfying (i) $S_0(x) = x$ for all $x \in X$, (ii) $S_t(S_{t'}(x)) = S_{t+t'}(x)$ for all $x \in X$, $t, t' \in \mathbb{R}$ and (iii) the map $(t, x) \to S_t(x)$ is continuous. We will not use this definition and leave it for the footnote.

[23] We do not give definitions of these elementary notions from (functional) analysis as well as of the Hilbert space mentioned later. See [Reed, Simon, Ch.1 and 2].

There is an important issue that in L^p spaces we cannot distinguish functions which differ on a set of measure 0. In fact, $f \in L^p$ is a representative of an equivalence class of functions, which are equal up to a set of measure 0. It follows that all (in)equalities of functions should be understood only as "almost everywhere" - a.e. - (in)equalities.

We will treat sets in a similar way. Since characteristic functions are equal in L^1 if and only if the underlying sets are equal up to a set of measure 0, we will call two sets A and B **different** only if $A \backslash B$ or $B \backslash A$ has positive measure. Equalities of sets should also be understood as "almost" equalities.

From all L^p-spaces, only L^2 can be made to a Hilbert space.

Definition 1.5.20. *Let $f, g \in L^2(X)$. Then the* **scalar product** *of f and g is*

$$\langle f, g \rangle = \int_X f g d\mu. \tag{1.5.16}$$

Finally, we state the Fubini theorem that will be used in Chapter 2.

Theorem 1.5.21 (Fubini)**.** *Let $(X_1, \mathcal{A}_1, \mu)$ and $(X_2, \mathcal{A}_2, \nu)$ be two σ-finite measure spaces and f be a nonnegative measurable function on $X_1 \times X_2$. Then*

$$\int_{X_1} \left(\int_{X_2} f(x_1, x_2) d\nu(x_2) \right) d\mu(x_1) < \infty \tag{1.5.17}$$

if and only if

$$\int_{X_2} \left(\int_{X_1} f(x_1, x_2) d\mu(x_1) \right) d\nu(x_2) < \infty \tag{1.5.18}$$

and if one (and thus both) of these integrals is finite, then they are equal.

For the proof, see [Teschl, topics, Th.A.21 and A.22].

1.5.4 Normal distribution, characteristic function and some notation

We will not mention here the preliminaries from probability theory, which can be found, e.g., in [Klenke, Ch.1].

Definition 1.5.22. *A matrix $M_{n\times n}$ is called* **positive definite**, *if $x^T M x > 0$ for all $x \in \mathbb{R}^n$.*

Definition 1.5.23. *Let Σ be a positive definite symmetric real $n \times n$ Matrix and $a \in \mathbb{R}^n$. A random vector $\xi = (\xi_1, \ldots, \xi_n)^T$ is said to have the* **(multivariate) normal distribution** *with mean a and covariance matrix Σ, if ξ has the probability density*

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2} \langle x - a, \Sigma^{-1}(x - a) \rangle \right) \tag{1.5.19}$$

for $x \in \mathbb{R}^n$. We write $\xi \sim \mathcal{N}(a, \Sigma)$.

Definition 1.5.24. *Let μ be a finite measure on $\mathbb{R}^n$. The function $\varphi_\mu : \mathbb{R}^n \to \mathbb{C}$ defined by*

$$\varphi(\lambda) = \int e^{\mathrm{i}\langle \lambda, x \rangle} \mu(dx), \tag{1.5.20}$$

where i *is the imaginary unit, is called the* **characteristic function** *of μ.*

For a normal distribution with mean a and covariance matrix Σ, the characteristic function has the form ([Klenke, Th.15.53])

$$\varphi(\lambda) = e^{\mathrm{i}\langle \lambda, a\rangle} e^{-\frac{1}{2}\langle \lambda, \Sigma\lambda\rangle} \tag{1.5.21}$$

for all $\lambda \in \mathbb{R}^n$.

At the end of this section we repeat some common notation that will be used in the thesis.

By $\bigtimes_{i\in I} A_i$ we denote the Cartesian product of the sets $\{A_i\}_{i\in I}$.

For two sets A and B, $A\Delta B$ denotes the symmetric difference:

$$A\Delta B = A\backslash B \cup B\backslash A. \tag{1.5.22}$$

The restriction of a function $f : D \to X$ to a subdomain $A \subset D$ is denoted by $f|_A$.

Chapter 2

Transitivity dimension and the impact of noise on dimensions

We start this chapter with the definition of transitivity as a measure in the complex network theory. As in introduction, we start with a time series $(x_t)_{t=1,\dots,N}$, $x_t \in \mathbb{R}^n$ for all $t = 1,\dots,N$ and some $n, N \in \mathbb{N}$, which represents states of a dynamical system, and construct the recurrence network. The vertices are all x_t irrespectively of possible equality of two states and the edges are defined by the adjacency matrix $A \in \mathbb{R}^{N\times N}$ with

$$A_{ij}(\epsilon) = 1_{\{||x_i - x_j|| \leqslant \epsilon\}} - 1_{\{i=j\}} \text{ for all } i,j \in \{1,\dots,n\}, \tag{2.0.1}$$

where ϵ is a fixed threshold and the norm is specified. Throughout this chapter, we will use the supremum norm, which is normally easier for computations, and write just $||\cdot||$ instead of $||\cdot||_\infty$.

Definition 2.0.25. *The* **(ϵ-)transitivity** *is the number*

$$\hat{\mathcal{T}} = \frac{\sum_{i,j,k=1}^{N} A_{ki}A_{kj}A_{ij}}{\sum_{i,j,k=1,\ i\neq j}^{N} A_{ki}A_{kj}}. \tag{2.0.2}$$

In words, we have that the transitivity[1] is

$$\frac{6\times \text{ number of triangles in the network}}{2\times \text{ number of connected triples in the network}} = \frac{3\times \text{ number of triangles}}{\text{number of connected triples}}, \tag{2.0.3}$$

where “triangle” means three different vertices connected pairwise through edges and “connected triple” is a vertex connected through edges to two other different ordered vertices. The factor 3 corresponds to the fact that each triangle contributes to 3 triples and thus ensures that $0 \leqslant \hat{\mathcal{T}} \leqslant 1$. This way, the transitivity can be interpreted as the mean probability that two neighbour vertices of one chosen vertex are themselves neighbours.

[1]According to [Newman, Sec.3.2], transitivity in this form was first proposed in physical literature by [Barrat and Weigt]. However, they called it “clustering coefficient” following [Watts and Strogatz], who gave a slightly different definition. In the recent literature (c.f. [Donges et al.]), the term “clustering coefficient” is used for this second definition, and the measure defined in [Barrat and Weigt] is called “transitivity”, as we do here for now. Later, we will call it the “transitivity estimator” in order to distinguish this quantity, which can be computed from a finite network, from another quantity defined using the invariant measure.

As all measures in complex network theory, the transitivity is computed for a finite discrete network. However, our theoretical interest lies in studying attractors of dynamical systems, which are often "continuous" objects. [Donner et al., 2011] proposed to "interpret an ϵ-recurrence network as a discrete subnetwork of a "continuous" graph with uncountably many vertices and edges corresponding to the system's attractor."[2] Accordingly, they redefined measures and came up with the following definition of transitivity:

Definition 2.0.26. *Let $\mathcal{X}$ be a dynamical system in $\mathbb{R}^n$ and μ the corresponding invariant density. We call*

$$\mathcal{T}(\epsilon) = \frac{\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} 1_{\{||x-y||\leqslant\epsilon\}} 1_{\{||x-z||\leqslant\epsilon\}} 1_{\{||y-z||\leqslant\epsilon\}} d\mu(z)d\mu(y)d\mu(x)}{\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} 1_{\{||x-y||\leqslant\epsilon\}} 1_{\{||x-z||\leqslant\epsilon\}} d\mu(z)d\mu(y)d\mu(x)} \tag{2.0.4}$$

the **ϵ-transitivity** *of $\mathcal{X}$.*

The notion of transitivity is this way uncoupled from the recurrence network and is solely a measure of the attractor of the system, more precise, of its ivariant measure. From now on, to avoid confusion, we use the term "transitivity" for this new object and call the network measure from definiton 2.0.25 the "transitivity estimator", though this name is probably not yet established.

2.1 Transitivity dimension and its properties

Transitivity is a measure corresponding to an attractor. As mentioned in Section 1.2, measures of attractors often come up in form of dimensions. Donner et al. observed close interrelation between the transitivity and the dimensionality of attractors, e.g., for the simplest examples of the attractor points uniformly distributed on an interval, a square or an n-dimensional hypercube, the transitivity converges to $(3/4)^n$ for the corresponding dimension n of the attractor as $\epsilon \to 0$. This encouraged them to define a new notion of dimension. Following [Donner et al., 2011, Sec.3.2], we define the transitivity dimension.

Definition 2.1.1. *Let $\mathcal{X}$ be a dynamical system in $\mathbb{R}^n$, μ the corresponding invariant density and $\mathcal{T}(\epsilon)$ the corresponding ϵ-transitivity. If the limit*

$$D_{\mathcal{T}} = \lim_{\epsilon\to 0} \frac{\log \mathcal{T}(\epsilon)}{\log(3/4)} \tag{2.1.1}$$

exists, it is called the **transitivity dimension** *of $\mathcal{X}$.*

Donner et al. point out that for self-similar sets, $D_{\mathcal{T}}$ does not exist in general. Instead of considering definition 2.1.1, they study the upper and lower transitivity dimensions.

Definition 2.1.2. *Let $\mathcal{X}$ be a dynamical system in $\mathbb{R}^n$, μ the corresponding invariant density and $\mathcal{T}(\epsilon)$ the corresponding ϵ-transitivity. The* **upper and lower transitivity dimensions** *of $\mathcal{X}$ are*

$$\begin{aligned} D^u_{\mathcal{T}} &= \limsup_{\epsilon\to 0} \frac{\log \mathcal{T}(\epsilon)}{\log(3/4)} \quad \textit{and} \\ D^l_{\mathcal{T}} &= \liminf_{\epsilon\to 0} \frac{\log \mathcal{T}(\epsilon)}{\log(3/4)} \end{aligned} \tag{2.1.2}$$

[2] [Donges et al., Abstract], which is another article from the same authors summarizing many issues from [Donner et al., 2011].

respectively.

The authors describe several properties of these new notions.

First, $D^u_{\mathcal{T}}$ and $D^l_{\mathcal{T}}$ can differ for one attractor, if it has fractal geometry. E.g., for the Cantor set, i.e., the set of real numbers in $[0,1]$ with ternary expansion which does not contain any "1" digit, $\mathcal{T}(\epsilon) = 1$ for all $\epsilon \in \{1/3^k\}_{k\in\mathbb{N}}$ and $\mathcal{T}(\epsilon) = 11/13$ for all $\epsilon \in \{5/3^k\}_{k>1}$, so $D^u_{\mathcal{T}} \approx 0.581$ and $D^l_{\mathcal{T}} = 0$.

Second, $D^u_{\mathcal{T}}$ can be smaller or larger than the established notions of dimension, in particular the Rényi entropy dimensions (see defintions in Section 2.1.2).

Third, $D^u_{\mathcal{T}}$ can even exceed the dimension of the phase space. This is supported by a Cantor-like example and is probably true only for pathological cases.

Transitivity dimension is a conceptually new notion of dimension, since it is the first one that is based on geometric three-point interdependencies. This novelty has practical implications, e.g., for detecting distinct spatial structures related with supertrack functions and the unstable periodic orbits. We do not discuss this, see [Donner et al., 2011, Sec.4].

Further, transitivity dimension has computational advantage in comparison to Rényi entropy dimensions: reasonable estimates can be obtained from rather short time series, i.e., with $\mathcal{O}(10^3 \ldots 10^4)$ points, at least for low-dimensional systems.

2.1.1 Transitivity dimension for absolutely continuous ivariant measures

We have managed to analytically prove another property of the transitivity dimension, which generalizes the motivating example of the uniform distribution on an n-dimensional hypercube. For invariant measures which are absoulutely conitnuous w.r.t. Lebesgue and have Radon-Nikodym derivatives with at least one point of continuity on its support, the transitivity dimension is always equal to the phase space dimension. After the proof of the proposition, we discuss how one could infer that the same properties of the invariant measure w.r.t. a proper smooth submanifold of the phase space would yield the transitivity dimension equal to the dimension of the submanifold.

Proposition 2.1.3. *Let $\mathcal{X}$ be a dynamical system in $\mathbb{R}^n$ such that its invariant density μ is absolutely continuous w.r.t. the Lebesgue measure λ, i.e., there exists a function $f \in L^1(\mathbb{R}^n)$ such that $\mu(A) = \int_A f(x)dx$ for all measurable sets $A \in \mathbb{R}^n$. If $f \in L^\infty(\mathbb{R}^n)$ and there exists at least one point $z \in \mathbb{R}^n$ such that f is continuous in z and $f(z) > 0$, then the transitivity dimension is equal to the space dimension n.*

Proof. We prove this theorem in 3 steps. First, we show that for every integrable step function, i.e., a function

$$\sum_{i=1}^{m} c_i 1_{R_i} \tag{2.1.3}$$

for some rectangle R_i and some constants $c_i \in \mathbb{R}$, the ϵ-transitivity converges to $(3/4)^n$ as $\epsilon \to 0$. Second, we show this for integrable simple functions, i.e., functions of the form

$$\sum_{i=1}^{m} c_i 1_{A_i} \tag{2.1.4}$$

for some measurable sets A_i of finite measure and some constants $c_i \in \mathbb{R}$. Finally, we infer the convergence for arbitrary nonnegative $L^1 \cap L^\infty$ functions. In the second and the third steps, density of the corresponding function sets is used (see, e.g., [Stein and Shakarchi, RA, Th.II.2.4]). Clearly, convergence of ϵ-transitivity to $(3/4)^n$ is equivalent to $D_{\mathcal{T}} = n$.

Step I. Let

$$f = \sum_{i=1}^{m} c_i 1_{\times_{j=1}^{n}[a_i^j, b_i^j]} \tag{2.1.5}$$

with all $c_i > 0$ and all intervals $[a_i^j, b_i^j]$ disjoint for a fixed j be an integrable step function (integrability implies that all $a_i^j, b_i^j \neq \pm\infty$). Define $c = \min_i\{c_i\}$ and $C = \max_i\{c_i\}$ and choose some

$$\epsilon < \min_{\{i,j:\ a_i^j, b_i^j \in \mathbb{R}\}} \left\{ \frac{|b_i^j - a_i^j|}{2}, \frac{1}{2} \right\}. \tag{2.1.6}$$

We will now consider the "inner" part of each rectangle $\times_{j=1}^{n}[a_i^j + \epsilon, b_i^j - \epsilon]$ and the rest separately. Denoting $\mathcal{T}_d(\epsilon)$ the denominator of $\mathcal{T}(\epsilon)$, we can write

$$\begin{aligned}
\mathcal{T}_d(\epsilon) &= \int_{\mathbb{R}^n} \int_{B_\epsilon(x)} \int_{B_\epsilon(x)} \sum_{i=1}^{m} c_i 1_{\times_{j=1}^{n}[a_i^j, b_i^j]}(x) \sum_{l=1}^{m} c_l 1_{\times_{j=1}^{n}[a_l^j, b_l^j]}(y) \sum_{k=1}^{m} c_k 1_{\times_{j=1}^{n}[a_k^j, b_k^j]}(z) dz dy dx \\
&= \sum_{i=1}^{m} c_i^3 \int_{\mathbb{R}^n} \int_{B_\epsilon(x)} \int_{B_\epsilon(x)} 1_{\times_{j=1}^{n}[a_i^j+\epsilon, b_i^j-\epsilon]}(x) 1_{\times_{j=1}^{n}[a_i^j, b_i^j]}(y) 1_{\times_{j=1}^{n}[a_i^j, b_i^j]}(z) dz dy dx + R_d \\
&= \sum_{i=1}^{m} c_i^3 \prod_{j=1}^{n} \int_{-\infty}^{\infty} \int_{x_j-\epsilon}^{x_j+\epsilon} \int_{x_j-\epsilon}^{x_j+\epsilon} 1_{[a_i^j+\epsilon, b_i^j-\epsilon]}(x_j) 1_{[a_i^j, b_i^j]}(y_j) 1_{[a_i^j, b_i^j]}(z_j) dz_j dy_j dx_j + R_d \\
&= \sum_{i=1}^{m} c_i^3 \cdot (4\epsilon^2)^n \prod_{j=1}^{n} (b_i^j - a_i^j - 2\epsilon) + R_d,
\end{aligned} \tag{2.1.7}$$

where we used the fact that all intervals $[a_i^j, b_i^j]$ disjoint for a fixed j in the first step and the Fubini theorem to compute the integral. The computation of R_d is much more complicated, but we can easily find the upper and the lower bounds for it. Any area in $\times_{j=1}^{n}[a_i^j, b_i^j] \setminus \times_{j=1}^{n}[a_i^j + \epsilon, b_i^j - \epsilon]$ has length equal to ϵ in at least one and at most n directions, so we get

$$k\epsilon^{3n} \leqslant R_d \leqslant K(4\epsilon^2)^n \cdot \epsilon \tag{2.1.8}$$

for some $k, K \in \mathbb{R}$ depending on c and C respectively. Using the Landau notation, $R_d = o(\epsilon^{2n})$, i.e., $\lim_{\epsilon\to 0} R_d/\epsilon^{2n} = 0$.

Analogously, for the numerator of $\mathcal{T}(\epsilon)$,

$$\begin{aligned}
\mathcal{T}_{num}(\epsilon) &= \sum_{i=1}^{m} c_i^3 \prod_{j=1}^{n} \int_{a_i^j+\epsilon}^{b_i^j-\epsilon} dx_j \left(\int_{x_j-\epsilon}^{x_j} \int_{x_j-\epsilon}^{y_j+\epsilon} dz_j dy_j + \int_{x_j}^{x_j+\epsilon} \int_{y_j-\epsilon}^{x_j+\epsilon} dz_j dy_j \right) + R_{num} \\
&= \sum_{i=1}^{m} c_i^3 \cdot (3\epsilon^2)^n \prod_{j=1}^{n} (b_i^j - a_i^j - 2\epsilon) + R_{num}
\end{aligned} \tag{2.1.9}$$

with $R_{num} = o(\epsilon^{2n})$. It follows that

$$\mathcal{T}(\epsilon) = \frac{\sum_{i=1}^{m} c_i^3 \cdot (3\epsilon^2)^n \prod_{j=1}^{n}(b_i^j - a_i^j - 2\epsilon) + R_{num}}{\sum_{i=1}^{m} c_i^3 \cdot (4\epsilon^2)^n \prod_{j=1}^{n}(b_i^j - a_i^j - 2\epsilon) + R_d} \to \left(\frac{3}{4}\right)^n \quad \text{as } \epsilon \to 0. \tag{2.1.10}$$

Step II. Now let f be an integrable simple function, i.e.,

$$f = \sum_{i=1}^{m} c_i 1_{A_i} \tag{2.1.11}$$

with measurable sets A_i of finite measure and all $c_i > 0$. W.l.o.g. assume that A_i's are disjoint and let $C = m \cdot \max_i\{c_i\}$ [3]. Since the set of step functions is dense in the set of simple functions, there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ of step functions such that

$$||f - f_k||_1 \to 0 \text{ as } k \to \infty. \tag{2.1.12}$$

We construct $\{f_k\}$ with special properties that we will use in the proof to follow. Fix some i. As a measurable set, A_i can be approximated from outside by open sets; every open set is a countable union of almost disjoint closed rectangles, more than that, all these rectangles have measure 2^{-ln} for some $l \in \mathbb{N}$, there are finitely many rectangles of each measure and they are ordered so that the measure is non-increasing (e.g., [Stein and Shakarchi, RA, Th.I.3.4 and Th.I.1.4]). So there exist almost disjoint closed rectangles $\{D_{i,j}\}_{j\in\mathbb{N}}$ with

$$A_i \subset \bigcup_{j=1}^{\infty} D_{i,j} \tag{2.1.13}$$

and $\lambda(\bigcup_{j=1}^{\infty} D_{i,j} \backslash A_i) \leqslant \frac{1}{4mk}$. Since for every $N \in \mathbb{N}$

$$\begin{aligned}\bigcup_{j=1}^{N} D_{i,j} \Delta A_i &= \left(\bigcup_{j=1}^{N} D_{i,j} \backslash A_i\right) \cup \left(A_i \backslash \bigcup_{j=1}^{N} D_{i,j}\right) \\ &\subset \left(\bigcup_{j=1}^{N} D_{i,j} \backslash A_i\right) \cup \bigcup_{j=N+1}^{\infty} D_{i,j} \end{aligned} \tag{2.1.14}$$

and $\{D_{i,j}\}$ has the properties discussed above, there exists a number $N_{i,k} \in \mathbb{N}$ such that for $U_{i,k} = \bigcup_{j=1}^{N_{i,k}} D_{i,j}$ it holds

$$\lambda(U_{i,k} \Delta A_i) \leqslant \frac{1}{2mk}. \tag{2.1.15}$$

Denote by Z_k the set of all points, where at least two of $D_{i,j}$, $j \leqslant N_k$, intersect. Define

$$f_k = \sum_{i=1}^{m} c_i 1_{U_{i,k}}. \tag{2.1.16}$$

Clearly, $f = f_k$ a.e. on $D_k = (\bigcup_{i=1}^{m} U_{i,k}) \backslash Z_k$. Further, since A_i's are disjoint,

$$\lambda(supp(f) \Delta D_k) \leqslant \frac{1}{2k}. \tag{2.1.17}$$

In addition, w.l.o.g. (by extracting a subsequence), assume that $\int |f - f_k| \leqslant \frac{1}{k}$. Finally, since f is continuous in z and $f(z) > 0$, w.l.o.g. there exists a rectangle $\times_{j=1}^{n}[\tilde{a}_j, \tilde{b}_j]$ such that $f_k|_{\times_{j=1}^{n}[\tilde{a}_j,\tilde{b}_j]} = \tilde{c} > 0$ for all $k \in \mathbb{N}$.

[3] We will need to use this upper bound due to the choice of approximating sequence we make below.

Now, denoting by $\mathcal{T}(f,\epsilon)$ the ϵ-transitivity for the measure fdx, we notice that if the following limits exist,

$$\begin{aligned}\lim_{\epsilon\to 0}\left|\mathcal{T}(f,\epsilon)-\left(\frac{3}{4}\right)^n\right| &= \lim_{k\to\infty}\lim_{\epsilon\to 0}\left|\mathcal{T}(f,\epsilon)-\left(\frac{3}{4}\right)^n\right| \\ &\leqslant \lim_{k\to\infty}\lim_{\epsilon\to 0}\left|\mathcal{T}(f,\epsilon)-\mathcal{T}(f_k,\epsilon)\right| + \lim_{k\to\infty}\lim_{\epsilon\to 0}\left|\mathcal{T}(f_k,\epsilon)-\left(\frac{3}{4}\right)^n\right|. \end{aligned} \tag{2.1.18}$$

The last limit is zero, so in order to prove that $\mathcal{T}(f,\epsilon)\to 3/4$ as $\epsilon\to 0$, it is now enough to show that $\lim_{k\to\infty}\lim_{\epsilon\to 0}|\mathcal{T}(f,\epsilon)-\mathcal{T}(f_k,\epsilon)|=0$, which is equivalent to $|\mathcal{T}(f,\epsilon)-\mathcal{T}(f_k,\epsilon)|\to 0$ as $n\to\infty$ uniformly in ϵ.

Denote the numerator and denominator of $\mathcal{T}(f,\epsilon)$ by $\mathcal{T}_{num}(f,\epsilon)$ and $\mathcal{T}_d(f,\epsilon)$ respectively. Since for all nonnegative numbers a,b,c,d

$$\begin{aligned}|ab-cd| &\leqslant |ab-ad|+|ad-cd| \\ &\leqslant a\cdot|b-d|+d\cdot|a-c|,\end{aligned} \tag{2.1.19}$$

we have

$$\begin{aligned}|\mathcal{T}(f,\epsilon)-\mathcal{T}(f_k,\epsilon)| &= \frac{|\mathcal{T}_{num}(f,\epsilon)\mathcal{T}_d(f_k,\epsilon)-\mathcal{T}_{num}(f_k,\epsilon)\mathcal{T}_d(f,\epsilon)|}{\mathcal{T}_d(f,\epsilon)\mathcal{T}_d(f_k,\epsilon)} \\ &\leqslant \frac{\mathcal{T}_{num}(f,\epsilon)}{\mathcal{T}_d(f,\epsilon)}\cdot\frac{|\mathcal{T}_d(f_k,\epsilon)-\mathcal{T}_d(f,\epsilon)|}{\mathcal{T}_d(f_k,\epsilon)}+\frac{|\mathcal{T}_{num}(f,\epsilon)-\mathcal{T}_{num}(f_k,\epsilon)|}{\mathcal{T}_d(f_k,\epsilon)}.\end{aligned} \tag{2.1.20}$$

Clearly,

$$0\leqslant\frac{\mathcal{T}_{num}(f,\epsilon)}{\mathcal{T}_d(f,\epsilon)}\leqslant 1 \tag{2.1.21}$$

and

$$|\mathcal{T}_{num}(f,\epsilon)-\mathcal{T}_{num}(f_k,\epsilon)|\leqslant|\mathcal{T}_d(f_k,\epsilon)-\mathcal{T}_d(f,\epsilon)|, \tag{2.1.22}$$

so it remains to show that

$$\frac{|\mathcal{T}_d(f,\epsilon)-\mathcal{T}_d(f_k,\epsilon)|}{\mathcal{T}_d(f_k,\epsilon)}\to 0 \tag{2.1.23}$$

as $k\to\infty$ uniformly in ϵ.

We aim at estimating this term from above and start with the numerator. By the triangle inequality,

$$|\mathcal{T}_d(f,\epsilon)-\mathcal{T}_d(f_k,\epsilon)|\leqslant\int_{\mathbb{R}^n}\int_{B_\epsilon(x)}\int_{B_\epsilon(x)}|f(x)f(y)f(z)-f_k(x)f_k(y)f_k(z)|\,dzdydx.$$

For all nonegative a,b,c,a',b',c'

$$\begin{aligned}|abc-a'b'c'| &\leqslant |ab|\cdot|c-c'|+|ab-a'b'|\cdot|c'| \\ &\leqslant a\cdot b\cdot|c-c'|+a\cdot c'\cdot|b-b'|+b'\cdot c'\cdot|a-a'|.\end{aligned} \tag{2.1.24}$$

Since both f and f_k are bounded by C, it holds

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{B_\epsilon(x)}\int_{B_\epsilon(x)}\left(f(x)f(y)|f(z)-f_k(z)|+f(x)f_k(z)|f(y)-f_k(y)|\right)dzdydx \\ &\qquad\leqslant 2C(2\epsilon)^n\int_{\mathbb{R}^n}f(x)\int_{B_\epsilon(x)}|f(y)-f_k(y)|dydx,\end{aligned} \tag{2.1.25}$$

and, since $\int |f - f_k| \leqslant \frac{1}{k}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{B_\epsilon(x)} \int_{B_\epsilon(x)} f_k(y) f(z) |f(x) - f_k(x)| dz dy dx \\
&\leqslant C^2 (4\epsilon^2)^n \int_{\mathbb{R}^n} |f(x) - f_k(x)| dx \leqslant C^2 (4\epsilon^2)^n \frac{1}{k}.
\end{aligned} \tag{2.1.26}$$

We will resume the estimation of the numerator later.

In order to estimate the denominator $\mathcal{T}_d(f_k, \epsilon)$ from below, we use the existence of a rectangle $\times_{j=1}^n [\tilde{a}_j, \tilde{b}_j]$ such that $f_k|_{\times_{j=1}^n [\tilde{a}_j, \tilde{b}_j]} = \tilde{c} > 0$:

$$\begin{aligned}
\mathcal{T}_d(f_k, \epsilon) &\geqslant \int_{\mathbb{R}^n} \int_{B_\epsilon(x)} \int_{B_\epsilon(x)} \tilde{c}^3 1_{\times_{j=1}^n [\tilde{a}_j, \tilde{b}_j]}(x) 1_{\times_{j=1}^n [\tilde{a}_j, \tilde{b}_j]}(y) 1_{\times_{j=1}^n [\tilde{a}_j, \tilde{b}_j]}(z) dz dy dx \\
&\geqslant \tilde{c}^3 (2\epsilon^2)^n \prod_{j=1}^n (\tilde{b}_j - \tilde{a}_j - 2\epsilon) \geqslant \tilde{c}^3 (2\epsilon^2)^n \prod_{j=1}^n \frac{\tilde{b}_j - \tilde{a}_j}{2},
\end{aligned} \tag{2.1.27}$$

where the two last inequalities hold for all ϵ small enough.

We can finally estimate

$$\frac{|\mathcal{T}_d(f, \epsilon) - \mathcal{T}_d(f_k, \epsilon)|}{\mathcal{T}_d(f_k, \epsilon)} \leqslant \frac{\epsilon^n 2^{n+1} C \int_{\mathbb{R}^n} f(x) \int_{B_\epsilon(x)} |f(y) - f_k(y)| dy dx + \epsilon^{2n} 4^n C^2 \frac{1}{k}}{\epsilon^{2n} 2^n \tilde{c}^3 \prod_{j=1}^n \frac{\tilde{b}_j - \tilde{a}_j}{2}}. \tag{2.1.28}$$

Imagine that for some constant $\tilde{K}$

$$\int_{\mathbb{R}^n} f(x) \int_{B_\epsilon(x)} |f(y) - f_k(y)| dy dx \leqslant \epsilon^n \tilde{K} \frac{1}{k} \tag{2.1.29}$$

holds for all $k \in \mathbb{N}$. Then we can further estimate

$$\frac{|\mathcal{T}_d(f, \epsilon) - \mathcal{T}_d(f_k, \epsilon)|}{\mathcal{T}_d(f_k, \epsilon)} \leqslant \frac{\epsilon^{2n} 2^{n+1} C \tilde{K} \frac{1}{k} + \epsilon^{2n} 4^n C^2 \frac{1}{k}}{\epsilon^{2n} 2^n \tilde{c}^3 \prod_{j=1}^n \frac{\tilde{b}_j - \tilde{a}_j}{2}} = \frac{1}{k} \frac{2^{n+1} C \tilde{K} + 4^n C^2}{2^n \tilde{c}^3 \prod_{j=1}^n \frac{\tilde{b}_j - \tilde{a}_j}{2}}, \tag{2.1.30}$$

where the last term goes to 0 as k goes to ∞ uniformly in ϵ.

To finish this step of the proof, we show (2.1.29).

Recall that there exists a measurable set D_k with $f = f_k$ a.e. on D_k and such that $\lambda(supp(f) \Delta D_k) \leqslant \frac{1}{2k}$. Using again the fact that every measurable set can be approximated from outside by open sets and every open set is a countable union of almost disjoint closed rectangles, suppose that

$$supp(f) \Delta D_k \subset \bigcup_{l=1}^\infty D_k^l(y), \tag{2.1.31}$$

where all D_k^l are closed rectangles and $\sum_{l=1}^\infty \lambda(D_k^l) \leqslant \frac{1}{k}$. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x) \int_{B_\epsilon(x)} |f(y) - f_k(y)| dy dx &\leqslant C \int_{\mathbb{R}^n} \int_{B_\epsilon(x)} |f(y) - f_k(y)| \cdot 1_{supp(f) \Delta D_k}(y) dy dx \\
&\leqslant 2C^2 \int_{\mathbb{R}^n} \int_{B_\epsilon(x)} \sum_{l=1}^\infty 1_{D_k^l}(y) dy dx \\
&= 2C^2 \sum_{l=1}^\infty \int_{\mathbb{R}^n} \int_{B_\epsilon(x)} 1_{D_k^l}(y) dy dx,
\end{aligned} \tag{2.1.32}$$

where the last equality holds by the monotone convergence theorem.

Let $D_k^l = \times_{j=1}^n [a_j, b_j]$. Then, by the Fubini theorem,

$$\int_{\mathbb{R}^n} \int_{B_\epsilon(x)} 1_{D_k^l}(y) dy dx = \prod_{j=1}^n \int_{\mathbb{R}} \int_{x-\epsilon}^{x+\epsilon} 1_{[a_j,b_j]}(y) dy dx. \tag{2.1.33}$$

Since there can be infinitely many different D_k^l's, we need to consider 3 cases:

Case 1. $2\epsilon \leqslant |b_j - a_j|$. Exploiting symmetry,

$$\begin{aligned}\int_{\mathbb{R}} \int_{x-\epsilon}^{x+\epsilon} 1_{[a_j,b_j]} dy dx &= 2 \int_{a_j-\epsilon}^{a_j+\epsilon} \int_{a_j}^{x+\epsilon} dy dx + \int_{a_j+\epsilon}^{b_j-\epsilon} \int_{x-\epsilon}^{x+\epsilon} dy dx \\ &= (2\epsilon)^2 + (b_j - a_j - 2\epsilon) \cdot 2\epsilon = 2\epsilon(b_j - a_j). \end{aligned} \tag{2.1.34}$$

Case 2. $\epsilon \leqslant |b_j - a_j| \leqslant 2\epsilon$.

$$\begin{aligned}\int_{\mathbb{R}} \int_{x-\epsilon}^{x+\epsilon} 1_{[a_j,b_j]} dy dx &= \int_{a_j-\epsilon}^{b_j-\epsilon} \int_{a_j}^{x+\epsilon} dy dx + \int_{b_j-\epsilon}^{a_j+\epsilon} \int_{a_j}^{b_j} dy dx + \int_{a_j+\epsilon}^{b_j+\epsilon} \int_{x-\epsilon}^{b_j} dy dx \\ &= \frac{(b_j - a_j)^2}{2} + 2\epsilon(b_j - a_j) - (b_j - a_j)^2 + \frac{(b_j - a_j)^2}{2} \\ &= 2\epsilon(b_j - a_j). \end{aligned} \tag{2.1.35}$$

Case 3. $|b_j - a_j| \leqslant \epsilon$. Exploiting symmetry again,

$$\begin{aligned}\int_{\mathbb{R}} \int_{x-\epsilon}^{x+\epsilon} 1_{[a_j,b_j]} dy dx &= 2 \int_{a_j-\epsilon}^{b_j-\epsilon} \int_{a_j}^{x+\epsilon} dy dx + \int_{b_j-\epsilon}^{a_j+\epsilon} \int_{a_j}^{b_j} dy dx \\ &= (b_j - a_j)^2 + (b_j - a_j - 2\epsilon) \cdot 2\epsilon = 2\epsilon(b_j - a_j). \end{aligned} \tag{2.1.36}$$

In all cases we get the same result and it follows that

$$\int_{\mathbb{R}^n} \int_{B_\epsilon(x)} 1_{D_k^l}(y) dy dx = (2\epsilon)^n \prod_{j=1}^n (b_j - a_j) = (2\epsilon)^n \lambda(D_k^l). \tag{2.1.37}$$

Together with (2.1.32), we get (2.1.29).

Step III. Finally, consider an arbitrary nonnegative function $f \in L^1 \cap L^\infty(\mathbb{R}^n)$ with $||f||_\infty = C$. This function can be approximated in L^1 by integrable simple functions such as (2.1.11). Let us construct a specific approximating sequence $\{f_k\}_{k \in \mathbb{N}}$. First, for each $k \in \mathbb{N}$, find a number $L_k > 0$, such that

$$\int_{([-L_k, L_k]^n)^c} f \leqslant \frac{1}{k}. \tag{2.1.38}$$

Define

$$f_k = \sum_{i=1}^{\lfloor Ck \rfloor + 1} \frac{i-1}{k} 1_{A_i} \tag{2.1.39}$$

with

$$A_i = f^{-1}\left(\frac{i-1}{k}, \frac{i}{k}\right) \cap [-L_k, L_k]^n, \tag{2.1.40}$$

ensuring that $|f - f_k| \leqslant \frac{1}{k}$ a.e. on $[-L_k, L_k]^n$. Second, slightly change f_k, ensuring that there exists a rectangle $\times_{j=1}^{n}[\tilde{a}_j, \tilde{b}_j]$ such that $f_k|_{\times_{j=1}^{n}[\tilde{a}_j,\tilde{b}_j]} \geqslant \tilde{c} > 0$ for all $k \in \mathbb{N}$.

Now we can repeat the argumentation of step II up to equation (2.1.27). Further, with the assumptions on $\{f_k\}$,

$$\begin{aligned}\int_{\mathbb{R}^n} f(x) \int_{B_\epsilon(x)} |f(y) - f_k(y)| dy dx &= \int_{[-L_k,L_k]^n \cup ([-L_k,L_k]^n)^c} f(x) \int_{B_\epsilon(x)} |f(y) - f_k(y)| dy dx \\ &\leqslant (2\epsilon)^n \frac{1}{k} + 2C(2\epsilon)^n \frac{1}{k}. \end{aligned} \tag{2.1.41}$$

We estimate

$$\begin{aligned}\frac{|\mathcal{T}_d(f,\epsilon) - \mathcal{T}_d(f_k,\epsilon)|}{\mathcal{T}_d(f_k,\epsilon)} &\leqslant \frac{\epsilon^n 2^{n+1} C \int_{\mathbb{R}^n} f(x) \int_{B_\epsilon(x)} |f(y) - f_k(y)| dy dx + \epsilon^{2n} 4^n C^2 \frac{1}{k}}{\epsilon^{2n} 2^n \tilde{c}^3 \prod_{j=1}^{n} \frac{\tilde{b}_j - \tilde{a}_j}{2}} \\ &\leqslant \frac{\epsilon^n 2^{n+1} C \left((2\epsilon)^n \frac{1}{k} + 2C(2\epsilon)^n \frac{1}{k} \right) + \epsilon^{2n} 4^n C^2 \frac{1}{k}}{\epsilon^{2n} 2^n \tilde{c}^3 \prod_{j=1}^{n} \frac{\tilde{b}_j - \tilde{a}_j}{2}} \\ &= \frac{1}{k} \frac{2^{n+1} C \left(2^n + 2^{n+1} C \right) + 4^n C^2}{2^n \tilde{c}^3 \prod_{j=1}^{n} \frac{\tilde{b}_j - \tilde{a}_j}{2}}, \end{aligned} \tag{2.1.42}$$

where the last term goes to 0 as k goes to ∞ uniformly in ϵ. □

While not having the complete proof, we believe that one could further generalize this proposition in the following way. Suppose that the invariant measure is supported on a proper smooth compact submanifold of dimension $m < n$.[4] Since this submanifold has Lebesgue measure zero in $\mathbb{R}^n$ and at the same time supports the invariant measure, this measure can not be absolutely continuous w.r.t. Lebesgue by definition. However, there can exist an atlas of the submanifold such that the pullback of the invariant measure is absoulutely continuous w.r.t. Lebesgue measure on every chart, which is in that case diffeomorphic to $\mathbb{R}^m$. Note that since the submanifold is compact, the atlas is finite. Suppose that — in line with the proposition — there is a point on the submanifold such that in a chart containig it the Radon-Nikodym derivative of the invariant measure is continuous. One can sligthly change the atlas in order to ensure that every chart of the submanifold contains this point. (This should be possible by finding a path from some point of a chart to the given point and enlarging the chart by a "tube" around this path.) If on every chart the Radon-Nikodym derivative is a function from $L^1 \cap L^\infty$, the proposition ensures that for this chart the ratio of two integrals from the definition of transitivity converges to $(3/4)^m$ as $\epsilon \to \infty$. Loosely speaking, the transitivity can be represented as a ratio of sums

$$\frac{\tau_{num}^1 + \tau_{num}^2 + \cdots + \tau_{num}^l}{\tau_d^1 + \tau_d^2 + \cdots + \tau_d^l}, \tag{2.1.43}$$

where $l \in \mathbb{N}$ is the number of charts and, as discussed,

$$\frac{\tau_{num}^i}{\tau_d^i} \to \left(\frac{3}{4}\right)^m \quad \text{as} \quad \epsilon \to 0 \tag{2.1.44}$$

[4] Here we use different terminology from the differential geometry. Please, refer to [Lee].

for every $i \in \{1, \dots, l\}$. Since all estimations in the proof of the proposition use polynomials in ϵ, the ratio (2.1.43) also coverges to $(3/4)^m$.

In order to prove this generalization, one has to carefully elaborate every step, especially the pullback of the measure and the splitting of the integral (using partition of unity), paying attention to the change of metric (and thus ϵ) in the differentiable chart. The generalization is useful since attractors are often submanifolds of the phase space, to describe which one needs less than n coordinates (to avoid the word "dimension").

Of course, these considerations do not suggest that the transitivity dimension is always an integer. Many attractors have fractal geometry: in this case, the corresponding invariant measure is not absolutely continuous w.r.t. Lebesgue, thus allowing the transitivity dimension to be non-integer. E.g., the attractor of the generalized baker's map (equation (4.1.10) with coefficients which may differ from 2 and 1/2) is the Cartesian product of the Cantor set and the $[0, 1]$ interval (cf. [Farmer et al., Sec.4.3]). As above, since the Cantor set has measure zero, this density can not be absolutely continuous w.r.t. Lebesgue by definition. In [Donner et al., 2011, Sec.3.2.2] it is stated that the upper transitivity dimension for the generalized baker's map is approximately 1.58.

It would be interesting to find an example of an attractor, whose invariant measure is absolutely continuous w.r.t. Lebesgue, but with a Radon-Nikodym derivative which is discontinuous in every point where it is non-zero. One should search among fat fractals - fractal sets with positive Lebesgue measure. At the same time, this fat fractal should be the support of the invariant measure of some dynamical system. Unfortunately, we could not find an appropriate example. If there is one, the proposition above would be sharp in the sense that it would clearly classify the attractors with respect to the property of having an integer transitivity dimension.

The statement of this proposition supports the idea that the definition of the transitivity dimension is meaningful. If the density f corresponding to the invariant measure of the attractor has a point of continuity, the attractor contains at least a small set which "continuously stretches in all directions of the phase space" (to avoid the word "dimension" in a different way). Thus any reasonable definition of dimension should assign to this set a number equal to the phase space dimension. However, it is not obvious that a global dimension, i.e., a dimension that takes into account the whole attractor and not only a part of it, will have this property. We see that the transitivity dimension has it. In the following subsection we prove that the same holds for a large class of established dimensions, called the Rényi entropy dimensions.

2.1.2 Rényi entropy dimensions for absolutely continuous ivariant measures

In 1983, [Grassberger] defined a family of dimensions based on Rényi entropies (see [Rényi, entropy]), which are now known as the Rényi entropy dimensions and are widely used in the modern literature for the characterization of attractors.

Definition 2.1.4. *Let $\mathcal{X}$ be a dynamical system in $\mathbb{R}^n$ and μ the corresponding invariant density. Let $\epsilon > 0$ and $\{B^i_\epsilon\}_{i \in \mathbb{Z}}$ be the set of boxes defined on the ϵ-coordinate mesh with* *elements*

$$[\epsilon k_1, \epsilon k_1 + \epsilon) \times \cdots \times [\epsilon k_n, \epsilon k_n + \epsilon) \tag{2.1.45}$$

for some $k_1, \ldots, k_n \in \mathbb{Z}$. *Let*

$$J = \{j \in \mathbb{Z} | \mu(B_\epsilon^j) > 0\}. \tag{2.1.46}$$

For every $q \in \{0\} \cup \mathbb{R}^+ \backslash \{1\}$, *if the limit*

$$\hat{D}_q = \lim_{\epsilon \to 0} \frac{1}{q-1} \frac{\log \sum_{j \in J} \mu(B_\epsilon^j)^q}{\log \epsilon} \tag{2.1.47}$$

exists, it is called the **Rényi entropy dimension of $\mathcal{X}$ of order** q.

If the limit

$$\hat{D}_1 = \lim_{\epsilon \to 0} \lim_{q \searrow 1} \frac{1}{q-1} \frac{\log \sum_{j \in J} \mu(B_\epsilon^j)^q}{\log \epsilon} \tag{2.1.48}$$

exists, it is called the **Rényi entropy dimension of $\mathcal{X}$ of order** 1.

The most widely used Rényi entropy dimensions are those of the orders 0, 1 and 2. For order 0, we have the box-counting dimension already discussed in the introduction. This is the only Rényi entropy dimension which does not depend on the invariant measure and reflects only the geometrical form of the attractor.

For order 1, it is shown in [Rényi, entropy] that, equivalently,

$$\hat{D}_1 = \lim_{\epsilon \to 0} \frac{\sum_{j \in J} \mu(B_\epsilon^j) \log \mu(B_\epsilon^j)}{\log \epsilon}, \tag{2.1.49}$$

which is well-known in the literature under the name **information dimension**.

The order 2 Rényi entropy dimension is the **correlation dimension** popularized by [Grassberger and Procaccia]. It is often used in chaos theory due to the considerable easiness of estimation. The correlation dimension is especially useful for characterizing data from very high dimensional systems (see [Grassberger and Procaccia, Sec.5 and 6]).

Clearly, the given definitions of the Rényi entropy dimensions use the invariant measure, but are written with sums and not integrals. In order to make statements about these dimensions for the invariant measures absolutely continuous w.r.t. Lebesgue, it is better to have definitions with integrals. The one for the correlation dimension can be found in [Grassberger and Procaccia, Sec.2]. For the case of absolutely continuous w.r.t. Lebesgue measure μ with $d\mu(x) = f(x)dx$, we have

$$D_2 = \lim_{\epsilon \to 0} \frac{\log \int_{\mathbb{R}^n} \int_{B_\epsilon(x)} f(x) f(y) dy dx}{\log \epsilon}. \tag{2.1.50}$$

Analogously, we are able to define

$$D_q = \lim_{\epsilon \to 0} \frac{1}{q-1} \frac{\log \int_{\mathbb{R}^n} f(x) \left(\int_{B_\epsilon(x)} f(y) dy \right)^{q-1} dx}{\log \epsilon} \tag{2.1.51}$$

for $q > 1$ and

$$D_1 = \lim_{\epsilon \to 0} \lim_{q \searrow 1} \frac{1}{q-1} \frac{\log \int_{\mathbb{R}^n} f(x) \left(\int_{B_\epsilon(x)} f(y) dy \right)^{q-1} dx}{\log \epsilon}. \tag{2.1.52}$$

This does not work for $q < 1$, since $\int_{B_\epsilon(x)} f(y)dy$ can be equal to zero.

Note that although we conjecture that dimensions defined with sums and with integrals are equal we denote them with different letters, since for now we cannot give proofs that they are exactly the same (and did not find them in the literature). Moreover, we see the definitions with sums as definitions of the dimension estimators (by analogy with the transitivity dimension estimator), since they are the ones applicable for numerical computations.

Now we are ready to prove the result similar to Proposition 2.1.3.

Proposition 2.1.5. *Let $\mathcal{X}$ be a dynamical system in $\mathbb{R}^n$ such that its invariant density μ is absolutely continuous w.r.t. the Lebesgue measure λ, i.e., there exists a function $f \in L^1(\mathbb{R}^n)$ such that $\mu(A) = \int_A f(x)dx$ for all measurable sets $A \in \mathbb{R}^n$. If $f \in L^\infty(\mathbb{R}^n)$ and there exists at least one point $z \in \mathbb{R}^n$ such that f is continuous in z and $f(z) > 0$, then the Rényi entropy dimension of any order $q \geqslant 1$ is equal to the space dimension n.*

The dimensions of order less than 1 are not considered here, since we do not have expressions for them that use invariant measures (and the interesing box-counting dimension does not depend on the invariant measure).

Proof. Let $f \in L^1 \cap L^\infty(\mathbb{R}^n)$ be a density with $||f||_\infty = M$. Since f is continuous in z and $f(z) > 0$, there exists a rectangle $\times_{j=1}^n [\tilde{a}_j, \tilde{b}_j]$ around z such that $f \geqslant \tilde{c} > 0$ a.e. on it and $\tilde{c}^2 \prod_{j=1}^n (\tilde{b}_j - \tilde{a}_j) < 1$.

We can estimate the integral in the definition of the dimension from both sides. From above,

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) \left(\int_{B_\epsilon(x)} f(y)dy \right)^{q-1} dx &\leqslant \int_{\mathbb{R}^n} (2\epsilon)^{n(q-1)} M^{(q-1)} f(x)dx \\ &= (2\epsilon)^{n(q-1)} M^{q-1}. \end{aligned} \tag{2.1.53}$$

From below,

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) \left(\int_{B_\epsilon(x)} f(y)dy \right)^{q-1} dx &\geqslant \int_{\mathbb{R}^n} \tilde{c} 1_{\times_{j=1}^n [\tilde{a}_j, \tilde{b}_j]}(x) \left(\int_{B_\epsilon(x)} \tilde{c} 1_{\times_{j=1}^n [\tilde{a}_j, \tilde{b}_j]}(y)dy \right)^{q-1} dx \\ &\geqslant \int_{\mathbb{R}^n} \tilde{c} 1_{\times_{j=1}^n [\tilde{a}_j+\epsilon, \tilde{b}_j-\epsilon]}(x) \left(\int_{B_\epsilon(x)} \tilde{c} 1_{\times_{j=1}^n [\tilde{a}_j, \tilde{b}_j]}(y)dy \right)^{q-1} dx \\ &\geqslant \tilde{c}^2 (2\epsilon)^{n(q-1)} \prod_{j=1}^n (\tilde{b}_j - \tilde{a}_j - 2\epsilon), \end{aligned} \tag{2.1.54}$$

in case

$$\epsilon \leqslant \min_{j=1,\ldots,n} \left\{ \frac{\tilde{b}_j - \tilde{a}_j}{2} \right\}. \tag{2.1.55}$$

For $q > 1$, it follows that

$$\begin{aligned} D_q &\leqslant \lim_{\epsilon \to 0} \frac{1}{q-1} \frac{\log[(2\epsilon)^{n(q-1)} M^{(q-1)}]}{\log \epsilon} \\ &= \lim_{\epsilon \to 0} \frac{1}{q-1} \left[\frac{n(q-1)\log \epsilon}{\log \epsilon} + \frac{\log(2^{n(q-1)} M^{(q-1)})}{\log \epsilon} \right] = n. \end{aligned} \tag{2.1.56}$$

On the other hand,

$$\begin{aligned} D_q &\geqslant \lim_{\epsilon\to 0}\frac{1}{q-1}\frac{\log\left[\tilde{c}^2(2\epsilon)^{n(q-1)}\prod_{j=1}^n(\tilde{b}_j-\tilde{a}_j-2\epsilon)\right]}{\log\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{1}{q-1}\left[\frac{n(q-1)\log\epsilon}{\log\epsilon}+\frac{\log\left(\tilde{c}^2 2^{n(q-1)}\prod_{j=1}^n(\tilde{b}_j-\tilde{a}_j-2\epsilon)\right)}{\log\epsilon}\right]=n, \end{aligned} \tag{2.1.57}$$

so $D_q = n$ for all $q \geqslant 2$.

For $q = 1$,

$$\begin{aligned} D_1 &\leqslant \lim_{\epsilon\to 0}\lim_{q\searrow 1}\frac{1}{q-1}\frac{\log\left((2\epsilon)^{n(q-1)}M^{(q-1)}\right)}{\log\epsilon} \\ &= \lim_{\epsilon\to 0}\lim_{q\searrow 1}\left[\frac{n\cdot\log\epsilon}{\log\epsilon}+\frac{\log(2^nM)}{\log\epsilon}\right]=n \end{aligned} \tag{2.1.58}$$

and

$$\begin{aligned} D_1 &\geqslant \lim_{\epsilon\to 0}\lim_{q\searrow 1}\frac{1}{q-1}\left[\frac{n(q-1)\log\epsilon}{\log\epsilon}+\frac{\log\left(\tilde{c}^2 2^{n(q-1)}\prod_{j=1}^n(\tilde{b}_j-\tilde{a}_j-2\epsilon)\right)}{\log\epsilon}\right] \\ &\geqslant \lim_{\epsilon\to 0}\lim_{q\searrow 1}\frac{1}{q-1}\frac{n(q-1)\log\epsilon}{\log\epsilon}=n, \end{aligned} \tag{2.1.59}$$

where the second inequality holds for ϵ and $q-1$ small enough, since $\tilde{c}^2\prod_{j=1}^n(\tilde{b}_j-\tilde{a}_j) < 1$. It follows that $D_1 = n$.

□

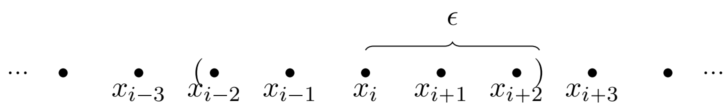
2.2 Convergence of the transitivity dimension estimator for data from attractors with absolutely continuous invariant measures

Now we found out what should be the theoretical value of the transitivity dimension for sufficiently nice invariant measures. In practice, one often does not know the invariant measure and is interested in the dimension estimator, which can be computed using a finite number of observations. In this section we study how the transitivity dimension estimator (2.0.2)

$$\hat{\mathcal{T}}(\epsilon)=\frac{\sum_{i,j,k=1}^N A_{ki}(\epsilon)A_{kj}(\epsilon)A_{ij}(\epsilon)}{\sum_{i,j,k=1,\ i\neq j}^N A_{ki}(\epsilon)A_{kj}(\epsilon)}$$

converges to its theoretical value. We consider the simple example of regular n-dimensional grids as representative case for absoulutely continuous w.r.t. Lebesgue invariant measures in n-dimensional phase spaces. Since the transitivity estimator and not the dimension is considered, we expect the values $(3/4)^n$ and not n. The main interests here are the bias and the rate of convergence.

Limit cycle (1 dimension). Suppose that the trajectory of the system is some closed curve with curvature that allows us to consider it locally as a straight line. Suppose that locally the observables look like the dots in the figure below with equal distances between neighbours, forming a regular 1-dimensional grid.



Fix ϵ so that there are $2k + 1$ points in an ϵ-ball with center in any observable and fix x_i as this center (in the figure, an ϵ corresponding to $k = 2$ is shown). First, consider the denominator of the transitivity, i.e., the number of triples with distancies shorter than ϵ. There are $2k\cdot(2k-1)$ different triples with x_i as the first vertex. In sum, there are $N(4k^2-2k)$ triples.

In order to compute the numerator, i.e., the number of triangles with sides shorter than ϵ, first rename the states so that the ϵ-ball around x_i contains points $x_{i-k}, x_{i-k+1}, \ldots, x_{i+k}$ in this order. Second, suppose that we count triangles with 2 or even 3 equal vertices (we will than substract additional triangles and apply this strategy, since it helps in further cases of higher dimensions). Consider x_{i-k}. There are $k + 1$ triangles inside the ϵ-ball with center in x_i, having x_i as the first and x_{i-k} as the second vertex, and the same result holds for x_{i+k}. For each x_{i-k+1} and x_{i+k-1} there are $k + 2$ corresponding triangles etc. x_i is the only point with $2k + 1$ corresponding triangles. Thus, we have

$$2[(k+1) + (k+1) + \cdots + 2k] + 2k + 1 = 3k^2 + 3k + 1 \tag{2.2.1}$$

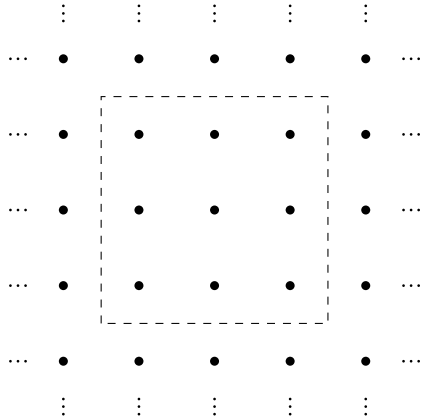
triangles with x_i as the first vertex. How many of them are in fact "not allowed"? First, these are $2k + 1$ triangles, where the second and the third vertices are the same. Second, these are $2 \cdot 2k$ triangles where x_i, which is the first vertex, is the second or the third one as well. It follows that there are $N(3k^2 + 3k + 1 - (2k + 1) - 4k)$ triangles and we get

$$\hat{\mathcal{T}}_1(k) = \frac{3k^2 - 3k}{4k^2 - 2k} = \frac{3}{4} - \frac{3k}{8k^2 - 4k}. \tag{2.2.2}$$

As expected, $\hat{\mathcal{T}}_1(k) \to \frac{3}{4}$ as $k \to \infty$. We see that the estimator converges from below and the rate of convergence is proportional to k^{-1}.

Here we express the convergence in terms of the changing k. Clearly, it is more natural to ask about convergence with the increase in the number of observations. In this simplified setting both ways are the same, since we may assume that, e.g., a doubling of the number of observations (uniformly distibuted on the attractor) doubles k in case that ϵ remains fixed.

Limit torus (2 dimensions). Suppose that the trajectory of the system is some closed surface with curvature that allows us to consider it locally as a plane (e.g., a torus). Suppose that locally the observables look like the dots in figure below with equal distances between neighbours, forming a regular two-dimensional grid.



Fix ϵ so that there are $(2k+1)^2$ points in an ϵ-ball with center in any observable and fix x_i as this center (recall that we use the supremum norm, the dashed square in the figure above show a ball for some ϵ corresponding to $k=1$). Clearly, there are $((2k+1)^2-1)\cdot((2k+1)^2-2)$ triples inside this ball.

In order to compute the number of triangles, we consider two dimensions of the surface separately. Again, count first all triangles, including those with 2 or 3 same vertices. By the computation for limit cycle, there are $3k^2+3k+1$ pairs of horizontal lines with the distance in y-direction less than ϵ and $3k^2+3k+1$ pairs of vertical lines with the distance in x-direction less than ϵ. Since all pairs are ordered, there are as many triangles as intersections of the lines (the second vertex is the intersection of the first horizontal line in its pair with the first vertical line in its pair and the third vertex is the intersection of the second lines), i.e., $(3k^2+3k+1)^2$ triangles. How many of them are in fact "not allowed"? First, these are $(2k+1)^2$ triangles, where the second and the third vertices are the same. Second, these are $2\left((2k+1)^2-1\right)$ triangles where x_i, which is the first vertex, is the second or the third one as well. We get

$$(3k^2+3k+1)^2-(2k+1)^2-2\left((2k+1)^2-1\right)=9k^4+18k^3+3k^2-6k \tag{2.2.3}$$

triangles inside the ball and

$$\begin{aligned}\hat{\mathcal{T}}_2(k) &= \frac{9k^4+18k^3+3k^2-6k}{((2k+1)^2-1)((2k+1)^2-2)} = \frac{9k^4+18k^3+3k^2-6k}{16k^4+32k^3+12k^2-4k} \\ &= \left(\frac{3}{4}\right)^2 - \frac{15}{16(4k^2+4k-1)}\end{aligned} \tag{2.2.4}$$

Again, as expected, $\hat{\mathcal{T}}_2(k) \to \left(\frac{3}{4}\right)^2$ as $k\to\infty$. The estimator converges from below and the rate of convergence is proportional to k^{-2}.

***n*-dimensional grid.** We are now ready to generalize the computations above and consider an n-dimensional grid with equal distancies between neighbours.

Consider ϵ such that there are $(2k+1)^n$ points in an ϵ-ball with center in some fixed x_i. There are $((2k+1)^n-1)\cdot((2k+1)^n-2)$ triples inside this ball.

For the triangles, we again consider dimensions of the grid separately and count first all triangles, including those with 2 or 3 same vertices. With the same argumentation as in the limit torus case, we get $(3k^2+3k+1)^n$ triangles. Out of them $(2k+1)^n$ have equal second and third vertices, and $2\left((2k+1)^n-1\right)$ have x_i not only as the first, but also as the second or the third vertex. This means that there are

$$(3k^2+3k+1)^n-3(2k+1)^n+2 \tag{2.2.5}$$

triangles inside the ϵ-ball with center in x_i and

$$\hat{\mathcal{T}}_n(k)=\frac{(3k^2+3k+1)^n-3(2k+1)^n+2}{((2k+1)^n-1)((2k+1)^n-2)}. \tag{2.2.6}$$

Clearly, $\hat{\mathcal{T}}_n(k)\to\left(\frac{3}{4}\right)^n$ as $k\to\infty$. In order to determine the rate of convergence, we need further computations. Consider $n\geqslant 3$. Computing polynomials in the fraction, we ignore all powers less than $2n-2$:

$$\hat{\mathcal{T}}_n(k)=\frac{(3k^2+3k)^n+n(3k^2)^{n-1}+P_1}{(4k^2+4k)^n+n(4k^2)^{n-1}+P_2}, \tag{2.2.7}$$

where P_1 and P_2 are some polynomials in k with the highest power $2n-3$. Expanding, we get

$$\hat{\mathcal{T}}_n(k)=\frac{3^nk^{2n}+3^nnk^{2n-1}+\left(3^{n-1}n+3^n\frac{n(n-1)}{2}\right)k^{2n-2}+Q_1+P_1}{4^nk^{2n}+4^nnk^{2n-1}+\left(4^{n-1}n+4^n\frac{n(n-1)}{2}\right)k^{2n-2}+Q_2+P_2}, \tag{2.2.8}$$

where Q_1 and Q_2 are again some polynomials in k with the highest power $2n-3$. Further,

$$\hat{\mathcal{T}}_n(k)=\left(\frac{3}{4}\right)^n+\frac{3^n\frac{n}{12}k^{2n-2}-\frac{3^n}{4^n}(Q_2+P_2)+Q_1+P_1}{4^nk^{2n}+4^nnk^{2n-1}+\left(4^{n-1}n+4^n\frac{n(n-1)}{2}\right)k^{2n-2}+Q_2+P_2}. \tag{2.2.9}$$

It follows that for $n\geqslant 3$ the estimator converges from above and the rate of convergence is proportional to k^{-2}.

Note that the bias of the transitivity estimator is different for low- (1 and 2) and high-dimensional systems and that the rate of convergence is k^{-2} for all $n\geqslant 2$ and not k^{-n} as one could have conjectured.

2.3 The impact of noise on dimensions

In this section we first come back to the original question of infering the coupling direction in two systems from numerical data. Suppose that the systems $\mathcal{X}$ and $\mathcal{Y}$ are coupled via diffusive coupling so that the system $\mathcal{Y}$ is the driver and consider the discrete time case:

$$\begin{cases} x_{t+1}=f(x_t)+k(y_t-x_t)=f(x_t)-kx_t+ky_t \\ y_{t+1}=g(y_t) \end{cases} \tag{2.3.1}$$

for some functions f,g and $k>0$.

Suppose that k is very small, i.e., the systems are weakly coupled. In this case one reasonable approach to study the effects of coupling on system $\mathcal{X}$ is to approximate the influence of system $\mathcal{Y}$ — the term ky_t — by stochastic noise ξ_t, leading to the equation

$$x_{t+1}=f(x_t)-kx_t+\xi_t=\tilde{f}(x_t)+\xi_t. \tag{2.3.2}$$

Of course, in general the most approprate noise term ξ_t would represent some complicated autocorrelated stochastic process. However, the theory of stochastic processes is a complex domain of mathematics (see, e.g., [Gikhman and Skorokhod]). Hence, in the limited space of this thesis, we will only consider the simplest case of uncorrelated Gaussian noise added to a discrete time system with a fixed point as the attractor. Even here interesting results come out.

To quantify the effect of noise, we will consider the difference in dimension of the attractor in cases with and without noise. We will speak of "dimension" in general - with the results of Section 2.1 one can substitute the word for the transitivity or any of the Rényi entropy dimensions of order $q \geqslant 1$. This is because the invariant measure induced by the Gaussian noise is clearly absolutely continuous w.r.t. Lebesgue and fulfills other requirements of the propositions 2.1.3 and 2.1.5.

So, consider two systems governed by the same deterministic dynamics, where one is disturbed by an additive noise:

$$\begin{cases} y_{t+1} = \tilde{f}(y_t) \\ x_{t+1} = \tilde{f}(x_t) + \xi_{t+1}, \end{cases} \tag{2.3.3}$$

where $\{\xi_t\}_{t\in\mathbb{N}}$ is a sequence of independent identically distributed random variables.

Let $\{\xi_t\}$ represent the Gaussian white noise, i.e., every ξ_t is normally distributed with zero mean and covariance matrix Σ (in the 1-dimensional case, $\Sigma = \sigma^2$):

$$\xi_t \sim \mathcal{N}(0, \Sigma) \quad \text{for all} \quad t \in \mathbb{N}, \tag{2.3.4}$$

whereas $\{\xi_t\}$ are mutually independent. Later we will assume that Σ is diagonal. "Pathological" cases, when some of the diagonal elements are 0, will be considered to represent cases where the actual coupling term $y - x$ lies in a proper submanifold of the phase space.

2.3.1 Fixed point in $\mathbb{R}$, Gaussian white noise

Fixed point in $\mathbb{R}$. Let $n = 1$ and $\tilde{f}(y) \equiv 0$. Clearly, the attractor of the system $\mathcal{Y}$ is just a one-point set, w.l.o.g. $\{0\}$, which has dimension 0. For every t, $x_t = \xi_t$, so $x_t \sim \mathcal{N}(0, \sigma^2)$, hence the invariant density is simply the Gaussian density of the noise and the dimension of the system $\mathcal{X}$ is 1. In other words, the noise increases the dimension of the system up to the space dimension.

Convergence to a fixed point in $\mathbb{R}$. Let $n = 1$, $a > 1$ and $\tilde{f}(y) = y/a$. We have

$$\lim_{t\to\infty} y_t = \lim_{t\to\infty} \frac{y_0}{a^t} = 0, \tag{2.3.5}$$

so $\mathcal{Y}$ has again 0-dimensional attractor $\{0\}$. On the other hand, one can show by induction that

$$x_t = \frac{x_0}{a^t} + \sum_{i=1}^{t} \frac{\xi_i}{a^{t-i}}, \tag{2.3.6}$$

so

$$x_\infty = \lim_{t\to\infty} x_t = \sum_{t=1}^{\infty} \frac{\xi_t}{a^{t-1}}, \tag{2.3.7}$$

where we inverted the sum, using the fact that ξ_t's are independent and identically distributed. Clearly, the last expression makes sense, if the sum converges, which it does:

The normal distribution is invariant under convolution and for every $n \in \mathbb{N}$

$$\sum_{t=1}^{n} \frac{\xi_t}{a^{t-1}} \sim \mathcal{N}\left(0, \sum_{t=1}^{n} \frac{\sigma^2}{a^{2(t-1)}}\right) \tag{2.3.8}$$

More than that, we can compute the limit of the characteristic function

$$\lim_{n\to\infty} \varphi_{\sum_{t=1}^{n} \frac{\xi_t}{a^{t-1}}}(\lambda) = \lim_{n\to\infty} \exp\left(-\frac{1}{2}\sum_{t=1}^{n} \frac{\sigma^2}{a^{2(t-1)}}\lambda^2\right) = \exp\left(-\frac{1}{2}\cdot\frac{a^2}{a^2-1}\sigma^2\lambda^2\right), \tag{2.3.9}$$

which itself turns out to be the characteristic function of a normally distributed random variable. It follows that

$$x_\infty = \sum_{t=1}^{\infty} \frac{\xi_t}{a^{t-1}} \sim \mathcal{N}\left(0, \frac{a^2}{a^2-1}\sigma^2\right), \tag{2.3.10}$$

so the invariant density is again Gaussian, but with a larger variance than that of the noise, and the dimension of the system $\mathcal{X}$ in its dynamical equilibrium is 1. Again, noise increases the dimension of the system up to the space dimension.

In fact, the result on dimension does not depend crucially on the assumption of the normal distribution. For example, since $\{\xi_t\}$ are independent, any symmetric distibution having whole $\mathbb{R}$ as domain will result in dimension 1 for the system $\mathcal{X}$.

2.3.2 Fixed point in $\mathbb{R}^n$, Gaussian white noise

Now let n be an arbitrary integer, $A \in \mathcal{M}_{n\times n}$ a square matrix with all eigenvalues smaller than 1 in absolute value and $\tilde{f}(y) = Ay$.

For any square matrix there exists a Jordan matrix $J \in \mathcal{M}_{n\times n}$ and an invertible matrix $P \in \mathcal{M}_{n\times n}$ such that $A = PJP^{-1}$. J has the eigenvalues of A on its principal diagonal and some 1's on the first diagonal (above the principal one), all other elements are zero. It is easy to show that, if the eigenvalues are smaller than 1 in absoulte value, then $J^n \to 0$ as $n \to \infty$. It follows that

$$y_t = A^t y_0 = PJ^tP^{-1}y_0 \to 0 \ \text{ as } \ t \to \infty, \tag{2.3.11}$$

so the attractor of $\mathcal{Y}$ is $\{0\}$ and has dimension 0.

For the system $\mathcal{X}$ we have

$$x_t = A^t x_0 + \sum_{i=1}^{t} A^{t-1}\xi_t, \tag{2.3.12}$$

and, inverting the sum,

$$x_\infty = \lim_{t\to\infty} x_t = \lim_{n\to\infty} \sum_{t=1}^{n} A^{t-1}\xi_t. \tag{2.3.13}$$

For $\xi_t \sim \mathcal{N}(0,\Sigma)$,

$$A^{t-1}\xi_t \sim \mathcal{N}(0, A^{t-1}\Sigma(A^{t-1})^T) \tag{2.3.14}$$

and, by continuity of the exponential function and the scalar product,

$$\begin{aligned}\lim_{n\to\infty}\varphi_{\sum_{t=1}^{n}A^{t-1}\xi_t}(\lambda) &= \lim_{n\to\infty}\exp\left(-\frac{1}{2}\left\langle\lambda,\sum_{t=1}^{n}A^{t-1}\Sigma(A^{t-1})^T\right\rangle\right)\\ &= \exp\left(-\frac{1}{2}\left\langle\lambda,\sum_{t=1}^{\infty}A^{t-1}\Sigma(A^{t-1})^T\right\rangle\right), \end{aligned}\tag{2.3.15}$$

if the sum converges, which it in fact does (see the proof after equation (2.3.31), which is there conducted for real λ_i's but works for the complex ones exactly in the same way). It follows that

$$x_\infty = \sum_{t=1}^{\infty}A^{t-1}\xi_t \sim \mathcal{N}\left(0,\sum_{t=1}^{\infty}A^{t-1}\Sigma(A^{t-1})^T\right), \tag{2.3.16}$$

so, once again, the invariant density is Gaussian.

In order to determine the dimension of x_∞, we again use the Jordan matrix, this time the *real* Jordan normal form. It is known (see, e.g., [Handbook of LA, Ch.6.3]) that any square matrix can be represented in its *real* Jordan normal form, i.e., there exists an invertible matrix $P\in\mathcal{M}_{n\times n}$ and a real Jordan matrix $J\in\mathcal{M}_{n\times n}$ with $A=PJP^{-1}$. The *real* Jordan matrix is a block diagonal matrix having real Jordan blocks as the diagonal blocks and all other elements equal to zero. A Jordan block can have two possible forms. For a real eigenvalue λ_i of A with the algebraic multiplicity k_i, the *real* Jordan block is a $k_i\times k_i$ matrix of the form

$$J_{\lambda_i} = \begin{pmatrix}\lambda_i & 1 & 0 & \cdots & 0\\ 0 & \lambda_i & 1 & & 0\\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & \lambda_i & 1\\ 0 & \cdots & 0 & 0 & \lambda_i\end{pmatrix}. \tag{2.3.17}$$

For a complex eigenvalue $\lambda_j=\alpha_j+\mathrm{i}\beta_j$ ($\alpha_i,\beta_i\in\mathbb{R}$, $\beta_i\neq 0$)[5] with the algebraic multiplicity k_j, the real Jordan block is a $2k_j\times 2k_j$ matrix of the form

$$J_{\lambda_j} = \begin{pmatrix}S_{\lambda_j} & I_2 & 0_2 & \cdots & 0_2\\ 0_2 & S_{\lambda_j} & I_2 & & 0_2\\ \vdots & \ddots & \ddots & \ddots & \\ 0_2 & \cdots & 0_2 & S_{\lambda_j} & I_2\\ 0_2 & \cdots & 0_2 & 0_2 & S_{\lambda_j}\end{pmatrix}, \tag{2.3.18}$$

where

$$S_{\lambda_j} = \begin{pmatrix}\alpha_j & \beta_j\\ -\beta_j & \alpha_j\end{pmatrix},\ I_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ \text{ and }\ 0_2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}. \tag{2.3.19}$$

Now we are ready to examine how different covariance matrices Σ influence the dimension of $\mathcal{X}$. Since the invertible matrix P can be seen as a basis transformation, we also assume w.l.o.g. that A itself is a real Jordan matrix. For simplicity, we assume that different components of ξ_t are independent, so Σ is a diagonal matrix. We can thus consider the influence of noise block-by-block. We will see how the structure of the covariance matrix of the noise

[5] i is here the imaginary unit, to distinguish it from the index i

affects the dimension of the system $\mathcal{X}$. The overall effect on dimension will be the sum of effects stemming from each block.

Case 1.1. λ_i is a real eigenvalue of multiplicity 1 and the corresponding block is $A_i = J_i = (\lambda_i)$. Then

$$\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T = \frac{1}{1-\lambda_i^2}\sigma_i^2, \tag{2.3.20}$$

where σ_i^2 is the corresponding element of the covariance matrix of ξ_t. In this case, the block contributes a value of 1 to the dimension if and only if $\sigma_i^2 \neq 0$, i.e., there exists non-zero noise in the corresponding direction.

Case 1.2. λ_i is a real eigenvalue of multiplicity 2. We have

$$A_i = \begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}, \quad A_i^{t-1} = \begin{pmatrix} \lambda_i^{t-1} & (t-1)\lambda_i^{t-2} \\ 0 & \lambda_i^{t-1} \end{pmatrix} \tag{2.3.21}$$

and

$$\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T = \sum_{t=1}^{\infty} \begin{pmatrix} \lambda_i^{t-1} & (t-1)\lambda_i^{t-2} \\ 0 & \lambda_i^{t-1} \end{pmatrix} \begin{pmatrix} \sigma_{i_1}^2 & 0 \\ 0 & \sigma_{i_2}^2 \end{pmatrix} \begin{pmatrix} \lambda_i^{t-1} & 0 \\ (t-1)\lambda_i^{t-2} & \lambda_i^{t-1} \end{pmatrix} \tag{2.3.22}$$

$$= \sum_{t=1}^{\infty} \begin{pmatrix} \lambda_i^{2(t-1)}\sigma_{i_1}^2 + (t-1)^2\lambda_i^{2(t-2)}\sigma_{i_2}^2 & (t-1)\lambda_i^{2t-3}\sigma_{i_2}^2 \\ (t-1)\lambda_i^{2t-3}\sigma_{i_2}^2 & \lambda_i^{2(t-1)}\sigma_{i_2}^2 \end{pmatrix}. \tag{2.3.23}$$

Here it is easy to see that for $|\lambda_i| < 1$ the sum converges, but we are not interested in the exact value. All elements on the principal diagonal are nonnegative, so the principal diagonal of the sum is also nonnegative and for $\lambda_i \neq 0$ we get a matrix of the following form:

$$\begin{pmatrix} a_1\sigma_{i_1}^2 + a_2\sigma_{i_2}^2 & b\sigma_{i_2}^2 \\ b\sigma_{i_2}^2 & a_1\sigma_{i_2}^2 \end{pmatrix} \tag{2.3.24}$$

with $a_1, a_2 > 0$. For the dimension of $\mathcal{X}$ this means that

if $\sigma_{i_1}^2 = \sigma_{i_2}^2 = 0$, then the block does not contribute to the dimension;
if $\sigma_{i_1}^2 \neq 0$, $\sigma_{i_2}^2 = 0$, then the block contributes a value of 1 to the dimension;
if $\sigma_{i_2}^2 \neq 0$, then the block contributes a value of 2 to the dimension.

The latter holds only if the matrix (2.3.24) has full rank, which is indeed the case. If both $\sigma_{i_1}^2$, $\sigma_{i_2}^2$ are positive, then each matrix in (2.3.22) has positive determinant, so the matrix in (2.3.23) has positive determinant. Moreover, it is clearly symmetric and thus Hermitian and positive semidefinite as a product of the form $U^T U$ (see, e.g., [Devos, Prop.1.1]) with

$$U^T = A_i^{t-1}\begin{pmatrix} |\sigma_{i_1}| & 0 \\ 0 & |\sigma_{i_2}| \end{pmatrix}. \tag{2.3.25}$$

For positive semidefinite Hermitian matrices $A \in \mathcal{M}_{n\times n}$ and $B \in \mathcal{M}_{n\times n}$, the Minkowski determinant theorem ([Minkowski]) states that

$$(\det(A+B))^{1/n} \geqslant (\det A)^{1/n} + (\det B)^{1/n}, \tag{2.3.26}$$

in particular,

$$\det(A+B) \geqslant \det A + \det B, \tag{2.3.27}$$

which implies that the matrix (2.3.24) has positive determinant and thus full rank.

In case only σ_{i_2} is positive, the determinants of all matrices in (2.3.23) are zero. However, already the sum of the first two matrices has positive determinant:

$$\det(\Sigma_i + A_i\Sigma_i A_i^T) = \begin{vmatrix} \sigma_{i_2}^2 & \lambda_i\sigma_{i_2}^2 \\ \lambda_i\sigma_{i_2}^2 & (1+\lambda_i^2)\sigma_{i_2}^2 \end{vmatrix} = \sigma_{i_2}^4 > 0. \tag{2.3.28}$$

Thus, it again follows from the Minkowski determinant inequality that the matrix (2.3.22) has positive determinant and thus full rank.[6]

Finally, if $\lambda_i = 0$, then

$$\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T = \Sigma_i + A_i\Sigma_i A_i^T = \begin{pmatrix} \sigma_{i_1}^2 + \sigma_{i_2}^2 & 0 \\ 0 & \sigma_{i_2}^2 \end{pmatrix}, \tag{2.3.29}$$

so we get the same result as for $\lambda_i \neq 0$.

Case 1.k_i. λ_i is a real eigenvalue of multiplicity k_i. The block A_i is a $k_i \times k_i$ matrix as in (2.3.17) and, by induction,

$$A_i^{t-1} = \begin{pmatrix} \lambda_i^{t-1} & (\lambda_i^{t-1})^{(1)} & \frac{1}{2!}(\lambda_i^{t-1})^{(2)} & \cdots & \frac{1}{(k_i-1)!}(\lambda_i^{t-1})^{(k_i-1)} \\ 0 & \lambda_i^{t-1} & (\lambda_i^{t-1})^{(1)} & \cdots & \frac{1}{(k_i-2)!}(\lambda_i^{t-1})^{(k_i-2)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_i^{t-1} & (\lambda_i^{t-1})^{(1)} \\ 0 & \cdots & 0 & 0 & \lambda_i^{t-1} \end{pmatrix}, \tag{2.3.30}$$

where $(\lambda_i^{t-1})^x$ denotes the x-th derivative of λ_i^{t-1}. It follows that if $\lambda_i \neq 0$, then $A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ is a symmetric matrix of the form

$$\begin{pmatrix} a_1\sigma_{i_1}^2 + \cdots + a_{k_i}\sigma_{i_{k_i}}^2 & & & * \\ & a_1\sigma_{i_2}^2 + \cdots + a_{k_i-1}\sigma_{i_{k_i}}^2 & & \\ & * & \ddots & \\ & & & a_1\sigma_{i_{k_i}}^2 \end{pmatrix}, \tag{2.3.31}$$

where every a_j, $j = 1, ..., k_i$, is positive. Consequently, if the sum $\sum_{t=1}^{n} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ converges as $n \to \infty$, it has the same form.

We first show the convergence of $\sum_{t=1}^{n} A_i^{t-1}$. An element in the upper triangle of this partial sum has the form

$$\sum_{t=0}^{n-1} \frac{1}{k!}(\lambda_i^t)^{(k)} \tag{2.3.32}$$

for some $0 \leqslant k \leqslant k_i - 1$. It holds

$$\sum_{t=0}^{n-1} \frac{1}{k!}(\lambda_i^t)^{(k)} = \frac{1}{k!}\left(\sum_{t=0}^{n-1} \lambda_i^t\right)^{(k)} \to \frac{1}{k!}\left(\frac{1}{1-\lambda_i}\right)^{(k)} = \frac{k+1}{(\lambda_i - 1)^{k+1}} \quad \text{for } n \to \infty,$$

where the last equality follows by induction. Since $|\lambda_i| < 1$, the sum $\sum_{t=1}^{n} A_i^{t-1}$ converges as $n \to \infty$.

[6] Note that in this case one could prove the same result more directly, but we wanted to show this method on a simple example, since it will be used later, where no other proofs are known.

Now consider an element of $A_i^{t-1}\Sigma_i(A_i^{t-1})^T$. Denoting $a_{jl}^{t-1} = (A_i^{t-1})_{jl}$, we have

$$\begin{aligned}\left(A_i^{t-1}\Sigma_i(A_i^{t-1})^T\right)_{jl} &= \sum_{m=1}^{k_i}(A_i^{t-1}\Sigma_i)_{jm}\left((A_i^{t-1})^T\right)_{ml}\\ &= \sum_{m=1}^{k_i} a_{jm}^{t-1}\sigma_m^2 a_{lm}^{t-1} = \sum_{m=\max\{j,l\}}^{k_i} a_{jm}^{t-1}\sigma_m^2 a_{lm}^{t-1}.\end{aligned} \tag{2.3.33}$$

Since $\sum_{t=1}^n A_i^{t-1}$ converges, there exsits $t^* \in \mathbb{N}$ such that $\left|a_{lm}^{t-1}\right| < 1$ for all $1 < l, m < k_i$, all $t > t^*$. It follows that

$$\left|\sum_{t=1}^{\infty}\left(A_i^{t-1}\Sigma_i(A_i^{t-1})^T\right)_{jl}\right| \leqslant \left|\sum_{t=1}^{t^*}\left(A_i^{t-1}\Sigma_i(A_i^{t-1})^T\right)_{jl}\right| + \sum_{t=t^*+1}^{\infty}\sum_{m=1}^{k_i}\left|a_{jm}^{t-1}\sigma_m^2\right|, \tag{2.3.34}$$

where the r.h.s is finite due to convergence of $\sum_{t=1}^n A_i^{t-1}$. With the Weierstrass M-test (which is a version of direct comparison test for series, but works both for real and complex summands, see, e.g., [Amann and Escher, Thm.V.1.6]) this implies that the sum $\sum_{t=1}^n A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ converges as $n \to \infty$.

For the dimension of $\mathcal{X}$ this means that

if $\sigma_{i_1}^2 = ... = \sigma_{i_{k_i}}^2 = 0$, then the block does not contribute to the dimension;

if $\sigma_{i_1}^2 \neq 0$ and $\sigma_{i_2}^2 = ... = \sigma_{i_{k_i}}^2 = 0$, then then the block contributes a value of 1 to the dimension;

if $\sigma_{i_2}^2 \neq 0$ and $\sigma_{i_3}^2 = ... = \sigma_{i_{k_i}}^2 = 0$, then then the block contributes a value of 2 to the dimension;

...

if $\sigma_{i_{k_i}}^2 \neq 0$, then then the block contributes a value of k_i to the dimension;

As in Case 1.2, all but the first two statements here hold only if the corresponding matrices have full ranks (for dimension increase by x, the upper left $x \times x$ submatrix of $\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ should have full rank). Unfortunately, we could not prove it, though after having computed several examples we strongly believe that this is indeed the case. We provide here the attempt of the proof by induction on k_i and point out the unsolved problem.

If in all cases all the submatrices of the $k_{i-1} \times k_{i-1}$ matrix have full ranks, it remains to show for $\sigma_{i_{k_i}}^2 \neq 0$ that the matrix $\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ has nonzero determinant irrespectively of the values of other $\sigma_{i_j}^2$'s.

If all the $\sigma_{i_j}^2$'s are nonzero, this can be done in the same way as in case 1.2. A_i^{t-1} for all $t \in \mathbb{N}$ and Σ_i have positive determinants, so $A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ has positive deteminant. Moreover, $A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ is symmetric and thus Hermitian. It is positive semidefinite as a product $U^T U$ with

$$U^T = A_i^{t-1}\begin{pmatrix}|\sigma_{i_1}| & & 0\\ & \ddots & \\ 0 & & |\sigma_{i_{k_i}}|\end{pmatrix}. \tag{2.3.35}$$

By the Minkowski inequality, $\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ has positive determinant and thus full rank.

This proof does not work in case some of $\sigma_{i_j}^2$'s are zero, since then $A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ has zero determinant. Instead, we may argue that the finite sum $\sum_{t=1}^{2k_i} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ has positive determinant and then use the Minkowski inequality adding to this matrix the other summands (with zero determinants). This positivity of the determinant of the finite sum is the unsolved problem. As already pointed out, the positivity was confirmed in all computed examples, but we could not prove it due to the complexity of the matrices involved.

Finally, if $\lambda_i = 0$, it is easy to compute that $\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T = \sum_{t=1}^{k_i} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ equals to

$$\begin{pmatrix} \sum_{j=1}^{k_i}\sigma_{i_j}^2 & & & 0 \\ & \sum_{j=2}^{k_i}\sigma_{i_j}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{i_{k_i}}^2 \end{pmatrix} \tag{2.3.36}$$

and it is evident that the consequences for the dimension of $\mathcal{X}$ are the same as in the $\lambda_i \neq 0$ case.

Case 2.1. $\lambda_i = \alpha_i + \mathrm{i}\beta_i$ with $\alpha_i, \beta_i \in \mathbb{R}$, $\beta_i \neq 0$ is a complex eigenvalue of multiplicity 1. We have

$$A_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix} \tag{2.3.37}$$

and A_i^{t-1} has the same form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{2.3.38}$$

for some $a, b \in \mathbb{R}$. It follows that

$$A_i^{t-1}\Sigma_i(A_i^{t-1})^T = \begin{pmatrix} a^2\sigma_{i_1}^2 + b^2\sigma_{i_2}^2 & c \\ -c & a^2\sigma_{i_1}^2 + b^2\sigma_{i_2}^2 \end{pmatrix} \tag{2.3.39}$$

for some $c \in \mathbb{R}$ depending on a, b, σ_{i_1} and σ_{i_2}. Consequently, $\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ has the same form. Here it is evident that $\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ has positive determinant in case $\sigma_{i_1}^2 \neq 0$ or $\sigma_{i_2}^2 \neq 0$ and

if $\sigma_{i_1}^2 = \sigma_{i_2}^2 = 0$, then the block does not contribute to the dimension;
if $\sigma_{i_1}^2 \neq 0$ or $\sigma_{i_2}^2 \neq 0$, then the block contributes a value of 2 to the dimension.

Case 2.k_i. $\lambda_i = \alpha_i + \mathrm{i}\beta_i$ with $\alpha_i, \beta_i \in \mathbb{R}$, $\beta_i \neq 0$ is a complex eigenvalue of multiplicity k_i. A_i has a form defined in (2.3.18) and, by induction,

$$A_i^{t-1} = \begin{pmatrix} S_{\lambda_i}^{t-1} & (t-1)S_{\lambda_i}^{t-2} & \binom{t-1}{2}S_{\lambda_i}^{t-3} & \cdots & \binom{t-1}{k_i-1}S_{\lambda_i}^{t-k_i} \\ 0_2 & S_{\lambda_i}^{t-1} & (t-1)S_{\lambda_i}^{t-2} & \cdots & \binom{t-1}{k_i-2}S_{\lambda_i}^{t-k_i+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_2 & \cdots & 0_2 & S_{\lambda_i}^{t-1} & (t-1)S_{\lambda_i}^{t-2} \\ 0_2 & \cdots & 0_2 & 0_2 & S_{\lambda_i}^{t-1} \end{pmatrix}, \tag{2.3.40}$$

where we assume that the binomial coefficient $\binom{n}{k}$ is zero, whenever $k > n$.

Note that the structure of this matrix is similar to the one in (2.3.30), but this time it is impossible to use notation with derivatives. Further, every $S_{\lambda_i}^t$ has the same rotation-scaling form as S_{λ_i}:

$$S_{\lambda_i}^t = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{2.3.41}$$

for some $a, b \in \mathbb{R}$.

Now, define the 2×2 matrix $\Sigma_{j(j+1)}$ rewriting Σ_i

$$\Sigma_i = \begin{pmatrix} \Sigma_{12} & & 0 \\ & \ddots & \\ 0 & & \Sigma_{(k_i-1)k_i} \end{pmatrix}. \tag{2.3.42}$$

$A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ is then a symmetric $2k_i \times 2k_i$-matrix with the j-th principal diagonal 2×2 block equal to

$$\begin{aligned} &S_{\lambda_i}^{t-1}\Sigma_{(2j-1)(2j)}(S_{\lambda_i}^{t-1})^T + (t-1)^2 S_{\lambda_i}^{t-2}\Sigma_{(2j+1)(2j+2)}(S_{\lambda_i}^{t-2})^T \\ &\qquad + \cdots + \binom{t-1}{k_i - j}^2 S_{\lambda_i}^{t-k_i+j-1}\Sigma_{(2k_i-1)(2k_i)}(S_{\lambda_i}^{t-k_i+j-1})^T, \end{aligned} \tag{2.3.43}$$

or, concentrating only on the principal diagonal, $A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ can be written as

$$\begin{pmatrix} \sum_{j=1}^{2k_i} a_{1j}\sigma_{i_j}^2 & & & * \\ & \sum_{j=3}^{2k_i} a_{2j}\sigma_{i_j}^2 & & \\ & & \ddots & \\ * & & & \sum_{j=2k_i-2}^{2k_i} a_{(2k_i)j}\sigma_{i_j}^2 \end{pmatrix}, \tag{2.3.44}$$

where all coefficients $\{a_{kj}\}$ are nonnegative and positive if α_i is nonzero. Consequently, $\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ has the same form. For the dimension of $\mathcal{X}$ in case $\alpha_i \neq 0$ this means that

if $\sigma_{i_1}^2 = \ldots = \sigma_{i_{k_i}}^2 = 0$, then the block does not contribute to the dimension;

if ($\sigma_{i_1}^2 \neq 0$ or $\sigma_{i_2}^2 \neq 0$) and $\sigma_{i_3}^2 = \ldots = \sigma_{i_{k_i}}^2 = 0$, then the block contributes a value of 2 to the dimension;

if ($\sigma_{i_3}^2 \neq 0$ or $\sigma_{i_4}^2 \neq 0$) and $\sigma_{i_5}^2 = \ldots = \sigma_{i_{k_i}}^2 = 0$, then the block contributes a value of 4 to the dimension;

...

if $\sigma_{i_{2k_i-1}}^2 \neq 0$ or $\sigma_{i_{2k_i}}^2 \neq 0$, then the block contributes a value of $2k_i$ to the dimension;

Again, these statements hold only if the corresponding matrices have full ranks, which is indeed the case. As in case $1.k_i$, we do not have the proof and provide the attempt of the proof by induction on k_i. If all σ_{i_j}'s are nonzero, the Minkowski inequality yields positivity of the determinant of $\sum_{t=1}^{\infty} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$. If some σ_{i_j}'s are zero, we may argue that the finite sum $\sum_{t=1}^{4k_i} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$ has positive determinant and then use the Minkowski inequality adding to this matrix the other summmands (with zero determinants).

In case $\alpha_i = 0$ not all coefficients in (2.3.44) are positive. However, one could show that the corresponding coefficients are positive already for the finite sum $\sum_{t=1}^{2k_i} A_i^{t-1}\Sigma_i(A_i^{t-1})^T$, which would imply that the same results as for $\alpha_i \neq 0$ hold. We strongly believe that this is true.

2.3.3 Summary and generalizations

We see that the Gaussian white noise with zero mean increases the dimension of the system. In general, an m-dimensional Gaussian white noise in $\mathbb{R}^n$ $(m \leqslant n)$ can increase dimension from 0 to any integer value from m to n. For an extreme example, suppose that ξ_t is a one dimensional Gaussian white noise in $\mathbb{R}^n$, i.e., its covariance matrix is zero up to a positive element in the lower right corner. If A is a diagonal matrix, the system $\mathcal{X}$ will yield dimension 1. If the whole A is a Jordan block for a real eigenvalue (see (2.3.17)), the system $\mathcal{X}$ will yield dimension n.

Though we considered only the case of diagonal Σ, it seems that the same result should hold for general covariance matrices: additional non-zero elements should not diminish the rank of matrices we encountered and the resulting dimension should again take any integer value from m to n.

This dimension effect depends on the direction in which the noise is nonzero. In our analysis, latter directions within one block influence all preceding directions in the same block. This is not surprising in view of the structure of the Jordan block. Geometrically, a Jordan block in its classical form represents a transformation, where latter directions have influence on the former ones, but not vice versa, and this effect spreads the noise from one direction into others.

It is not surprising that the additive noise increases the dimension of the attractor. Physical intuition suggests that the attractor of the disturbed system should be more irregular than that of the undisturded system and could stretch in more directions of the phase space, thus having a higher dimension. We conjecture that also for more complex attractors of the undisturbed systems, the additive noise of some kind, absolutely continuous w.r.t. Lebesgue, can increase the attractor's dimension to any intger up to the phase space dimension independently of the dimension of the noise itself.

Chapter 3

Invariant measures for the Rényi transformations

As we have discussed in the introduction, the invariant measure of a system describes its attractor fully enough, but it is often very difficult to find. Even so, studying different dynamical systems, one wants to know, whether there exists an invariant measure and whether it is unique. Further question is the stability, i.e., which measures, describing the distribution of the initial states of the system, will eventually converge to the invariant measure. If there is no invariant measure, one can ask, whether the measure of the system becomes after several iterations periodic, or at least asymptotically periodic.

In this chapter we focus on the invariant measures of dynamical systems described by Rényi transformations. The measure space is $([0,1], \mathcal{B}([0,1]), \lambda|_{[0,1]})$, where $\lambda|_{[0,1]}$ is the Lebesgue measure constrained to $[0,1]$. The independent system $\mathcal{Y}$ with initial condition y_0 and trajectory $\{y_t\}_{t\in\mathbb{N}}$ and the driven system $\mathcal{X}$ with initial condition x_0 and trajectory $\{x_t\}_{t\in\mathbb{N}}$ are described by the following transformations:

$$\begin{cases} y_{t+1} = by_t \bmod 1 \\ x_{t+1} = a[x_t + k(y_t - x_t)] \bmod 1, \end{cases} \tag{3.0.1}$$

where $a, b > 1$ and $k \in [0,1]$. Systems $\mathcal{X}$ and $\mathcal{Y}$ are thus coupled through a diffusive coupling $\mathcal{Y} \to \mathcal{X}$.

System $\mathcal{Y}$ is the standard Rényi transformation, first studied by Alfréd Rényi in [Rényi]. In case $b = 2$ it is the well-known Bernoulli map, and for all $b \in \mathbb{N}$ one can easily find the invariant density which is 1 (see example 3.1.7). In order to find this density, one makes use of the Perron-Frobenius operator, whose fixed point coincides with the invariant density. Perron-Frobenius operator is a strong tool in the theory of invariant measures and this chapter is devoted to the discussion of its properties.

For a general $b > 1$, the invariant density was first found by [Parry, Th.2]. Defining $S^0(y) = y$ and inductively $S^t(y) = S^{t-1}(by \bmod 1)$ for all $t \in \mathbb{N}$, we can write this invariant density as

$$c \cdot \sum_{t=0}^{\infty} \frac{1}{b} 1_{\{y<S^t(1)\}}, \tag{3.0.2}$$

where 1_A is the indicator function of the set A and c is the normalising factor. The invariant measure is thus a step function with an (in general) infinite number of steps.

While finding the invariant density, Parry did not make use of the Perron-Frobenius operator. However, the theory of this operator allows to state that the invariant density (3.0.2) is unique and stable. Indeed, if the initial state of the system is distributed according to some arbitrary density $f \geqslant 0$, $\int_0^1 f dx = 1$, then after sufficiently many iterations the density of the states will converge to the invariant density. The precise definition and the proof can be found in [Lasota and Mackey, 5.6 and Th.6.2.1].

So it turns out that we know quite a lot about the invariant density for the Rényi transformation. The driven Rényi transformation[1], governing the system $\mathcal{X}$, yields a much more complex dynamics, since in each iteration it depends on y_t, which is itself a state of another system, governed by a Rényi transformation. To our knowledge, the driven Rényi transformation has not been studied in the literature. However, existing theory on Perron-Frobenius operators allows us to state some result on the invariant measure of the driven Rényi transformation: if the driver system $\mathcal{Y}$ has a periodic trajectory, then the density of the states of the driven system $\mathcal{X}$ will eventually become asymptotically periodic, where the periodic sequence of densities is independent from the density of the initial states.

The rest of the chapter is devoted to the proof of this fact and is organized as follows. In Section 3.1 we define Markov and Perron-Frobenius operators (the latter is a special case of the former) and establish their basic properties, above all the connection to the invariant measure. In Section 3.2 we establish conditions under which the Perron-Frobenius operator corresponding to the driven Rényi transformation is constrictive, which turns out to be the essential property for the asymptotic periodicity of the density. The asymptotic periodicity itself is discussed in Section 3.3, where the theorem about asymptotic periodicity of constrictive Markov operators is proved. The consequences for a driven Rényi transformation are summarized in the last section. This chapter is based on [Lasota and Mackey, Chapters 3-6] and summarizes the existing material, providing some novelty only in the application to the driven Rényi transformation.

3.1 Preliminaries

Here we define Markov and Perron-Frobenius operators and give their basic properties which will be used throughout this chapter. We also discuss notation issues.

We will always consider the σ-finite measure space $(X, \mathcal{A}, \mu)$ unless another assumption is made explicitly. $||f||$ will be written for $||f||_1$, since the L^1 norm will be used most frequently. Sometimes we consider L^1 spaces over X with two different measures μ and $\bar{\mu}$. In this case we distinguish $||f||_{L^1(\mu)}$ and $||f||_{L^1(\bar{\mu})}$.

You may now want to recall some basic definitions and facts listed in Section 1.4.

Definition 3.1.1. *A linear operator $P : L^1(X) \to L^1(X)$ satisfying for all $f \in L^1$, $f \geqslant 0$*

(i) $Pf \geqslant 0$

(ii) $||Pf|| = ||f||$

is called a **Markov operator***.*

For every function f, we denote

$$f^+(x) = \max\{f(x), 0\} \text{ and } f^-(x) = \max\{-f(x), 0\}. \tag{3.1.1}$$

[1]We use this term for shortness, though it is neither established nor exact.

Proposition 3.1.2. *For every Markov operator $P : L^1(X) \to L^1(X)$, every $f, g \in L^1$,*

(i) $Pf(x) \leqslant Pg(x)$, *whenever* $f(x) \leqslant g(x)$

(ii) $(Pf(x))^+ \leqslant Pf^+(x)$

(iii) $(Pf(x))^- \leqslant Pf^-(x)$

(iv) $|Pf(x)| \leqslant P|f(x)|$ *and*

(v) $||Pf|| \leqslant ||f||$.

In particular, Markov operators are (i) monotonic and (v) contractions.

Proof. (i) $g - f \geqslant 0$ implies $P(g - f) = Pg - Pf \geqslant 0$.
(ii) By definition of f^+ and f^-,

$$(Pf)^+ = (Pf^+ - Pf^-)^+ = \max\{Pf^+ - Pf^-, 0\} \leqslant \max\{Pf^+, 0\} = Pf^+. \tag{3.1.2}$$

(iii) Analogously to (ii),

$$(Pf)^- = (Pf^+ - Pf^-)^- = \max\{Pf^- - Pf^+, 0\} \leqslant \max\{Pf^-, 0\} = Pf^-. \tag{3.1.3}$$

(iv) From (ii) and (iii),

$$|Pf| = (Pf)^+ + (Pf)^- \leqslant Pf^+ + Pf^- = P(f^+ + f^-) = P|f|. \tag{3.1.4}$$

(v) From (iv) and by definition of the Markov operator,

$$||Pf|| = \int_X |Pf| d\mu \leqslant \int_X P|f| d\mu = \int_X |f| d\mu = ||f||. \tag{3.1.5}$$

□

From the general concept of Markov operators we now turn to Perron-Frobenius operator which is the basic tool for determining invariant measures.

Definition 3.1.3. *Let $S : X \to X$ be a nonsingular trasnformation. An operator $P : L^1(X) \to L^1(X)$ such that for every $A \in \mathcal{A}$*

$$\int_A Pf d\mu = \int_{S^{-1}(A)} f d\mu \tag{3.1.6}$$

is called a **Perron-Frobenius operator** *corresponding to S.*

In fact, the Perron-Frobenius operator is unique.

Proposition 3.1.4. *For any nonsingular transformation $S : X \to X$ there exists a unique Perron-Frobenius operator.*

Proof. First, consider nonnegative $f \in L^1$.

Existence. For every $A \in \mathcal{A}$, define

$$\nu(A) = \int_{S^{-1}(A)} f d\mu, \tag{3.1.7}$$

which is a finite measure, since $f \in L^1$, integral is linear and

$$S^{-1}\left(\bigcup_i A_i\right) = \bigcup_i S^{-1}(A_i). \tag{3.1.8}$$

By the Radon-Nikodym theorem, there exists a nonnegative measurable function, which we denote by Pf, such that for all $A \in \mathcal{A}$

$$\int_A Pf d\mu = \int_{S^{-1}(A)} f d\mu. \tag{3.1.9}$$

Clearly, $Pf \in L^1$.

Uniqueness. For any other $\tilde{P}f$ with the same property we have

$$\int_A Pf d\mu = \int_A \tilde{P}f d\mu \quad \text{for all } A \in \mathcal{A}. \tag{3.1.10}$$

Define

$$A_1 = \{x : Pf(x) > \tilde{P}f(x)\} \quad \text{and } A_2 = \{x : Pf(x) \leqslant \tilde{P}f(x)\}. \tag{3.1.11}$$

A_1 and A_2 are measurable sets, $X = A_1 \cup A_2$ and

$$0 = \int_{A_1} (Pf - \tilde{P}f) d\mu - \int_{A_2} (Pf - \tilde{P}f) d\mu = \int_X |Pf - \tilde{P}f| d\mu, \tag{3.1.12}$$

which implies that $Pf = \tilde{P}f$ a.e. Thus Pf is unique up to a set of measure 0.

For an arbitrary $f \in L^1$, write $f = f^+ - f^-$ and define

$$Pf = Pf^+ - Pf^-. \tag{3.1.13}$$

Then for all $A \in \mathcal{A}$, we have

$$\int_A Pf d\mu = \int_{S^{-1}(A)} f^+ d\mu - \int_{S^{-1}(A)} f^- d\mu = \int_{S^{-1}(A)} f d\mu. \tag{3.1.14}$$

Uniqueness follows with the same arguments as above. □

Further,

Proposition 3.1.5. *The Perron-Frobenius operator is a Markov operator.*

Proof. It is linear, since

$$\begin{aligned}\int_A P(\lambda_1 f_1 + \lambda_2 f_2) d\mu &= \int_{S^{-1}(A)} (\lambda_1 f_1 + \lambda_2 f_2) d\mu \\ &= \lambda_1 \int_{S^{-1}(A)} f_1 d\mu + \lambda_2 \int_{S^{-1}(A)} f_2 d\mu = \int_A (\lambda_1 Pf_1 + \lambda_2 Pf_2) d\mu. \end{aligned} \tag{3.1.15}$$

For every $A \in \mathcal{A}$ with $\mu(A) > 0$, every nonnegative $f \in L^1$,

$$\int_A Pf d\mu = \int_{S^{-1}(A)} f d\mu \geqslant 0, \tag{3.1.16}$$

so Pf is nonnegative almost everywhere. Finally,

$$||Pf|| = \int_X Pf d\mu = \int_X f d\mu = ||f||. \tag{3.1.17}$$

□

Having explored the basic properties of the Perron-Frobenius operator, we now want to discuss its connection to the invariant measure of the transformation.

Perron-Frobenius operator describes the evolution of the distribution of the initial states of a dynamical system. Consider a transformation $S : X \to X$ and suppose that initial states are distributed according to some density function $f_0 \in D$, i.e., for every $A \in \mathcal{A}$,

$$\mathbb{P}(x_0 \in A) = \int_A f_0 d\mu, \tag{3.1.18}$$

where $\mathbb{P}$ denotes probability. What is now the distribution of $x_1 = S(x_0)$?

Clearly, $x_1 \in A$ if and only if $x_0 \in S^{-1}(A)$, so

$$\mathbb{P}(x_1 \in A) = \mathbb{P}(x_0 \in S^{-1}(A)) = \int_{S^{-1}(A)} f_0 d\mu \tag{3.1.19}$$

and, if $f_1 \in D$ is the density function for the distribution of x_1, it should hold

$$\int_A f_1 d\mu = \int_{S^{-1}(A)} f_0 d\mu \tag{3.1.20}$$

for all $A \in \mathcal{A}$. This way $Pf_0 = f_1$ describes the evolution of densities.

Moreover, any invariant density is a fixed point of the corresponding Perron-Frobenius operator and vice versa:

Proposition 3.1.6. *Let $S : X \to X$ be a nonsingular transformation and P the corresponding Perron-Frobenius operator. A measure μ_f given by*

$$\mu_f(A) = \int_A f d\mu \tag{3.1.21}$$

for some nonnegative $f \in L^1$ is invariant under S if and only if f is a fixed point of P, i.e., $Pf = f$.

Proof. Assume that μ_f is invariant. Then $\mu_f(A) = \mu_f(S^{-1}(A))$ implies

$$\int_A Pf d\mu = \int_{S^{-1}(A)} f d\mu = \int_A f d\mu. \tag{3.1.22}$$

Conversely, by definition of the Perron-Frobenius operator, $Pf = f$ implies

$$\mu_f(A) = \int_A f d\mu = \int_{S^{-1}(A)} f d\mu = \mu_f(S^{-1}(A)). \tag{3.1.23}$$

□

Finally, we note that in case $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with Lebesgue measure λ, the Perron-Frobenius operator can be defined as follows (cf. (3.1.6)):

$$Pf(x) = \frac{d}{dx} \int_{S^{-1}([-\infty,x])} f(t)dt \tag{3.1.24}$$

We give one

Example 3.1.7 (Invariant measure for Rényi transformation in case $b \in \mathbb{N}$)**.** *In order to find an invariant measure for the transformation*

$$y_{t+1} = S(y_t) = by_t \ mod\ 1 \tag{3.1.25}$$

with $b \in \mathbb{N}$, we determine a fixed point of the corresponding Perron-Frobenius operator. Clearly, for every $x \in (0, 1]$,

$$S^{-1}([0, x]) = \bigcup_{i=0}^{b-1} \left[\frac{i}{b}, \frac{i+x}{b}\right], \tag{3.1.26}$$

so

$$Pf(x) = \sum_{i=0}^{b-1} \frac{d}{dx} \int_{i/b}^{(i+x)/b} f(t)dt = \frac{1}{b} \sum_{i=0}^{b-1} f\left(\frac{i+x}{b}\right). \tag{3.1.27}$$

$f(x) \equiv 1$ is a fixed point of P and thus a measure invariant under S.

In sections to follow, we denote the complement of a set A by A^c.

At some point (proposition 3.3.13), we will use the notion of weak convergence. If a sequence of functions $\{f_i\}$ converges weakly to a function f, we will write $f_i \xrightarrow{w} f$.

Finally, we give one more definition that we will use already in the proof of proposition 3.2.3.

Definition 3.1.8. *A set $A \subset X$ is called* **weakly sequentially compact** *if every sequence $\{a_n\} \subset A$ contains a subsequence which converges weakly to a point in X.*

3.2 Constrictivness of a sequence of Perron-Frobenius operators for driven Rényi transformations

If we have a sequence of transformations $\{S_t\}$, as it is the case for the driven Rényi system, we get a sequence of Perron-Frobenius operators $\{P_t\}$. Since for the density $f \in D$ of the initial states, $P_t P_{t-1} \cdots P_1 f$ corresponds to the density of the states at the t-th iterate, we are interested in its behavior. One important notion here is the asymptotic periodicity.

Definition 3.2.1. *A sequence of Markov operators $\{P_t : L^1(X) \to L^1(X)\}_{t \in \mathbb{N}}$ is called* **asymptotically periodic** *if there exists a periodic operator $\bar{P} : L^1(X) \to L^1(X)$, such that for every $f \in L^1$*

$$\lim_{t \to \infty} ||P_t P_{t-1} \cdots P_1 f - \bar{P}^t f|| = 0. \tag{3.2.1}$$

If for a Markov operator P, the sequence $\{P_t = P\}$ is asymptotically periodic, then this operator is called **asymptotically periodic**.

Asymptotic periodicity of a sequence of Perron-Frobenius operators implies that the sequence of densities, describing the distribution of the system states, will eventually become asymptotically periodic. As we will see in Section 3.3 (theorem 3.3.1), in order to prove asymptotic periodicity of a Markov operator, one needs to verify another property - that of constrictivness.

In this section we define the property of constrictivness for a sequence of Markov operators and show that the sequence of Perron-Frobenius operators corresponding to the sequence of driven Rényi transformations has this property under certain conditions.

Definition 3.2.2. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. A sequence of Markov operators $\{P_n : L^1(X) \to L^1(X)\}_{n\in\mathbb{N}}$ is called* **constrictive**[2] *if there exists a measurable set B of finite measure and two constants $\delta > 0$, $\kappa < 1$, such that for every density $f \in D$, there is an $n_0(f) \in \mathbb{N}$ for which*

$$\int_{(X\setminus B)\cup E} P_n P_{n-1} \dots P_1 f(x)\mu(dx) \leqslant \kappa \quad \text{for all } n \geqslant n_0(f)$$
$$\text{and every set } E \text{ with } \mu(E) \leqslant \delta. \tag{3.2.2}$$

If for a Markov operator P the sequence $\{P^n = \underbrace{P \cdot P \cdots P}_{n}\}_{n\in\mathbb{N}}$ is constrictive, then this operator is called **constrictive***.*

In case of a finite measure space, it is easier to verify constrictivness using the following

Proposition 3.2.3. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space. A sequence of Markov operators $\{P_n : L^1(X) \to L^1(X)\}_{n\in\mathbb{N}}$ is constrictive if there is a $q > 1$ and $K > 0$, such that for every density $f \in D$ there is an $n_0(f) \in \mathbb{N}$ for which*

$$\|P_n P_{n-1} \dots P_1 f\|_q \leqslant K \quad \text{for all } n \geqslant n_0(f). \tag{3.2.3}$$

Proof. We use [Dunford and Schwartz, Cor. IV.8.4 and IV.8.11]. The first corollary states that a set in L^q-space is weakly sequentially compact if and only if it is bounded. Thus, it follows from (3.2.3) that the set $\{P_n P_{n-1} \dots P_1 f\}_{n\geqslant n_0(f)}$ is weakly sequentially compact in L^q for all $f \in D$.

The second corollary states that for a weakly sequentially compact set of functions G in L^1 it holds
$$\lim_{\mu(E)\to 0} \int_E g(x)\mu(dx) = 0$$
uniformly for $g \in G$. Since for a finite measure space $L^q \subset L^1$, $\{P_n P_{n-1} \dots P_1 f\}_{n\geqslant n_0(f)}$ is weakly sequentially compact in L^1 and for a fixed $\kappa < 1$ there exists a $\delta > 0$, such that for every density $f \in D$, there is an $n_0(f) \in \mathbb{N}$ for which

$$\int_E P_n P_{n-1} \dots P_1 f(x)\mu(dx) \leqslant \kappa \quad \text{for all } n \geqslant n_0(f) \quad \text{if } \mu(E) \leqslant \delta.$$

Thus the condition of definition 3.2.2 is fulfilled with $B = X$. $\square$

[2] We generalize the definition of constrictivness of a Markov operator given in [Lasota and Mackey, Def. 5.3.2]. In the earlier literature, cf. [Komornik and Lasota], the term "quasi-constrictivness" was used for the same notion, whereas "constrictivness" meant a stronger property.

Now we turn us to the main result of this section: the sequence of Perron-Frobenius operators for the transformations

$$x_{t+1} = a(x_t + k(y_t - x_t)) \bmod 1 \tag{3.2.4}$$

with $a(1-k) > 1$ and $k, y_t \in [0,1)$ for all $t \in \mathbb{N}$, is constrictive if $\{y_t\}_{t\in\mathbb{N}}$ is a periodic sequence or if $a \geqslant 2$ and $k < 1 - \frac{\lfloor a \rfloor}{a}$, where $\lfloor a \rfloor$ is the integer part of a. We will see in Corollary 3.2.9 how these conditions evolve and now state a more general

Proposition 3.2.4. *Let $\{S_t : [0,1) \to [0,1)\}_{t\in\mathbb{N}}$ be a sequence of transformations, satisfying*

(i) For each $t \in \mathbb{N}$, there is a partition $0 = s_0^t < s_1^t < \dots < s_{r_t}^t = 1$ of $[0,1]$, such that for each $i = 1,..,r_t$ the restriction of S_t to (s_{i-1}^t, s_i^t) is a linear function. Furthermore, $\inf_{t,i} S_t(s_{i-1}^t, s_i^t) > 0$.

(ii) $S_t'(x) = \lambda > 1$ for all $x \neq s_i^t$, $i = 0,..,r_t$.

(iii) In case $\lambda \in (1,2]$, the sequence of transformations $\{\tilde{S}_t = S_{tm}S_{tm-1}\dots S_{(t-1)m+1}\}_{t\in\mathbb{N}}$ with $m = \min\{l : \lambda^l > 2\}$, satisfy conditions (i) and (ii) for some new $\{\tilde{s}_i^t\}_{i=0,\dots,\tilde{r}}$ and $\tilde{\lambda}$.

Let $\{P_t\}_{t\in\mathbb{N}}$ be a sequence of Perron-Frobenius operators associated with $\{S_t\}$. Then, for all $f \in D$, $\{P_t\}$ is constrictive.

Note that λ does not depend on t and condition *(iii)* is only important in order to ensure for the new sequence the infimum condition from *(i)*, since other properties always hold.

To prove this proposition, we first need to introduce some basic facts about variations of functions.

Definition 3.2.5. *Let $f : [a,b] \to \mathbb{R}$ be a real-valued function and $a = s_0 < s_1 < \dots < s_n = b$ a partition of $[a,b]$. Then*

$$\bigvee_a^b f = \sup_{\mathcal{S}} \sum_{i=1}^{n} |f(s_i) - f(s_{i-1})|, \tag{3.2.5}$$

where supremum is taken over the set $\mathcal{S}$ of all possible partitions of $[a,b]$, is called the **variation** *of f on $[a,b]$. If the variation is finite, f is said to be of* **bounded vartiation** *on $[a,b]$.*

Lemma 3.2.6. *Let $a < b < c$ be three real numbers, $f, f_1,..,f_n : [a,c] \to \mathbb{R}$ - functions of bounded variation and $g : [\alpha,\gamma] \to [a,c]$ - a monotone function. Then*

$$\bigvee_a^c (f_1 + \ldots + f_n) \leqslant \bigvee_a^c f_1 + \dots + \bigvee_a^c f_n, \tag{3.2.6}$$

$$\bigvee_\alpha^\gamma f \circ g \leqslant \bigvee_a^c f \quad \text{and} \tag{3.2.7}$$

$$\bigvee_a^b f + \bigvee_b^c f = \bigvee_a^c f. \tag{3.2.8}$$

The proof is straightforward and can be found in [Lasota and Mackey, Section 6.1]. However, we copy here the proofs of two other results.

Lemma 3.2.7 (Variation of the product). *Let $f : [a, b] \to \mathbb{R}$ be of bounded variation and $g : [a, b] \to \mathbb{R}$ be in C^1. Then*

$$\bigvee_a^b fg \leqslant (\sup |g|) \bigvee_a^b f + \int_a^b |f(x)g'(x)|dx. \tag{3.2.9}$$

Proof. Define for every $h : [a, b] \to \mathbb{R}$ and an arbitrary partition $0 = s_0 < s_1 < \cdots < s_n = 1$ of $[a, b]$

$$v_n(h) = \sum_{i=1}^n |h(s_i) - h(s_{i-1})|. \tag{3.2.10}$$

Then

$$\begin{aligned} v_n(fg) &= \sum_{i=1}^n |f(s_i)g(s_i) - f(s_{i-1})g(s_i) + f(s_{i-1})g(s_i) - f(s_{i-1})g(s_{i-1})| \\ &\leqslant \sum_{i=1}^n (|g(s_i)||f(s_i) - f(s_{i-1})| + |f(s_{i-1})||g(s_i) - g(s_{i-1})|) \\ &\leqslant (\sup |g|)v_n(f) + \sum_{i=1}^n |f(s_{i-1})g'(t_i)|(s_i - s_{i-1}) \\ &\leqslant (\sup |g|) \bigvee_a^b f + \sum_{i=1}^n |f(s_{i-1})g'(t_i)|(s_i - s_{i-1}), \end{aligned} \tag{3.2.11}$$

where $t_i \in (s_{i-1}, s_i)$ and in the last but one step we used the mean value theorem.

Now, since the last term is an approximation for the integral of $|f(x)g'(x)|$ on $[a, b]$, taking on both sides $\limsup_i |s_i - s_{i-1}| \to 0$, we get equation (3.2.9). □

Lemma 3.2.8 (Yorke inequality). *Let $f : [0, 1] \to \mathbb{R}$ be of bounded variation on $[a, b] \subset [0, 1]$. Then*

$$\bigvee_a^b f1_{[a,b]} \leqslant 2 \bigvee_a^b f + \frac{2}{b-a} \int_a^b |f(x)|dx. \tag{3.2.12}$$

Proof. We assume w.l.o.g. that partitions of $[0, 1]$ contain points a and b and use v_n as defined in (3.2.10), specifying the interval of the partition where needed. Then

$$\begin{aligned} v_n^{[0,1]}(f1_{[a,b]}) &\leqslant v_n^{[a,b]}(f) + |f(a)| + |f(b)| \\ &\leqslant v_n^{[a,b]}(f) + |f(b) - f(c)| + |f(c) - f(a)| + 2|f(c)| \\ &\leqslant 2 \bigvee_a^b f + 2|f(c)|, \end{aligned} \tag{3.2.13}$$

where $c \in [a, b]$ is an arbitrary point.

We get equation (3.2.12), choosing c with

$$|f(c)| \leqslant \frac{1}{b-a} \int_a^b |f(x)|dx, \tag{3.2.14}$$

which is always possible. □

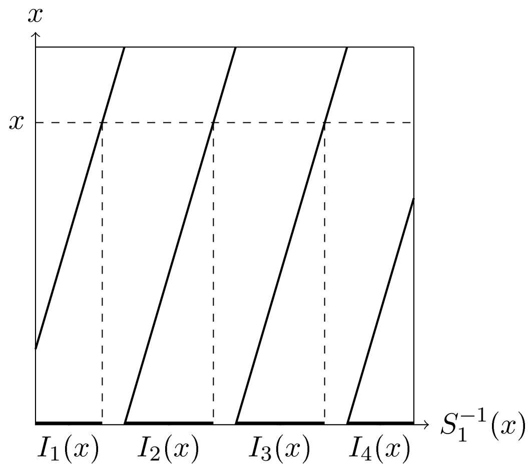


Figure 3.1: Example of a driven Rényi transformation with $a = 3.4$, $k = 0.2$ and $y_t = 0.7$. Intervals $I_1, \dots, I_4$ denote the preimage of $[0, x]$

Proof of Proposition 3.2.4. The proof is organized as follows: first, we compute the Perron Frobenius operator P_t for the transformation S_t, and then follow constrictivness from the fact that $\limsup_{t\to\infty} \bigvee_0^1 P_t \dots P_1 f$ is bounded. To simplify notation, we omit index t in the beginning; it will reappear later on.

Step I. For every $x \in [0, 1)$

$$S^{-1}([0, x]) = \bigcup_{i=1}^{r} I_i(x), \tag{3.2.15}$$

where

$$I_i(x) = \begin{cases} [s_{i-1}, g_i(x)], & \text{if } x \in S(s_{i-1}, s_i) \\ \varnothing \text{ or } [s_{i-1}, s_i] & \text{otherwise.} \end{cases} \tag{3.2.16}$$

Here $g_i(x) = (S^i)^{-1}(x)$ and S^i is the restriction of S to (s_{i-1}, s_i). An example for I_i's is shown in Figure 3.1.

The Perron-Frobenius operator is thus

$$\begin{aligned} Pf(x) &= \frac{d}{dx} \int_{S^{-1}[0,x]} f(t)dt = \sum_{i=1}^{r} \frac{d}{dx} \int_{I_i(x)} f(t)dt \\ &= \sum_{i=1}^{r} (g_i)'(x) f(g_i(x)) 1_{S(s_{i-1},s_i)}(x) = \frac{1}{\lambda} \sum_{i=1}^{r} f(g_i(x)) 1_{S(s_{i-1},s_i)}(x). \end{aligned} \tag{3.2.17}$$

Step II. Now we are interested in the variation of Pf for an arbitrary $f \in D$ of bounded variation. It holds

$$
\begin{aligned}
\bigvee_0^1 Pf(x) &\leqslant \sum_{i=1}^{r} \bigvee_0^1 \frac{1}{\lambda} f(g_i(x)) 1_{S(s_{i-1},s_i)}(x) \\
&\leqslant 2 \sum_{i=1}^{r} \bigvee_{S(s_{i-1},s_i)} \frac{1}{\lambda} f(g_i(x)) + \sum_{i=1}^{r} \frac{2}{\lambda |S(s_{i-1},s_i)|} \int_{S(s_{i-1},s_i)} f(g_i)(x) \\
&\leqslant \frac{2}{\lambda} \sum_{i=1}^{r} \bigvee_{s_{i-1}}^{s_i} f(x) + \frac{2}{\lambda} \sum_{i=1}^{r} \frac{1}{|S(s_{i-1},s_i)|} \int_{s_{i-1}}^{s_i} f(x) dx \\
&\leqslant \frac{2}{\lambda} \bigvee_0^1 f(x) + \frac{2}{\lambda} L,
\end{aligned} \tag{3.2.18}
$$

where $L = \max_i \frac{1}{|S(s_{i-1},s_i)|}$. Here we used Lemma 3.2.6 several times, Variation of the product lemma to pull out λ^{-1} and the Yorke inequality in the second step.

For a fixed t, straightforward induction yields

$$
\bigvee_0^1 P_t P_{t-1} \dots P_1 f(x) \leqslant \left(\frac{2}{\lambda}\right)^t \bigvee_0^1 f(x) + \sum_{i=1}^{t} \left(\frac{2}{\lambda}\right)^i L_{t+1-i}, \tag{3.2.19}
$$

where $L_t = \max_i \frac{1}{|S_t(s^t_{i-1},s^t_i)|}$.

Recall that $\{S_t\}$ is requested to fulfill $\inf_{t,i} S_t(s^t_{i-1}, s^t_i) > 0$. We thus can define

$$
L = \sup_t L_t < \infty \tag{3.2.20}
$$

and get

$$
\bigvee_0^1 P_t P_{t-1} \dots P_1 f(x) \leqslant \left(\frac{2}{\lambda}\right)^t \bigvee_0^1 f(x) + \sum_{i=1}^{t} \left(\frac{2}{\lambda}\right)^i L. \tag{3.2.21}
$$

Step III. $\limsup_{t\to\infty} \bigvee_0^1 P_t \dots P_1 f$ is bounded. This is obvious for $\lambda > 2$: from equation (3.2.21) one gets

$$
\limsup_{t\to\infty} \bigvee_0^1 P_t \dots P_1 f \leqslant \frac{\lambda}{\lambda - 2} L \tag{3.2.22}
$$

For $\lambda \in (1, 2]$, by assumption, $\{\tilde{S}_t = S_{tm} S_{tm-1} \dots S_{(t-1)m+1}\}_{t\in\mathbb{N}}$ with $m = \min\{l : \lambda^l > 2\}$, satisfy conditions *(i)* and *(ii)*. Moreover, $\tilde{S}'_t = \lambda^m > 2$, wherever it is defined, since the linear coefficient of a composition of linear transformations is equal to the product of their linear coefficients. It follows that for $\{\tilde{S}_t\}$ and the corresponding Perron-Frobenius operators $\{\tilde{P}_t\}$, we can repeat all arguments of the proof until equation (3.2.22).

Now, for a fixed $t \in \mathbb{N}$, find $d, q \in \mathbb{N}$ with $t = dm + q$ and $0 \leqslant q < m$. Consider t so large, that d is sufficiently large for

$$
\bigvee_0^1 \tilde{P}_{d_1} \tilde{P}_{d_1-1} \dots \tilde{P}_1 \leqslant \tilde{K} \tag{3.2.23}
$$

to hold for all $d_1 \geqslant d$ and some $\tilde{K} < \infty$. Then, by virtue of (3.2.21) and (3.2.23), we have

$$\begin{aligned}\bigvee_0^1 P_t P_{t-1} \dots P_1 f &= \bigvee_0^1 P_{dm+q} P_{dm+q-1} \dots P_{dm+1} \tilde{P}_d \tilde{P}_{d-1} \dots \tilde{P}_1 f \\ &\leqslant \left(\frac{2}{\lambda}\right)^q \bigvee_0^1 \tilde{P}_d \tilde{P}_{d-1} \dots \tilde{P}_1 f + \sum_{i=1}^q \left(\frac{2}{\lambda}\right)^i L \\ &\leqslant \left(\frac{2}{\lambda}\right)^m \tilde{K} + \sum_{i=1}^m \left(\frac{2}{\lambda}\right)^i L \end{aligned} \tag{3.2.24}$$

for all t large enough.

It follows that for all $\lambda > 1$ there is some $K < \infty$ such that

$$\limsup_{t\to\infty} \bigvee_0^1 P_t \dots P_1 f < K. \tag{3.2.25}$$

Step IV. We finally follow constrictivness of $\{P_t\}$. Define

$$\mathcal{F} = \{g \in D : \bigvee_0^1 g \leqslant K\}. \tag{3.2.26}$$

For every $g \in D$ defined on $[0,1]$ it holds

$$g(x) - g(y) \leqslant \bigvee_0^1 g$$

for all $x, y \in [0,1]$. It follows that for all $g \in \mathcal{F}$

$$g(x) \leqslant K + 1 \quad \text{for all } x \in [0,1],$$

since $g \in D$ and there is some $y \in [0,1]$ with $g(y) \leqslant 1$.

Now, for every $f \in D$, it follows from (3.2.25) that $\bigvee_0^1 P_t \dots P_1 f \in \mathcal{F}$ for all t large enough. Consequently, for all t large enough, $P_t \dots P_1 f \leqslant K + 1$ pointwise and

$$||P_t \dots P_1 f||_q \leqslant K_q + 1 \tag{3.2.27}$$

with $K_q < \infty$ for all $q \geqslant 1$. Constrictivness follows by Proposition 3.2.3. □

The main result of the section is now

Corollary 3.2.9. *Let $\{y_t\}_{t\in\mathbb{N}}$ be a sequence of real numbers with values in $[0,1)$. Let $k \in [0,1)$ and $a(1-k) > 1$. If $\{y_t\}$ is periodic or if $a \geqslant 2$ and $k < 1 - \frac{\lfloor a \rfloor}{a}$, then the sequence of Perron-Frobenius operators associated with the sequence of transformations*

$$S_t(x) = a(x + k(y_t - x)) \ mod\ 1 \tag{3.2.28}$$

is constrictive.

Proof. We have to show that $\{S_t\}$ satisfies conditions *(i)-(iii)* from Proposition 3.2.4. For convenience, rewrite

$$S_t(x) = (a(1-k)x + aky_t) \bmod 1. \tag{3.2.29}$$

The sequence $\{s_i^t\}_{i=0,\dots,r_t}$ can be determined explicitly. $s_0^t = 0$, $s_{r_t}^t = 1$ with

$$r_t = \lfloor a(1-k+ky_t)\rfloor - \lfloor aky_t\rfloor + 1_{a(1-k)+aky_t \notin \mathbb{N}} \tag{3.2.30}$$

and, for all $i = 1, .., r_t - 1$,

$$s_i^t = \frac{\lfloor aky_t\rfloor + i - aky_t}{a(1-k)}. \tag{3.2.31}$$

Clearly, for all $x \in (s_{i-1}^t, s_i^t)$, $i = 1, .., r_t$, the function $S_t(x)$ is linear and $S_t'(x) = a(1-k)$, so *(ii)* and the first part of *(i)* are proved.

Further, $S_t[s_{i-1}^t, s_i^t) = [0,1)$ for all $i = 2, .., r_t - 1$,

$$S_t[0, s_1^t) = [aky_t \bmod 1, 1) \qquad \text{and} \tag{3.2.32}$$

$$S_t[s_{r_t-1}^t, s_{r_t}^t) = \begin{cases} [0, a(1-k+ky_t) \bmod 1), & \text{if } a(1-k+ky_t) \notin \mathbb{N} \\ [0,1) & \text{otherwise.} \end{cases} \tag{3.2.33}$$

First, consider the case where $\{y_t\}$ is periodic. Clearly, for all $t \in \mathbb{N}$, there are only finitely many different $S_t[0, s_1^t)$ and $S_t[s_{r-1}^t, s_r^t)$, so $\inf_{t,i} S_t(s_{i-1}^t, s_i^t) > 0$ holds. It remains to show *(iii)*.

We use induction over m and suppose for fixed $t, m \in \mathbb{N}$ that $\tilde{S}_{t,m} = S_{tm}S_{tm-1}\dots S_{(t-1)m+1}$ satisfies conditions *(i)-(ii)* with partition $0 = s_0^m < \dots < s_{r_m}^m = 1$ and $\tilde{S}_{t,m}'(x) = a^m(1-k)^m$ for all $x \in (s_{i-1}^m, s_i^m)$, $i = 1, \dots, r_{t,m}$. The partition for $\tilde{S}_{t,m+1} = S_{tm+1}\tilde{S}_{t,m}$ can be constructed using the following algorithm (see Figure 3.2 for intuition):

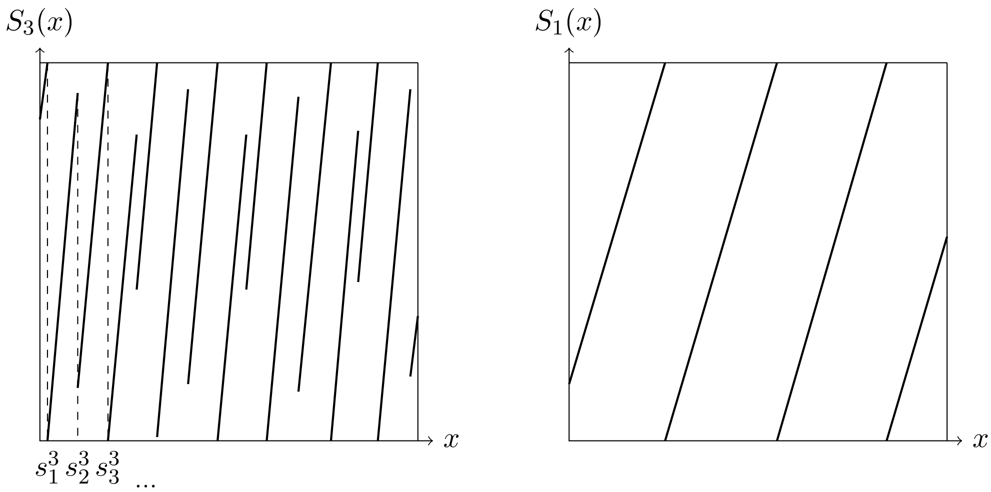


Figure 3.2: Example of a composition of three Rényi transformations (left). Together with a simple transformation (right), one can forsee how a compostion of four transformations looks like

0. $s_0^{m+1} = 0$

1. $s_1^{m+1} = \min\{s_1^m, \min\{s > 0 : S_{tm+1}\tilde{S}_{t,m}(s) = 0\}\}$

2. If $s_1^{m+1} = s_1^m$, then $s_2^{m+1} = \min\{s_2^m, \min\{s > s_1^m : S_{tm+1}\tilde{S}_{t,m}(s) = 0\}\}$.
   Else $s_2^{m+1} = \min\{s_1^m, \min\{s > s_1^{m+1} : S_{tm+1}\tilde{S}_{t,m}(s) = 0\}\}$

i+1. For j such that $s_i^{m+1} \in [s_j^m, s_{j+1}^m)$,

$$s_{i+1}^{m+1} = \min\{s_{j+1}^m, \min\{s > s_i^{m+1} : S_{tm+1}\tilde{S}_{t,m}(s) = 0\}\}. \tag{3.2.34}$$

Further, for all $x \in (s_i^{m+1}, s_{i+1}^{m+1})$, $i = 1, \dots, r_{t,m+1}$, where $r_{t,m+1}$ is defined through the algorithm, $\tilde{S}'_{t,m+1} = a^{m+1}(1-k)^{m+1}$. Finally, if $\{y_t\}$ is periodic, then $\{\tilde{S}_{t,m}\}_{t\in\mathbb{N}}$ is also periodic for any fixed m, so $\inf_{t,i} \tilde{S}_{t,m}(s_{i-1}^{t,m}, s_i^{t,m}) > 0$ with the same argument as in the last paragraph. We proved *(iii)* for an arbitrary $m \in \mathbb{N}$, in particular, for $m = \min\{l : a^l(1-k)^l > 2\}$.

Now, consider the case where $\{y_t\}$ is not periodic and can in general be dense in $[0, 1)$. We want to find conditions on a and k, which ensure that $\inf_{t,i} S_t(s_{i-1}^t, s_i^t)$ nevertheless remains positive. The necessary and sufficient conditions can be determined from equations (3.2.32) and (3.2.33): for some $\epsilon > 0$ and all $y_t \in [0, 1]$ it should hold

$$\begin{cases} aky_t \mod 1 \leqslant 1 - \epsilon \\ a(1 - k + ky_t) \mod 1 \geqslant \epsilon \end{cases} \tag{3.2.35}$$

Clearly, the first inequality holds for all $y_t \in [0, 1]$ if and only if $ak < 1$. For the second one, notice that $a(1 - k + ky_t)$ lies in the interval $[a(1 - k), a]$, which should be fully contained in some $[i, i + 1]$. Since $a \in [\lfloor a\rfloor, \lfloor a\rfloor + 1]$, (3.2.35) is equivalent to

$$\begin{cases} k < \frac{1}{a} \\ \frac{\lfloor a\rfloor}{1-k} < a < \lfloor a\rfloor + 1 \end{cases} \iff \begin{cases} k < \frac{1}{\lfloor a\rfloor+1} \\ \frac{\lfloor a\rfloor}{1-k} < a \end{cases} \iff \begin{cases} k < \frac{1}{\lfloor a\rfloor+1} \\ k < 1 - \frac{\lfloor a\rfloor}{a} \end{cases}, \tag{3.2.36}$$

where the first condition is weaker for $a > 1$, so that both are equivalent to

$$k < 1 - \frac{\lfloor a\rfloor}{a}. \tag{3.2.37}$$

Again, this simple condition ensures that $\inf_{t,i} S_t(s_{i-1}^t, s_i^t) > 0$. However, if $a(1 - k) \leqslant 2$, we need that a similar condition is satisfied by a sequence of finite compositions $\{\tilde{S}_t\}$ of S_t's (recall assumption *(iii)* of Proposition 3.2.4). But one can easily imagine a sequence $\{y_t\}$ such that $\inf_{t,i} \tilde{S}_t(\tilde{s}_{i-1}^t, \tilde{s}_i^t) = 0$. It follows that we need condition $a(1 - k) > 2$ to ensure constrictivness.

Finally, together with (3.2.37), it is enough to require $a \geqslant 2$. □

3.3 Asymptotic periodicity of constrictive Markov operators

Until now, we have focused on the property of constrictivness of a sequence of Markov operators. In the special case of a constant sequence, we can show that Markov operator has a stronger property - that of asymptotic periodicity (recall definition 3.2.1).

Theorem 3.3.1. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Let $P : L^1(X) \to L^1(X)$ be a Markov operator such that $\{P^n\}$ is a constrictive sequence. Then there exist finitely many densities $g_1, \dots, g_r \in D$ and bounded linear functionals $\lambda_1, \dots, \lambda_r : L^1(X) \to \mathbb{R}$ such that for every $f \in L^1$*

$$\lim_{n\to\infty} \left\| P^n \left(f - \sum_{i=1}^{r} \lambda_i(f) g_i \right) \right\| = 0. \tag{3.3.1}$$

Furthermore,

(i) g_i's have mutually almost disjoint supports, i.e., $\mu(supp(g_i) \cap supp(g_j)) = 0$ for all $i \neq j$.

(ii) There exists a permutation $\sigma : \{1, \dots, r\} \to \{1, \dots, r\}$ such that $Pg_i = g_{\sigma(i)}$.

We will later see that $g_1, \dots, g_r$ are determined by their almost disjoint supports $A_1, \dots, A_r$, which are chosen to have positive measure. It holds

$$g_i = \frac{1_{A_i}}{\mu(A_i)} \tag{3.3.2}$$

and

$$\lambda_i(f) = \int_{A_i} f d\mu. \tag{3.3.3}$$

Theorem 3.3.1 guarantees asymptotic periodicity of a constrictive Markov operator. Indeed, equation (3.3.1) implies that for all $f \in L^1$

$$P^n f = \sum_{i=1}^{r} \lambda_i(f) g_{\sigma^n(i)} + \epsilon_n(f), \tag{3.3.4}$$

where $||\epsilon_n(f)|| \to 0$ as $n \to 0$. Since $\{\sigma^n\}$ is periodic with period $\tilde{r} \leqslant r!$, P is asymptotically periodic with $\bar{P}^n f = \sum_{i=1}^{r} \lambda_i(f) g_{\sigma^n(i)}$.

The rest of the section will be devoted to the proof of this theorem. The sketch of the proof is given in [Komornik and Lasota] with more details provided in [Lasota, Li and Yorke], [Komornik] as well as in [Lasota and Socala].

First, the special case of a probability measure space $(X, \mathcal{A}, \mu)$ and $P1 = 1$ is considered. Subsection 3.3.4 handles the general case.

3.3.1 The σ-algebra of nice sets

Definition 3.3.2. *A measurable set $A \in \mathcal{A}$ is called a* **nice set** *if $P^n 1_A$ is a characteristic function for all $n \in \mathbb{N}$. For a nice set A, the function 1_A is called a* **nice function***.*

We aim to show that the set of nice sets $\tilde{\mathcal{A}}$ is a σ-algebra. We first prove

Lemma 3.3.3. *The set of nice sets $\tilde{\mathcal{A}}$ is a Dynkin-system, i.e.,*

(i) $X \in \tilde{\mathcal{A}}$

(ii) for all $A_1, A_2 \in \tilde{\mathcal{A}}$ with $A_1 \subset A_2$, it holds $A_2 \backslash A_1 \in \tilde{\mathcal{A}}$

(iii) for countably many mutually disjoint sets $A_1, A_2, \dots \in \tilde{\mathcal{A}}$, it holds $\bigcup_{i=1}^{\infty} A_i \in \tilde{\mathcal{A}}$

Proof. (i) By assumption, $P1_X = 1_X$, so $P^n 1_X = 1_X$ for all $n \in \mathbb{N}$.

(ii) Suppose that $A_1, A_2 \in \tilde{\mathcal{A}}$, $A_1 \subset A_2$ and choose an arbitrary $n \in \mathbb{N}$. Then there exist two sets $B_1, B_2 \in X$ with $P^n 1_{A_1} = 1_{B_1}$ and $P^n 1_{A_2} = 1_{B_2}$. By linearity of P,

$$P^n 1_{A_2 \backslash A_1} = P^n 1_{A_2} - P^n 1_{A_1} = 1_{B_2} - 1_{B_1}. \tag{3.3.5}$$

Now, by monotonicity of P (recall proposition (3.1.2)), it follows from $1_{A_1} \leqslant 1_{A_2}$ that

$$1_{B_1} = P^n 1_{A_1} \leqslant P^n 1_{A_2} = 1_{B_2}. \tag{3.3.6}$$

So, $B_1 \subset B_2$ and (3.3.5) becomes

$$P^n 1_{A_2 \backslash A_1} = 1_{B_2 \backslash B_1}. \tag{3.3.7}$$

(iii) Let $\{A_i\}_{i \in \mathbb{N}}$ be a sequence of disjoint nice sets. We first show that

$$P^n \left(\sum_{i=1}^{\infty} 1_{A_i} \right) = \sum_{i=1}^{\infty} P^n 1_{A_i}. \tag{3.3.8}$$

Indeed, the sequence $\{f_k = \sum_{i=1}^{k} 1_{A_i}\}$ converges to $f = \sum_{i=1}^{\infty} 1_{A_i}$ in L^1, as it is bounded by the constant 1 function and $||1|| = 1 < \infty$ by assumption on μ. Since $f, f_k \in L^1$ and $f - f_k \geqslant 0$, we get

$$||P^n f - P^n f_k|| = ||f - f_k|| \to 0 \text{ in } L^1, \tag{3.3.9}$$

which together with $P^n f_k = \sum_{i=1}^{k} P^n 1_{A_i}$ yields (3.3.8).

Now, fix $n \in \mathbb{N}$ and for all $i \in \mathbb{N}$ define B_i as a set with $1_{B_i} = P^n 1_{A_i}$. Clearly,

$$1_{A_j} \leqslant 1_{\cup_{i=1}^{\infty} A_i} \leqslant \sum_{i=1}^{\infty} 1_{A_i} \tag{3.3.10}$$

for all $j \in \mathbb{N}$. By monotonicity of P and equation (3.3.8),

$$1_{B_j} \leqslant P^n 1_{\cup_{i=1}^{\infty} A_i} \leqslant P^n \left(\sum_{i=1}^{\infty} 1_{A_i} \right) = \sum_{i=1}^{\infty} P^n 1_{A_i} = \sum_{i=1}^{\infty} 1_{B_i} \tag{3.3.11}$$

for all $j \in \mathbb{N}$. It follows from the right inequality that $P^n 1_{\cup_{i=1}^{\infty} A_i}(x) = 0$ whenever $x \notin \cup_{i=1}^{\infty} B_i$. Since $P^n 1_{\cup_{i=1}^{\infty} A_i} \leqslant 1$ by assumption, left inequalities imply $P^n 1_{\cup_{i=1}^{\infty} A_i}(x) = 1$ whenever $x \in \cup_{i=1}^{\infty} B_i$. So, $P^n 1_{\cup_{i=1}^{\infty} A_i}$ is a characteristic function. □

Proposition 3.3.4. *The set of nice sets $\tilde{\mathcal{A}}$ is a σ-algebra.*

Proof. Since $\tilde{\mathcal{A}}$ is a Dynkin-system, it remains to show that for every $A_1, A_2 \in \tilde{\mathcal{A}}$, it holds $A_1 \cap A_2 \in \tilde{\mathcal{A}}$ (see, e.g., [Klenke, Th.1.18]). Instead, we can show that the complements and finite unions of nice sets are also nice, as $A_1 \cap A_2 = (A_1^c \cup A_2^c)^c$.

Fix $n \in \mathbb{N}$, denote $1_{B_1} = P^n 1_{A_1}$ and $1_{B_2} = P^n 1_{A_2}$. Then

$$P^n 1_{A_1^c} = P^n 1_X - P^n 1_{A_1} = 1 - 1_{B_1} = 1_{B_1^c}, \tag{3.3.12}$$

so A_1^c is a nice set.

For the union, we repeat the argument from the proof of part (iii) of the previous lemma. For $j \in \{1, 2\}$,

$$1_{A_j} \leqslant 1_{A_1 \cup A_2} \leqslant 1_{A_1} + 1_{A_2}. \tag{3.3.13}$$

By monotonicity of P, for $j \in \{1, 2\}$,

$$1_{B_j} \leqslant P^n 1_{A_1 \cup A_2} \leqslant 1_{B_1} + 1_{B_2}. \tag{3.3.14}$$

By the right inequality, $P^n 1_{A_1 \cup A_2}(x) = 0$ whenever $x \notin B_1 \cup B_2$ and, by $P^n 1_{A_1 \cup A_2} \leqslant 1$ together with the left inequalities, $P^n 1_{A_1 \cup A_2}(x) = 1$ whenever $x \in B_1 \cup B_2$. So, $P^n 1_{A_1 \cup A_2} = 1_{B_1 \cup B_2}$, which completes the proof. □

Our next step is to show that $\tilde{\mathcal{A}}$, though possibly infinitely large, contains only a finite number of interesting sets. To be more precise, there are only finitely many different atoms in $\tilde{\mathcal{A}}$ [3].

Definition 3.3.5. *A set $A \in \bar{\mathcal{A}} \subset \mathcal{A}$ is called an* **atom** *of $\bar{\mathcal{A}}$ if $\mu(A) > 0$ and for all measurable $B \subset A$ with $\mu(B) < \mu(A)$, it holds $\mu(B) = 0$.*

We have the following

Lemma 3.3.6. *Two atoms of a set $\bar{\mathcal{A}}$ are either almost disjoint, i.e., intersect on a set of measure* 0*, or almost the same.*

Proof. Let A_1, A_2 be two atoms in $\bar{\mathcal{A}}$. Define $A_3 = A_1 \cap A_2$, which is clearly a measurable set. If $\mu(A_3) > 0$, then $\mu(A_3) = \mu(A_1)$ by definition of an atom, since $A_3 \subset A_1$, and analogously $\mu(A_3) = \mu(A_2)$. So, either $\mu(A_3) = 0$, or $\mu(A_1) = \mu(A_2)$ and A_1 is almost the same as A_2. □

Now we can prove this important

Proposition 3.3.7. *$\tilde{\mathcal{A}}$ has only finitely many different atoms.*

Proof. Let $A \in \tilde{\mathcal{A}}$ and $\mu(A) > 0$. Define

$$f_A = \frac{1}{\mu(A)} 1_A \tag{3.3.15}$$

and B_n as a set with $1_{B_n} = P^n 1_A$. Note that B_n is a nice set by definition and $\mu(A) = \mu(B_n)$ for all $n \in \mathbb{N}$, since P is Markov.

Now, by constrictivness of P, there exist some $n_0(f_A) \in \mathbb{N}$, $\delta > 0$ and $\kappa < 1$ such that

$$\int_E P^n f_A d\mu \leqslant \kappa \;\text{ for all } n \geqslant n_0(f_A) \;\text{ whenever } \mu(E) \leqslant \delta. \tag{3.3.16}$$

If $\mu(B_n) \leqslant \delta$ and n is large enough, then

$$1 = \frac{1}{\mu(A)} \int_{B_n} 1_{B_n} d\mu = \int_{B_n} P^n f_A d\mu \leqslant \kappa < 1, \tag{3.3.17}$$

which is impossible. It follows that $\mu(A) = \mu(B_n) > \delta > 0$, i.e., the measure of every nice set is either 0 or bigger than δ.

Consider all atoms of $\tilde{\mathcal{A}}$. By the preceding lemma, two atoms can be either almost disjoint or almost the same. The measure of the union of all mutually almost disjoint atoms can not exceed $\mu(X) = 1$, so there are only finitely many almost disjoint atoms. □

[3] Recall that we call sets A and B different if $A \backslash B$ or $B \backslash A$ has positive measure

Note that every nice set either has measure 0 or is a union of atoms. The whole space X is a nice set and a union of all atoms.

To complete this subsection, we prove that P permutes atoms of positive measure.

Proposition 3.3.8. *Let $\{A_0, A_1, \ldots, A_r\}$ be the set of atoms. Then there is a permutation $\sigma : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\}$, such that $P1_{A_i} = 1_{A_{\sigma(i)}}$ for all $i = 1, \ldots, r$.*

Since P permutes finitely many atoms, there exists some integer $\tilde{r} \leqslant r!$ such that $\sigma^{\tilde{r}} = id$ and $P^{\tilde{r}} 1_{A_i} = 1_{A_i}$.

Proof. For every A_j, denote $1_{B_j} = P1_{A_j}$. For $i \neq j$, we have

$$1 \geqslant P1_{A_i \cup A_j} = P1_{A_i} + P1_{A_j} - P1_{A_i \cap A_j} = 1_{B_i} + 1_{B_j} - P1_{A_i \cap A_j}, \tag{3.3.18}$$

so $B_i \cap B_j \subset supp(P1_{A_i \cap A_j})$. Since $\mu(supp(P1_{A_i \cap A_j})) = 0$, it follows that B_i and B_j are almost disjoint. The sets $B_1, B_2, \ldots, B_r$ are mutually almost disjoint nice sets and have measure $> \delta$, since P is a Markov operator. Clearly, they are atoms and form a permutation of $\{A_1, A_2, \ldots, A_r\}$, which is already the statement of the proposition. □

3.3.2 Weak convergence in case $P1_X = 1_X$, $\mu(X) = 1$

In this subsection we prove some convergence similar to that in equation (3.3.1), but in the weak sense. As also the coming subsections are, this one is quite technical. The main result here is proposition 3.3.13, but in order to prove it we need some other results. We first aim at lemma 3.3.11, which treating P as an operator from $L^2(X)$ to $L^2(X)$, establishes equality between the linear subspace of L^1 spanned by nice functions and the kernel of some operator. This equality will be used in the proof of proposition 3.3.13, which first treats the $f \in L^2$ case and then generalizes by a density argument.

Lemma 3.3.9. *If $f_1, f_2 \in L^1$ are nonnegative and have the same support, then Pf_1 and Pf_2 have the same support.*

Proof. We show that $supp(Pf_1) \subset supp(Pf_2)$. Define $B_2 = supp(Pf_2)$ and write

$$Pf_1 = Pf_1 \cdot 1_{B_2} + Pf_1 \cdot 1_{B_2^c}. \tag{3.3.19}$$

For $c > 2$, define $f_c = \min cf_2, f_1$. Since $f_c \leqslant f_1 \in L^1$ and $f_c(x) \to f_1(x)$ as $c \to \infty$, we can use the dominated convergence theorem to obtain

$$||Pf_1 - Pf_2|| = \int_X P(f_1 - f_c)d\mu = \int_X (f_1 - f_c)d\mu \to 0 \quad \text{as } c \to \infty. \tag{3.3.20}$$

Since $supp(Pf_c) \subset supp(Pf_2) = B_2$,

$$||Pf_1 - Pf_c|| \geqslant \int_{B_2^c} (Pf_1 - Pf_c)d\mu = \int_{B_2^c} Pf_1 d\mu = ||Pf_1 \cdot 1_{B_2^c}||. \tag{3.3.21}$$

Thus, $Pf_1 \cdot 1_{B_2^c} = 0$, which implies $supp(Pf_1) \subset B_2$.

The proof of the other inclusion is completely analogous. □

Lemma 3.3.10. *1) The operator P preserves mean values, i.e., for all $f \in L^1$*

$$E[Pf] = E[f]. \tag{3.3.22}$$

2) For all $1 \leqslant p \leqslant \infty$, the subspace L^p is P-invariant and, for all $f \in L^p$,

$$||Pf||_p \leqslant ||f||_p. \tag{3.3.23}$$

Proof. 1) For $f \in L^1$, write $f = f^+ - f^-$. Clearly, $f^+, f^- \in L^1$ and, since P is Markov,

$$\begin{aligned} E[Pf] &= E[Pf^+ - Pf^-] = E[Pf^+] - E[Pf^-] \\ &= E[f^+] - E[f^-] = E[f^+ - f^-] = E[f]. \end{aligned} \tag{3.3.24}$$

2) We know from Propostion 3.1.2 that for all $f \in L^1$,

$$||Pf||_1 \leqslant ||f||_1. \tag{3.3.25}$$

Since P is also monotonic and $\mu(X) = 1$, for all $f \in L^\infty$,

$$||Pf||_\infty = \sup_{x\in X} Pf(x) \leqslant \sup_{x\in X} P(||f||_\infty 1_X(x)) = ||f||_\infty \sup_{x\in X} P1_X(x) = ||f||_\infty. \tag{3.3.26}$$

So, P is bounded as an $L^1 \to L^1$ operator and as an $L^\infty \to L^\infty$ operator both with norm $\leqslant 1$. Then, by the Riesz-Thorin interpolation theorem (cf. [Stein and Shakarchi, FA, Chapter 2, Th. 2.1]), for every $1 \leqslant p \leqslant \infty$, P is bounded as an $L^p \to L^p$ operator with norm $\leqslant 1$. Clearly, this implies P-invariance of L^p spaces. □

In particular, $||P||_2 \leqslant 1$, so one can consider P as an operator on L^2 and define its adjoint $U = P^*$.

Let Q be the linear subspace of L^1 spanned by nice functions.

Lemma 3.3.11. *Their exists a symmetric operator $A : L^2(X) \to L^2(X)$ such that for every $f \in L^2$,*

$$\lim_{n\to\infty} ||Af - U^n P^n f||_2 = 0 \quad \textit{and} \tag{3.3.27}$$

$$\langle Af, f\rangle = \lim_{n\to\infty} ||P^n f||_2^2. \tag{3.3.28}$$

Moreover,

$$Q = Ker(I - A), \tag{3.3.29}$$

where I is the identity operator.

Proof. First, as for all adjoint operators, we have $||U||_2 = ||P||_2$, so, by lemma 3.3.10, both P and U are contractions. It follows that for every $f \in L^2$,

$$||f||_2 \geqslant ||UPf||_2 \geqslant \cdots \geqslant ||U^n P^n f||_2 \geqslant \cdots \geqslant 0, \tag{3.3.30}$$

so

$$I \geqslant UP \geqslant \cdots \geqslant U^n P^n \geqslant \cdots \geqslant 0, \tag{3.3.31}$$

which implies (3.3.27). Equation (3.3.28) follows, since $\langle P^n f, P^n f\rangle = \langle U^n P^n f, f\rangle$.

Now, we show that

$$Ker(I-A) = \{f \in L^2 : \langle Af, f\rangle = ||f||_2^2\} \tag{3.3.32}$$
$$= \{f \in L^2 : ||P^n f||_2 = ||f||_2 \ \text{ for all } n \in \mathbb{N}\}, \tag{3.3.33}$$

from which $Q = Ker(I-A)$ will follow. The second equality follows from (3.3.28) and the fact that P is a contraction. For the first one, if $(I-A)f = 0$, then $\langle Af, f\rangle = \langle f, f\rangle$. Conversely, if $\langle Af, f\rangle = \int_X f^2 d\mu$, then

$$||(I-A)^{1/2} f||_2^2 = \int |f^2 - Af \cdot f| d\mu = 0, \tag{3.3.34}$$

so $f - Af \equiv 0$.

Further, for all $n \in \mathbb{N}$, an arbitrary nice set A and a set B with $1_B = P^n 1_A$,

$$||P^n 1_A||_2 = ||1_B||_2 = ||1_B||_1 = ||1_A||_1 = ||1_A||_2, \tag{3.3.35}$$

so by (3.3.32), $1_A \in Ker(I-A)$, which implies $Q \subset Ker(I-A)$, since any kernel is a linear subspace.

It remains to show that $Ker(I-A) \subset Q$. Since $P^n 1_X = 1_X$, it follows from (3.3.33) that $1_X \in Ker(I-A)$. Accordingly, for all $c \in \mathbb{R}$, $c = c1_X \in Ker(I-A)$ and $f - c \in Ker(I-A)$, if $f \in Ker(I-A)$.

Take arbitrary $f \in Ker(I-A)$, $f = f^+ - f^-$. By lemma 3.3.10,

$$\begin{aligned} ||f||_2^2 = ||Pf||_2^2 &= ||Pf^+||_2^2 + ||Pf^-||_2^2 - 2\langle Pf^+, Pf^-\rangle \\ &\leqslant ||f^+||_2^2 + ||f^-||_2^2 - 2\int Pf^+ \cdot Pf^- d\mu \\ &= ||f||_2^2 - 2\int Pf^+ \cdot Pf^- d\mu, \end{aligned} \tag{3.3.36}$$

so $Pf^+ \cdot Pf^- = 0$, as both Pf^+ and Pf^- are nonnegative. Hence, Pf^+ and Pf^- have disjoint supports.

Analogously, for all $c \in \mathbb{R}$ and all $n \in \mathbb{N}$, $P^n(f-c)^+$ and $P^n(f-c)^-$ have disjoint supports.

Take an arbitrary $c \in \mathbb{R}$ and suppose first that $\mu(f^{-1}(c)) = 0$. Define

$$h_1 = 1_{f^{-1}(-\infty, c)}, \quad h_2 = 1_{f^{-1}[c, +\infty)}. \tag{3.3.37}$$

Clearly,

$$supp(h_1) = supp((f-c)^-), \quad supp(h_2) = supp((f-c)^+), \tag{3.3.38}$$

so, by lemma 3.3.9, $P^n h_1$ and $P^n h_2$ have disjoint supports for every $n \in \mathbb{N}$. On the other hand,

$$P^n 1_{h_1} + P^n 1_{h_2} = P^n 1 = 1, \tag{3.3.39}$$

so $P^n h_1$ is a characteristic function and $f^{-1}(-\infty, c)$ is a nice set.

Finally, suppose that $\mu(f^{-1}(c)) > 0$. Since $\mu(X) = 1 < \infty$ and, as for every function, $f^{-1}(c)$ are disjoint for different $c > 0$, there are at most countably many $c > 0$ with $\mu(f^{-1}(c)) \neq 0$. It follows that $\{c \in \mathbb{R} : \mu(f^{-1}(c)) = 0\}$ is dense in $\mathbb{R}$ and contains an increasing sequence $\{c_i\}$ with $c_i \to c$. Then

$$f^{-1}(-\infty, c) = \bigcup_{i=1}^{\infty} f^{-1}(-\infty, c_i) \tag{3.3.40}$$

is a nice set, since nice sets form a σ-algebra.

Thus, $f^{-1}(-\infty, c)$ is a nice set for every $c \in \mathbb{R}$ $\square$

Recall that $\tilde{r}$ is defined as a number for which $P^{\tilde{r}} 1_{A_i} = 1_{A_i}$, $i = 1, \ldots, r$.

Corollary 3.3.12. *For all $f \in Q$,*

$$U^{\tilde{r}} f = f. \tag{3.3.41}$$

Proof. Take an arbitrary $f \in Q = Ker(I - A)$. Using (3.3.31) in the first and third steps, we get

$$0 \leqslant \left\| \left(I - U^{\tilde{r}} P^{\tilde{r}}\right)^{1/2} f \right\|_2^2 = \left\langle f, \left(I - U^{\tilde{r}} P^{\tilde{r}}\right) f \right\rangle \leqslant \left\langle f, (I - A) f \right\rangle = 0, \tag{3.3.42}$$

so $(I - U^{\tilde{r}} P^{\tilde{r}}) f = 0$, which implies $If = U^{\tilde{r}} f$. $\square$

Now we are ready to state the main result of this section. Define $R = P^{\tilde{r}}$ and, for all $i = 1, \ldots, r$,

$$L_i = \{f 1_{A_i} : f \in L^1\}. \tag{3.3.43}$$

Proposition 3.3.13. *1) L_i's are R-invariant and*

2) for all $f \in L_i$, $i = 1, \ldots, r$,

$$R^n f \xrightarrow{w} \lambda_i(f) \frac{1_{A_i}}{\mu(A_i)} \quad \text{as } n \to \infty, \tag{3.3.44}$$

where

$$\lambda_i(f) = \int_{A_i} f d\mu \tag{3.3.45}$$

where the weak convergence is considered with respect to L_i space.

Proof. Fix an arbitrary $1 \leqslant i \leqslant r$.

1) The operator R is defined in such a way that $R 1_{A_i} = 1_{A_i}$. For all $f \in L^1 \cap L^\infty$, $f 1_{A_i} \leqslant ||f||_\infty 1_{A_i}$ and

$$R(f 1_{A_i}) \leqslant R(||f||_\infty 1_{A_i}) = ||f||_\infty 1_{A_i}, \tag{3.3.46}$$

which implies that $R(f 1_{A_i}) = g 1_{A_i} \in L_i$ for some $g \in L^1$.

Now, $L^1 \cap L^\infty$ is dense in L^1 (since $C_c^\infty \subset L^1 \cap L^\infty$ and C_c^∞ is dense in L^1 - see, e.g. [Teschl, Th.0.36]). For every $f \in L^1$, there is a sequence $\{f_j\} \in L^1 \cap L^\infty$ with $||f - f_j||_1 \to 0$ as $j \to \infty$. It follows that

$$\left\| R\left(f 1_{A_i}\right) - R\left(f_j 1_{A_i}\right) \right\|_1 \leqslant ||f 1_{A_i} - f_j 1_{A_i}||_1 \to 0 \quad \text{as } j \to \infty, \tag{3.3.47}$$

which implies that $R\left(f 1_{A_i}\right)$ is 0 almost everywhere on A_i^c for and is thus equal to $g 1_{A_i} \in L_i$ for some $g \in L^1$.

2) Let now $f \in L_i \cap L^2$. As $\{P^n f\}$, $\{R^n f\}$ is bounded and thus weakly sequentially compact in L^2 (cf. [Dunford and Schwartz, Cor. IV.8.4]). By definition, there exist a function $g \in L^2$ and a subsequence $\{n_k\}_{k \in \mathbb{N}}$ such that $R^{n_k} f \xrightarrow{w} g$ as $k \to \infty$.

Moreover, $g \in L_i \cap L^2$. Indeed, L_i is weakly-closed, since it is convex and closed (see, e.g., [Werner, Th. III.3.8]). It is convex as a linear subspace of L^1. To see why it is closed, consider $\{f_l 1_{A_i}\} \in L_i$ and $h \in L^1$ with $||f_l 1_{A_i} - h||_1 \to 0$. Then

$$||h 1_{A_i^c}||_1 = ||(f_l 1_{A_i} - h) 1_{A_i^c}||_1 \to 0 \quad \text{as } l \to \infty, \tag{3.3.48}$$

so h is supported almost only on A_i. It follows, first, that

$$\lim_{n\to\infty}\langle R^{n_k}f,g\rangle = \langle g,g\rangle = ||g||_2^2, \tag{3.3.49}$$

so

$$\begin{aligned}\lim_{k\to\infty}||R^{n_k}f||_2^2 &= \lim_{k\to\infty}||R^{n_k}f-g||_2^2 + 2\lim_{k\to\infty}\langle R^{n_k}f,g\rangle - \langle g,g\rangle\\ &= \lim_{k\to\infty}||R^{n_k}f-g||_2^2 + ||g||_2^2.\end{aligned} \tag{3.3.50}$$

Second, for every fixed $m\in\mathbb{N}$ and $h\in L^2$,

$$\langle R^{n_k+m}f,h\rangle = \langle R^{n_k}f,R^{*m}h\rangle \to \langle g,R^{*m}h\rangle = \langle R^m g,h\rangle \ \text{ as } k\to\infty, \tag{3.3.51}$$

so $R^{n_k+m}f \xrightarrow{w} R^m g$ and, using (3.3.50) and lemma 3.3.10,

$$\begin{aligned}\lim_{k\to\infty}||R^{n_k+m}f||_2^2 &= \lim_{k\to\infty}||R^{n_k+m}f-R^m g||_2^2 + ||R^m g||_2^2\\ &\leqslant \lim_{k\to\infty}||R^{n_k}f-g||_2^2 + ||R^m g||_2^2\\ &= \lim_{k\to\infty}||R^{n_k}f||_2^2 - ||g||_2^2 + ||R^m g||_2^2,\end{aligned} \tag{3.3.52}$$

which, by virtue of (3.3.28), is equivalent to

$$\langle Af,f\rangle \leqslant \langle Af,f\rangle - ||g||_2^2 + ||R^m g||_2^2. \tag{3.3.53}$$

By lemma 3.3.10, for every $m\in\mathbb{N}$, we have

$$||R^m g||_2 = ||g||_2 \tag{3.3.54}$$

and, together with (3.3.33), it yields

$$g\in Q\cap L_i. \tag{3.3.55}$$

It follows that g is constant on A_i, $g=\tilde{\lambda}_i(f)1_{A_i}$ for some $\tilde{\lambda}_i(f)\in\mathbb{R}$. For all $m\in\mathbb{N}$, $R^m g = g$, so it follows from the fact that $R^{n_k+m}f \xrightarrow{w} R^m g$ for all $m\in\mathbb{N}$, that $R^n f\xrightarrow{w} g$ as $n\to\infty$. Further,

$$\begin{aligned}\tilde{\lambda}_i(f)\cdot\mu(A_i) &= \int_X \tilde{\lambda}_i(f)1_{A_i}d\mu = \langle g,1_{A_i}\rangle = \lim_{k\to\infty}\langle R^{n_k}f,1_{A_i}\rangle\\ &= \lim_{k\to\infty}\langle f,R^{*n_k}1_{A_i}\rangle = \langle f,1_{A_i}\rangle = \int_{A_i} f d\mu,\end{aligned} \tag{3.3.56}$$

where we used corollary 3.3.12 in the last but one step. Thus, for all $f\in L_i\cap L^2$, we have

$$R^n f \xrightarrow{w} \tilde{\lambda}_i(f)1_{A_i} = \lambda_i(f)\frac{1_{A_i}}{\mu(A_i)} \quad \text{as } n\to\infty. \tag{3.3.57}$$

Finally, we show that this weak convergence holds for all $f\in L_i$. Define for every $f\in L^1$,

$$Sf = \lambda_i(f)\frac{1_{A_i}}{\mu(A_i)} \tag{3.3.58}$$

and $T_n f = R^n f - Sf$ for all $n \in \mathbb{N}$. Since $||R^n|| \leqslant 1$ and $||S|| = 1$, $||T_n|| \leqslant 2 < \infty$.

Now, similarly to part 1), we use the density argument. Since L^2 is dense in L^1, $L_i \cap L^2$ is dense in L_i. For every $f \in L_i$, there is a sequence $\{f_j\} \in L_i \cap L^2$ with $||f - f_j||_1 \to 0$ as $j \to \infty$. By definition of the weak convergence in the L^2 space, for every $j \in \mathbb{N}$,

$$\int_X |v \cdot T_n f_j| d\mu \to 0 \quad \text{as } n \to \infty \tag{3.3.59}$$

for all $v \in L^2$, in particular for all $v \in L^\infty$, since $\mu(X) = 1 < \infty$ and so $L^\infty(X) \in L^2(X)$.

For the weak convergence in the L^1 space, we need to consider all $v \in L^\infty$. It holds

$$\begin{aligned}\int_X |v \cdot T_n f| d\mu &\leqslant \int_X |v \cdot T_n(f - f_j)| d\mu + \int_X |v \cdot T_n f_j| d\mu \\ &\leqslant ||v||_\infty \cdot ||T_n(f - f_j)||_1 + \int_X |v \cdot T_n f_j| d\mu \\ &\leqslant 2||v||_\infty \cdot ||(f - f_j)||_1 + \int_X |v \cdot T_n f_j| d\mu \to 2||v||_\infty \cdot ||(f - f_j)||_1 \end{aligned} \tag{3.3.60}$$

as $n \to \infty$ by (3.3.59). Since j can be chosen so that $||(f - f_j)||_1$ is arbitrarily small, we have that $T_n f = R^n f - Sf \xrightarrow{w} 0$ by definition. □

3.3.3 Strong convergence in case $P1_X = 1_X$, $\mu(X) = 1$

In this subsection, we prove Theorem 3.3.1 in case $P1_X = 1_X$ and $\mu(X) = 1$. We start with lemma 3.3.16 that will be used in the proof.

Definition 3.3.14. *For two functions $f, g : X \to \mathbb{R}$ we define*

$$(f \wedge g)(x) = \min\{f(x), g(x)\}. \tag{3.3.61}$$

Definition 3.3.15. *Nonnegative functions $f_1, \dots, f_k \in L^1$ are called* ρ-**orthogonal** *if there exist nonngeative functions $h_1, \dots, h_k \in L^1$ with mutually disjoint supports such that $||f_i - h_i||_1 \leqslant \rho$ for all $i = 1, \dots, n$.*

Lemma 3.3.16. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Let $P : L^1 \to L^1$ be a Markov operator such that $P1_X = 1_X$ and for every $f \in L^1$,*

$$P^n f \xrightarrow{w} \int_X f d\mu. \tag{3.3.62}$$

If for some $\tilde{f} \in L^1$ the strong limit of $P^n \tilde{f}$ does not exist, then for every $\rho > 0$ and every $N \in \mathbb{N}$, there exists a sequence of densities $f_1, \dots, f_N$ such that $P^n f_1, \dots, P^n f_N$ are ρ-orthogonal for every fixed $n \in \mathbb{N}$.

The proof of this lemma is a compilation of different results from [Komornik]. First, we summarize some helpful statements in

Lemma 3.3.17. *(i) Let $f_1, \dots, f_k \in L^1$ be ρ-orthogonal and $||f_i|| \geqslant \epsilon > 0$. Then*

$$\frac{f_1}{||f_1||}, \dots, \frac{f_k}{||f_k||} \tag{3.3.63}$$

are ρ/ϵ-orthogonal.

(ii) *Let* $f_{1,1}, \ldots, f_{1,k_1} \in L^1$ *be* ρ_1*-orthogonal and* $f_{2,1}, \ldots, f_{2,k_1} \in L^1$ *be* ρ_2*-orthogonal. Then* $k_1 \cdot k_1$ *functions* $f_{1,i_1} \wedge f_{2,i_2}$ *with* $i_1 \in \{1, \ldots, k_1\}$ *and* $i_2 \in \{1, \ldots, k_1\}$ *are* $\rho_1 + \rho_2$*-orthogonal.*

(iii) *Let* $\{f_{i,1}, f_{i,2}\}_{i \in \{1,\ldots,s\}} \subset L^1$ *be* ρ*-orthogonal pairs of functions. Then the* 2^s *functions* $f_{1,i_1} \wedge f_{2,i_2} \wedge \cdots \wedge f_{s,i_s}$ *with* $i_l \in \{1, 2\}$ *for all* $l \in \{1, \ldots, s\}$ *are* $s \cdot \rho$*-orthogonal.*

(iv) *Let* $f_1, \ldots, f_k$ *be nonnegative* L^1 *functions with* $||f_i||_\infty \leqslant M_0$ *for some* $M_0 > 0$ *and all* $i \in \{1, \ldots, k\}$*. Then*

$$E[f_1 \wedge f_2 \wedge \cdots \wedge f_k] \geqslant E[f_1 \cdot f_2 \cdots f_k]/M_0^{s-1} \tag{3.3.64}$$

and the sequence

$$\{E[P^n f_1 \wedge P^n f_2 \wedge \cdots \wedge P^n f_k]\}_{n=1}^{\infty}, \tag{3.3.65}$$

is nondecreasing in n*.*

Proof. (i) Let $h_1, \ldots, h_k$ be nonnegative L^1 functions with mutually disjoint supports and such that $||f_i - h_i||_1 \leqslant \rho$ for all $i = 1, \ldots, n$. Clearly, $h_1/||f_1||, \ldots, h_k/||f_k||$ have disjoint supports and

$$\left\| \frac{f_i}{||f_i||} - \frac{h_i}{||f_i||} \right\| = \frac{||f_i - h_i||}{||f_i||} \leqslant \frac{\rho}{\epsilon}. \tag{3.3.66}$$

(ii) Let $h_{1,1}, \ldots, h_{1,k_1}$ and $h_{2,1}, \ldots, h_{2,k_1}$ be two groups of L^1 functions with disjoint supports corresponding to $f_{1,1}, \ldots, f_{1,k_1}$ and $f_{2,1}, \ldots, f_{2,k_1}$ respectively by the definition of ρ-orthogonality. Clearly, the $k_1 \cdot k_1$ functions $h_{1,i_1} \wedge h_{2,i_2}$ with $i_1 \in \{1, \ldots, k_1\}$ and $i_2 \in \{1, \ldots, k_1\}$ have disjoint supports.

Further, for all $a, b, c \in \mathbb{R}$,

$$|a \wedge c - b \wedge c| \leqslant |a - b|, \tag{3.3.67}$$

so

$$\begin{aligned} ||f_{1,i_1} \wedge f_{2,i_2} - h_{1,i_1} \wedge h_{2,i_2}|| &\leqslant ||f_{1,i_1} \wedge f_{2,i_2} - f_{1,i_1} \wedge h_{2,i_2}|| + ||f_{1,i_1} \wedge h_{2,i_2} - h_{1,i_1} \wedge h_{2,i_2}|| \\ &\leqslant ||f_{2,i_2} - h_{2,i_2}|| + ||f_{1,i_1} - h_{1,i_1}|| \leqslant \rho_1 + \rho_2. \end{aligned} \tag{3.3.68}$$

(iii) follows from (ii) by induction.

(iv) Since $0 \leqslant f_i/M_0 \leqslant 1_X$ for all $i \in \{1, \ldots, n\}$,

$$\frac{f_1}{M_0} \cdot \frac{f_2}{M_0} \cdots \frac{f_k}{M_0} \leqslant \frac{f_1}{M_0} \wedge \frac{f_2}{M_0} \wedge \ldots \frac{f_k}{M_0} = \frac{f_1 \wedge f_2 \wedge \ldots f_k}{M_0}. \tag{3.3.69}$$

Further, for all $m \in \mathbb{N}$ and all $i = 1, \ldots, n$,

$$P^m(P^n f_1 \wedge \cdots \wedge P^n f_k) \leqslant P^{m+n} f_i \tag{3.3.70}$$

so

$$P^m(P^n f_1 \wedge \cdots \wedge P^n f_k) \leqslant P^{m+n} f_1 \wedge \cdots \wedge P^{m+n} f_k \tag{3.3.71}$$

and the second statement follows from the fact that P preserves mean values (see Lemma 3.3.10). □

Proof of Lemma 3.3.16. W.l.o.g. we may assume that $\tilde{f} \in L^\infty$. If it is not, there exists another function $\tilde{f}' \in L^\infty$ for which the strong limit of $P^n\tilde{f}'$ does not exist. Indeed, L^∞ is dense in L^1 (see the proof of proposition 3.3.13), so there exists a sequence of functions $\{f_i\}_{i\in\mathbb{N}} \subset L^\infty$ with $||\tilde{f} - f_i||_1 \to 0$. If for all i the strong limit of $P^n f_i$ exists, then for some fixed i,

$$\begin{aligned}\lim_{n\to\infty}\left\|P^n\tilde{f} - \int_X \tilde{f}d\mu\right\| &\leqslant \lim_{n\to\infty}\left(\left\|P^n\tilde{f} - P^n f_i\right\| + \left\|P^n f_i - \int_X \tilde{f}d\mu\right\|\right)\\ &\leqslant \lim_{n\to\infty}\left(\left\|P^n f_i - \int_X f_i d\mu\right\| + \left\|\int_X f_i d\mu - \int_X \tilde{f}d\mu\right\|\right)\\ &= \left\|\int_X f_i d\mu - \int_X \tilde{f}d\mu\right\| \qquad (3.3.72)\end{aligned}$$

is arbitrarily small which contradicts the absence of the strong limit for $P^n\tilde{f}$.

Let

$$M_0 = ||\tilde{f}||_\infty. \qquad (3.3.73)$$

Let $\rho > 0$ and $N \in \mathbb{N}$ be given. We will construct the sequence $f_1, \dots, f_N$ in 3 steps.

Step I. Define

$$\lambda = \int_X \tilde{f}d\mu \qquad (3.3.74)$$

and

$$M_1 = \frac{1}{2}\lim_{n\to\infty}||R^n\tilde{f} - \lambda||, \qquad (3.3.75)$$

where $R = P^{\tilde{r}}$ as in the previous section and the limit always exists, since

$$||R^n\tilde{f} - \lambda|| = ||R^n(\tilde{f} - \lambda)|| \qquad (3.3.76)$$

is nonincreasing in n and bounded from below by 0. We have $M_1 > 0$, since otherwise $P^n f$ would have the strong limit λ.

Define $m \in \mathbb{N}$ so that

$$M_1 \leqslant \frac{1}{2}||R^m\tilde{f} - \lambda|| \leqslant M_1 + \frac{\rho}{x} \qquad (3.3.77)$$

for some fixed $x > 0$, whose exact value will be defined in the end of the proof.

For all $l \in \mathbb{N}_0$, define

$$e_{1,l} = \left(R^{m+l}\tilde{f} - \lambda\right)^+ \quad \text{and} \quad e_{2,l} = \left(R^{m+l}\tilde{f} - \lambda\right)^- \qquad (3.3.78)$$

as well as

$$g_{1,l} = R^l e_{1,0} = R^l\left(\left(R^m\tilde{f} - \lambda\right)^+\right) \quad \text{and} \quad g_{2,l} = R^l e_{2,0} = R^l\left(\left(R^m\tilde{f} - \lambda\right)^-\right). \qquad (3.3.79)$$

Let

$$s = \lfloor \log_2 N \rfloor + 1. \qquad (3.3.80)$$

For $i \in \{1,2\}^s$ and $l \in \mathbb{N}_0^s$ with $l_j < l_{j+1}$ for all $j = 1, \dots, s-1$, define

$$h_{i,l} = e_{i_1,l_1} \wedge e_{i_2,l_2} \wedge \cdots \wedge e_{i_s,l_s} \qquad (3.3.81)$$

and

$$f_{i,l} = g_{i_1,l_1} \wedge g_{i_2,l_2} \wedge \cdots \wedge g_{i_s,l_s}. \tag{3.3.82}$$

We now show that $\{f_{i,l}\}_{i\in\{1,2\}^s}$ are $s \cdot \rho/x$-orthogonal for a fixed l. By lemma 3.3.17(iii), it is enough to show that $g_{1,l}$ and $g_{2,l}$ are ρ/x-orthogonal for a fixed $l \in \mathbb{N}$.

Clearly, $e_{1,l}$ and $e_{2,l}$ have disjoint supports. Further,

$$\begin{aligned} 0 = E\left[\tilde{f} - \lambda\right] &= E\left[R^m(\tilde{f} - \lambda)\right] = E\left[R^m\tilde{f} - \lambda\right] \\ &= E[e_{1,l} - e_{2,l}] = E[e_{1,l}] - E[e_{2,l}] = ||e_{1,l}|| - ||e_{2,l}||, \end{aligned} \tag{3.3.83}$$

so

$$||e_{1,l}|| = ||e_{1,l}|| = \frac{1}{2}||R^m\tilde{f} - \lambda||. \tag{3.3.84}$$

By monotonicity of P and, thus, R,

$$\left(R^{m+l}\tilde{f} - \lambda\right) = R^l\left(R^m\tilde{f} - \lambda\right) \leqslant R^l\left(\left(R^m\tilde{f} - \lambda\right)^+\right), \tag{3.3.85}$$

so

$$e_{1,l} \leqslant g_{1,l}. \tag{3.3.86}$$

Analogously,

$$\left(\lambda - R^{m+l}\tilde{f}\right) = R^l\left(\lambda - R^m\tilde{f}\right) \leqslant R^l\left(\left(R^m\tilde{f} - \lambda\right)^-\right), \tag{3.3.87}$$

so

$$e_{2,l} \leqslant g_{2,l}. \tag{3.3.88}$$

Together with inequality (3.3.77), we get for $j \in \{1,2\}$

$$||g_{j,l} - e_{j,l}|| = E[g_{j,l}] - E[e_{j,l}] \leqslant \rho/x. \tag{3.3.89}$$

Step II. Recall that $f_1, \ldots, f_N$ should be densities. The functions $\{f_{i,l}\}_{i\in\{1,2\}^s}$, divided by their L^1 norms, are good candidates, but we need to be sure that these norms are bounded from below. For this, we will specify $l \in \mathbb{N}_0^s$.

If there exists an $\tilde{l} \in \mathbb{N}_0^s$ such that for all $i \in \{1,2\}$

$$E\left[g_{i_1,\tilde{l}_1} \cdot g_{i_2,\tilde{l}_2} \cdots g_{i_s,\tilde{l}_s}\right] \geqslant M_1^s, \tag{3.3.90}$$

then, by lemma 3.3.17, (iv) and (i),

$$\left\{\frac{f_{i,\tilde{l}}}{||f_{i,\tilde{l}}||}\right\}_{i\in\{1,2\}^s} \quad \text{are} \quad \frac{M_0^s \cdot s \cdot \rho}{M_1^s \cdot x}\text{-orthogonal}. \tag{3.3.91}$$

Such an $\tilde{l}$ indeed exists. Since all $\{g_{i_j,\tilde{l}_j}\}_{j=1,\ldots,s}$ are in L^∞ (and $\mu(X) = 1 < \infty$), they are also in L^2. Proposition 3.3.13 implies that, for a fixed $l \in \mathbb{N}_0^s$,

$$\begin{aligned} \lim_{n\to\infty} E\left[g_{i_1,l_1} \cdot g_{i_2,l_2+n}\right] &= \lim_{n\to\infty} E\left[g_{i_1,l_1} \cdot R^n g_{i_2,l_2}\right] = \lim_{n\to\infty} E\left[g_{i_1,l_1} \cdot \sum_{j=1}^r R^n\left(g_{i_2,l_2} \cdot 1_{A_j}\right)\right] \\ &= \sum_{j=1}^r E\left[g_{i_1,l_1} \cdot 1_{A_j} \frac{\int_{A_j} g_{i_2,l_2} d\mu}{\mu(A_j)}\right] = E\left[g_{i_1,l_1}\right] \cdot E\left[g_{i_2,l_2}\right] \end{aligned} \tag{3.3.92}$$

and, by induction,

$$\lim_{n_1,\ldots,n_s\to\infty} E\left[g_{i_1,l_1+n_1}\cdot g_{i_2,l_s+n_2}\cdots g_{i_s,l_s+n_s}\right] = E\left[g_{i_1,l_1}\right]\cdot E\left[g_{i_2,l_2}\right]\cdots E\left[g_{i_s,l_s}\right]$$
$$= \left(\frac{||R^m\tilde{f}-\lambda||}{2}\right)^s \geqslant M_1^s. \tag{3.3.93}$$

It follows that for all $l \geqslant \tilde{l}$, i.e., l with $l_j \geqslant \tilde{l}_j$ for all $j \in \{1,\ldots,s\}$,

$$\left\{\frac{f_{i,l}}{||f_{i,l}||}\right\}_{i\in\{1,2\}^s} \quad \text{are} \quad \frac{M_0^s\cdot s\cdot\rho}{M_1^s\cdot x}\text{-orthogonal.} \tag{3.3.94}$$

Step III. Now we need to ensure that $P^n f_1,\ldots,P^n f_N$ are ρ-orhtogonal for all $n\in\mathbb{N}$. We fix n and write $\tilde{l}+n$ for $(\tilde{l}_1+n,\ldots,\tilde{l}_s+n)$. Clearly, $\{h_{j,\tilde{l}}\}_{j\in\{1,2\}^s}$ have disjoint supports. Further, for all $i\in\{1,2\}^s$,

$$\left\|P^n f_{i,\tilde{l}} - h_{i,\tilde{l}+n}\right\| \leqslant \left\|P^n f_{i,\tilde{l}} - f_{i,\tilde{l}+n}\right\| + \left\|f_{i,\tilde{l}+n} - h_{i,\tilde{l}+n}\right\|. \tag{3.3.95}$$

It was already shown in step I that the second norm on the r.h.s. is not larger than $s\cdot\rho/x$. For the first norm, notice that

$$P^n f_{i,\tilde{l}} \leqslant P^n g_{i_j,\tilde{l}_j} \quad \text{for all } j=1,\ldots,s, \tag{3.3.96}$$

so

$$P^n f_{i,\tilde{l}} \leqslant P^n g_{i_1,\tilde{l}_1} \wedge\cdots\wedge P^n g_{i_s,\tilde{l}_s} = f_{i,\tilde{l}+n} \tag{3.3.97}$$

and

$$\left\|P^n f_{i,\tilde{l}} - f_{i,\tilde{l}+n}\right\| = E\left[f_{i,\tilde{l}+n}\right] - E\left[P^n f_{i,\tilde{l}}\right]$$
$$= E\left[f_{i,\tilde{l}+n}\right] - E\left[f_{i,\tilde{l}}\right]. \tag{3.3.98}$$

From lemma 3.3.17(iv) we know that $\left\{E\left[f_{i,\tilde{l}+n}\right]\right\}_{n\in\mathbb{N}}$ is nondecreasing in n. Since $\mu(X)=1$, this sequence is bounded by M_0. Define $\tilde{m}\in\mathbb{N}$ so that

$$E\left[f_{i,\tilde{l}+\tilde{m}+n}\right] - E\left[f_{i,\tilde{l}+\tilde{m}}\right] \leqslant s\cdot\rho/x \tag{3.3.99}$$

and, consequently,

$$\left\|P^n f_{i,\tilde{l}+\tilde{m}} - h_{i,\tilde{l}+\tilde{m}+n}\right\| \leqslant \frac{2s\cdot\rho}{x}. \tag{3.3.100}$$

We can finally define

$$x = \frac{M_1^s}{2M_o^s\cdot s} \tag{3.3.101}$$

and

$$f_i = \frac{f_{i,\tilde{l}+\tilde{m}}}{||f_{i,\tilde{l}+\tilde{m}}||} \quad \text{for all } i\in\{1,2\}^s. \tag{3.3.102}$$

These are $2^s \geqslant N$ functions, which are ρ-orthogonal. □

Now we give the

Proof of Theorem 3.3.1 in case $P1_X = 1_X$ and $\mu(X) = 1$. For all $i = 1, \ldots, r$ and $f \in L^1$, define

$$\lambda_i(f) = \int_{A_i} f d\mu \quad \text{and} \tag{3.3.103}$$

$$g_i = \frac{1_{A_i}}{\mu(A_i)}. \tag{3.3.104}$$

Note that we use the same definition of $\lambda_i(f)$ as in proposition 3.3.13.

Clearly, g_i's are densities and statements (i) and (ii) were already proved in Subsection 3.3.1 (remember that $P1_{A_i} = 1_{A_{\sigma(i)}}$ implies $\mu(A_i) = \mu(A_{\sigma(i)})$).

Fix $n \in \mathbb{N}$ and find $k, m \in \mathbb{N}$, $0 \leqslant m < \tilde{r}$, with $n = k\tilde{r} + m$. Then, by propositions 3.1.2, 3.3.13 and the triangle inequality,

$$\begin{aligned}\left\| P^n \left(f - \sum_{i=1}^{r} \lambda_i(f) g_i \right) \right\| &= \left\| P^m R^k \left(\sum_{i=1}^{r} f \cdot 1_{A_i} - \sum_{i=1}^{r} \lambda_i(f) 1_{A_i} \right) \right\| \\ &\leqslant \sum_{i=1}^{r} \left\| R^k \left(f 1_{A_i} - \lambda_i(f) 1_{A_i} \right) \right\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.3.105}$$

Thus, it remains to show that convergence in (3.3.44) is strong.

Suppose that for some $i \in \{1, \ldots, r\}$ and some function $f_i \in L_i$, convergence in (3.3.44) is not strong. Clearly, since P is constrictive, so is R with the same $\delta > 0$ and $\kappa < 1$. Fix some $0 < \rho < (1 - \kappa)/2$ and $N > 1/\delta$. Setting $X = A_i$, we can apply lemma 3.3.16 by virtue of (3.3.44). Thus, there exist N densities $f_1, \ldots, f_N \in L_i$ such that $R^n f_1, \ldots, R^n f_N$ are ρ-orthogonal for every fixed $n \in \mathbb{N}$. By constrictivness, for every $k = 1, \ldots, N$, there exist an $n_0(f_k) \in \mathbb{N}$ with

$$\int_E R^n f_k d\mu \leqslant \kappa \quad \text{for all } n \geqslant n_0(f_k) \quad \text{if } \mu(E) \leqslant \delta. \tag{3.3.106}$$

Fix some $n \geqslant n_0$. By ρ-orhogonality, there exist nonnegative functions $h_1, \ldots, h_N \in L_i$ with disjoint supports and $||R^n f_i - h_i|| \leqslant \rho$ for all $i = 1, \ldots, N$. For every $k = 1, \ldots, N$, $\mu(E) \leqslant \delta$,

$$\int_E h_k d\mu = \int_E |h_k| d\mu = \int_E |R^n f_k| d\mu + \int_E |R^n f_k - h_k| d\mu \leqslant \kappa + \rho. \tag{3.3.107}$$

Since $N > 1/\delta$, μ is a probability measure and h_k's have disjoint supports, there is some $k \in \{1, \ldots, N\}$ with $\mu(supp(h_k)) \leqslant \delta$. For $E = supp(h_k)$ we get

$$||h_k|| \leqslant \kappa + \rho < 1 - \rho. \tag{3.3.108}$$

At the same time, since $R^n f_k$ is a density,

$$1 - ||h_k|| = ||R^n f_k|| - ||h_k|| \leqslant ||R^n f_k - h_k|| \leqslant \rho. \tag{3.3.109}$$

We get a contradiction, so for all $f \in L_i$, $i = 1, \ldots, r$, the convergence is strong. $\square$

3.3.4 Proof of the general case

Here we release both condition $\mu(X) = 0$ and condition $P1_X = 1_X$. The strategy will be to reduce the general case to the one already proved in the previous subsection. We first prove lemma 3.3.20, which states the existence of an invariant density for a constrictive Markov operator.

Recall that D is a set of densities on X.

Definition 3.3.18. *Let $P : L^1 \to L^1$ be a Markov operator on a σ-finite measure space $(X, \mathcal{A}, \mu)$. An invariant density $g \in D$ of P is said to have* **maximal support** *if*

$$\mu(supp(f)\backslash supp(g)) = 0 \tag{3.3.110}$$

for every invariant density $f \in L^1$ for P.

Lemma 3.3.19. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Let $P : L^1 \to L^1$ be a Markov operator. If there exist a set $\tilde{B} \in \mathcal{A}$ of finite measure and a number $\delta > 0$ such that for every $E \subset \tilde{B}$ with $\mu(E) < \delta$*

$$\limsup_{n\to\infty} \int_{(X\backslash\tilde{B})\cup E} P^n f d\mu < 1, \tag{3.3.111}$$

then P has an invariant density nonvanishing on $\tilde{B}$.

The proof of this lemma can be found in [Socala, Th.1].

Lemma 3.3.20. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. For every constrictive Markov operator $P : L^1 \to L^1$, there exists an invariant density with maximal support.*

Proof. First, we note that a constrictive Markov operator satisfies condition (3.3.111), since it is just weaker than constrictivness. Thus, P has at least one invariant density.

Since X is σ-finite, there exists a density f_0 with $f_0(x) > 0$ for all $x \in X$. Denote $A_f = supp(f)$, define a new measure

$$\mu_0(A) = \int_A f_0 d\mu \tag{3.3.112}$$

for every $A \in \mathcal{A}$, and a number

$$M = \sup\left\{\mu_0(A_f) : f \in D, Pf = f\right\}. \tag{3.3.113}$$

Choose a sequence $\{f_n\} \in D$ with $\mu_0(A_{f_n}) \to M$ and define

$$g = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n. \tag{3.3.114}$$

By the monotone convergence theorem, g is an invariant density and then $\mu_0(A_g) = M$.

Now, take some arbitrary invariant density f. Clearly, $h = \frac{1}{2}(f + g)$ is also invariant and $A_h = A_g \cup A_f$. At the same time $\mu_0(A_h) \leqslant M$, so $A_h = A_g$. We get $A_f \subset A_h = A_g$, which yields that the support of g is maximal. $\square$

Clearly, the maximal support corresponding to the operator P is unique up to a set of measure 0. Denote it by $G(P)$.

Lemma 3.3.21. *For a constrictive Markov operator $P : L^1 \to L^1$ on a σ-finite measure space $(X, \mathcal{A}, \mu)$, it holds*

$$\lim_{n\to\infty} \int_{X\backslash G(P)} P^n f d\mu = 0 \quad \textit{for every } f \in D. \tag{3.3.115}$$

Proof. Let B and κ come from the definition of constrictivness, i.e., $\mu(B) < \infty$, $\kappa < 1$ and for some $\delta > 0$ and every density $f \in D$, there is an $n_0(f) \in \mathbb{N}$ for which

$$\int_{(X\backslash B)\cup E} P^n f d\mu \leqslant \kappa \quad \text{for all } n \geqslant n_0(f),\ \mu(E) \leqslant \delta \tag{3.3.116}$$

First, we show that for every $f \in D$

$$\liminf_{n\to\infty} \int_{X\backslash G(P)} P^n f d\mu \leqslant \kappa. \tag{3.3.117}$$

Define $\tilde{B} = B\backslash G(P)$ and fix some $f \in D$. If condition (3.3.111) would hold for this $\tilde{B}$, there would exist an invariant density nonvanishing on $\tilde{B}$, which is impossible, since $G(P)$ is the maximal support. It follows that there exists some $E \subset B$ with $\mu(E) \leqslant \delta$ such that

$$\limsup_{n\to\infty} \int_{(X\backslash\tilde{B})\cup E} P^n f d\mu = 1 \tag{3.3.118}$$

or

$$\liminf_{n\to\infty} \int_{\tilde{B}\backslash E} P^n f d\mu = 0. \tag{3.3.119}$$

Since $X\backslash G(P) \subset ((X\backslash B) \cup E) \cup (\tilde{B}\backslash E)$, (3.3.116) and (3.3.119) imply (3.3.117).

Now define

$$a = \inf\left\{ b \in \mathbb{R} : \liminf_{n\to\infty} \int_{X\backslash G(P)} P^n f d\mu \leqslant b \quad \text{for } f \in D \right\}. \tag{3.3.120}$$

We already know that $b < 1$. Choose some $0 < \epsilon < 1 - a$ and fix $f \in D$. For some $k \in \mathbb{N}$, we have

$$\int_{P^k} f d\mu \leqslant a + \epsilon. \tag{3.3.121}$$

Further, since $supp\left(1_{G(P)}P^k f\right) \subset supp(g)$ and P is monotonic, it holds for all $n \in \mathbb{N}$

$$supp\left[P^n\left(1_{G(P)}P^k f\right)\right] \subset supp(P^n g) = supp(g) = G(P). \tag{3.3.122}$$

It follows

$$\begin{aligned} \limsup_{n\to\infty} \int_{X\backslash G(P)} P^{n+k} f d\mu &= \liminf_{n\to\infty} \int_{X\backslash G(P)} P^n\left(1_{X\backslash G(P)}P^k f\right) d\mu \\ &= \left\|1_{X\backslash G(P)}P^k f\right\| \cdot \liminf_{n\to\infty} \int_{X\backslash G(P)} P^n h_k d\mu \leqslant (a+\epsilon)a, \end{aligned} \tag{3.3.123}$$

where $h_k = (1_{X\backslash G(P)}P^k f)/\left\|1_{X\backslash G(P)}P^k f\right\|$. Since f was arbitrary, we get $a \leqslant (a+\epsilon)a$ and, thus, $a = 0$. Since every Markov operator is a contraction on L^1, the sequence in (3.3.117) is nonincreasing and limit inferior is just a limit. $\square$

Proof of theorem 3.3.1. The proof is organized as follows: first, we find the space $(X, \mathcal{A}, \bar{\mu})$ and the operator $\bar{P}$ for which $\bar{\mu}(X) = 1$ and $\bar{P}1_X = 1_X$ and the desired convergence holds. This implies similar convergence under P for a special class of functions. Second, we follow the convergence for arbitrary $f \in L^1$, but some unknown $\lambda_i(f)$'s and special g_i's. Finally, we show that these $\lambda_i(f)$'s and g_i's have the properties stated in the theorem.

Step I. By lemma 3.3.20, there exists an invariant density $g \in D$ for P with maximal support G. By lemma 3.3.21, for every $f \in L^1$,

$$\lim_{n\to\infty} \int_{X\setminus G} P^n f d\mu = 0. \tag{3.3.124}$$

Define a new measure $\bar{\mu}$ on X through

$$\bar{\mu}(A) = \int_A g d\mu \tag{3.3.125}$$

for all $A \in \mathcal{A}$, and an operator $\bar{P}$ through

$$\bar{P}h(x) = \begin{cases} P(hg)(x)/g(x), & \text{if } x \in G \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.126}$$

Clearly, $(X, \mathcal{A}, \bar{\mu})$ is a probability space and $\bar{P}$ is a Markov operator on $(X, \mathcal{A}, \bar{\mu})$ with $\bar{P}1_X = 1_X$. Moreover, $\bar{P}$ is constrictive. To see this, notice first that the measure μ restricted to G is absolutely continuous with respect to $\bar{\mu}$, since $g(x) > 0$ for $x \in G$. Now let B, κ and δ come from the definition of constrictivness of the operator P. Choose $\bar{\delta} > 0$ such that $\bar{\mu}(E) \leqslant \bar{\delta}$ would imply $\mu(E \cap G) \leqslant \delta$. It follows that there exists some $n_0(hg) \in \mathbb{N}$ such that for all $h \in L^1$

$$\int_{(X\setminus B)\cup E} \bar{P}^n h d\bar{\mu} = \int_{(X\setminus B)\cup(E\cap G)} P^n(hg) d\mu \leqslant \kappa \;\; \text{for all } n \geqslant n_0(hg),\;\; \bar{\mu}(E) \leqslant \bar{\delta}, \tag{3.3.127}$$

which is constrictivness of $\bar{P}$.

By the proof in Subsection 3.3.3, we know that equation (3.3.1) holds for $\bar{P}$ on $(X, \mathcal{A}, \bar{\mu})$. Together with (3.3.104), it follows that for all $h \in L^1$

$$\bar{P}^n h = \sum_{i=1}^{r} \lambda_i(h) \frac{1_{A_{\sigma^n(i)}}}{\bar{\mu}(A_{\sigma^n(i)})} + \epsilon_n(h), \tag{3.3.128}$$

for the corresponding nice sets A_i, $i = 1, \ldots, r$, and some $\{\epsilon_n\} \in L^1$ with $||\epsilon_n(h)||_{L^1(\bar{\mu})} \to 0$ as $n \to 0$.

Now, straightforward induction yields

$$\bar{P}^n h(x) = \begin{cases} P^n(hg)(x)/g(x), & \text{if } x \in G \\ 0 & \text{otherwise,} \end{cases} \tag{3.3.129}$$

which implies

$$P^n(hg) = \sum_{i=1}^{r} \lambda_i(h) g_{\sigma(i)} + g\epsilon_n(h), \tag{3.3.130}$$

where $g_i = g\frac{1_{A_i}}{\bar{\mu}(A_i)}$ and

$$||g\epsilon_n(h)||_{L^1(\mu)} = \int_X g\epsilon_n(h)d\mu = \int_X \epsilon_n(h)d\bar{\mu} = ||\epsilon_n(h)||_{L^1(\bar{\mu})} \to 0, \tag{3.3.131}$$

so for all functions of the form $hg \in L^1$ we get a decomposition similar to the one we need.

Step II. We want to consider an arbitrary $f \in L^1$. Fix $\epsilon > 0$ and take the nonnegative density f_0 from the proof of lemma 3.3.20. Choose a number $c_1 > 0$ large enough for

$$\left\|(c_1 f_0 - |f|)^+\right\| < \frac{\epsilon}{4} \tag{3.3.132}$$

to hold. Then for $q_1 = (c_1 f_0 - |f|)^+ \in L^1$ we have that $|f| \leqslant c_1 f_0 + q_1$ pointwise. Further, by lemma 3.3.21, there exists an $m \in \mathbb{N}$ such that

$$\int_{X\setminus G} P^m f d\mu \leqslant \frac{\epsilon}{4c_1}. \tag{3.3.133}$$

Then, by monotonicity of P,

$$\begin{aligned}\int_{X\setminus G} |P^m f| d\mu &\leqslant \int_{X\setminus G} P^m |f| d\mu \\ &\leqslant c_1 \int_{X\setminus G} P^m f_0 d\mu + \int_{X\setminus G} P^m q_1 d\mu \leqslant \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.\end{aligned} \tag{3.3.134}$$

Similarly, choose a number $c_2 > 0$ large enough for

$$\left\|(c_2 g - 1_G |P^m f|)^+\right\| < \frac{\epsilon}{4} \tag{3.3.135}$$

to hold. Then for $q_2 = (c_2 g - 1_G|P^m f|)^+ \in L^1$ we have that $supp(q_2) \subset G$ and $1_G|P^m f| \leqslant c_2 g + q_2$ pointwise. Define

$$h(x) = \begin{cases} (1_G \cdot P^m f - q_2)/g(x), & \text{if } x \in G \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.136}$$

Then $1_G \cdot P^m f = hg + q_2$ and

$$\begin{aligned} P^m f &= 1_{G^c} P^m f + 1_G P^m f \\ &= 1_{G^c} P^m f + hg + q_2 = hg + q_3, \end{aligned} \tag{3.3.137}$$

where

$$||q_3|| \leqslant ||1_{G^c} P^m f|| + ||q_2|| < \epsilon/2 + \epsilon/4 = 3\epsilon/4. \tag{3.3.138}$$

Coming back to equation (3.3.130), find $n \in \mathbb{N}$ large enough for $||g\epsilon_n(h)|| < \epsilon/4$ to hold. Then

$$\begin{aligned} P^{n+m} f &= P^n(hg) + P^n q_3 \\ &= \sum_{i=1}^{r} \lambda_i(h) g_{\sigma^n(i)} + g\epsilon_n(h) + P^n q_3, \end{aligned} \tag{3.3.139}$$

where

$$\left\| P^{n+m} f - \sum_{i=1}^{r} \lambda_i(h) g_{\sigma^n(i)} \right\| < \frac{3}{4}\epsilon + \frac{1}{4}\epsilon = \epsilon. \tag{3.3.140}$$

It remains to show that this is equivalent to the asymptotic decomposition (3.3.1).

First, notice that $\lambda_i(h)$ depends on h and thus on ϵ, whereas g_i does not. Second, $\epsilon > 0$ was arbitrary, and for every ϵ one could choose m with $\sigma^m = \sigma$ and thus $\sigma^{n+m} = \sigma^n$. It follows that there exist sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{C_i^k(f)\}_{k\in\mathbb{N}}$, such that

$$\lim_{k\to\infty} \left\| P^{n_k} f - \sum_{i=1}^{r} C_i^k(f) g_{\sigma^{n_k}(i)} \right\| = 0. \tag{3.3.141}$$

We show that for all $i = 1, \dots, r$, $\{C_i^k(f)\}_{k\in\mathbb{N}}$ is a bounded sequence. Suppose that it is unbounded for some i. Then, using triangular inequality in the first step and the fact that g_i's have almost disjoint supports in the second one, we get

$$\begin{aligned} \left\| P^{n_k} f - \sum_{i=1}^{r} C_i^k(f) g_{\sigma^{n_k}(i)} \right\| &\geqslant \left(\left\| \sum_{i=1}^{r} C_i^k(f) g_{\sigma^{n_k}(i)} \right\| - ||P^{n_k} f|| \right) \\ &= \left(\sum_{i=1}^{r} |C_i^k(f)| \cdot \left\| g_{\sigma^{n_k}(i)} \right\| - ||P^{n_k} f|| \right) \to \infty \end{aligned} \tag{3.3.142}$$

at least for some subsequence $\{k_l\} \subset \{k\}$, since $0 \leqslant ||P^{n_k} f|| \leqslant ||f|| < \infty$ and $||g_i|| = 1 > 0$ (cf. (3.3.146)). This contradicts (3.3.141). As bounded sequences in $\mathbb{R}$, $\{C_i^k(f)\}_{k\in\mathbb{N}}$ converge to some $\lambda_i(f)$, $i = 1, \dots, r$, and, for simplification of notation, assume w.l.o.g. that we don't need to extract convergent subsequences.

Further, $Pg_i = g_{\sigma(i)}$ for all $i = 1, \dots, r$. Indeed, in this proof σ first appeared in equation (3.3.128) and has the property

$$\bar{P}\left(\frac{1_{A_i}}{\bar{\mu}(A_i)} \right) = \frac{1_{A_{\sigma(i)}}}{\bar{\mu}(A_{\sigma(i)})}, \tag{3.3.143}$$

from which one gets

$$Pg_i = P\left(g \frac{1_{A_i}}{\bar{\mu}(A_i)} \right) = \bar{P}\left(\frac{1_{A_i}}{\bar{\mu}(A_i)} \right) \cdot g = \frac{1_{A_{\sigma(i)}}}{\bar{\mu}(A_{\sigma(i)})} \cdot g = g_{\sigma(i)} \tag{3.3.144}$$

Now, define

$$\epsilon_n = \left\| P^n \left(f - \sum_{i=1}^{r} \lambda_i(f) g_i \right) \right\|. \tag{3.3.145}$$

From (3.3.141) and (3.3.145), we know that $\epsilon_{n_k} \to 0$ as $k \to \infty$. Moreover, ϵ_n is nonincreasing by monotonicity of P. It follows that $\epsilon_n \to 0$ as $n \to \infty$.

Step III. Since, by construction, g_i's have almost disjoint supports and we proved the permutation condition (ii) from the theorem in (3.3.144), it remains to show that g_i is a density and $\lambda_i(f)$ is linear and bounded, $i = 1, \dots, r$.

For g_i, we have

$$||g_i||_{L^1(\mu)} = \int_X g \frac{1_{A_i}}{\bar{\mu}(A_i)} d\mu = \frac{1}{\bar{\mu}(A_i)} \int_{A_i} d\bar{\mu} = 1. \tag{3.3.146}$$

For $\lambda_i(f)$, observe that $\epsilon_n \to 0$ and the fact that g_i's are almost disjoint imply

$$||P^{\tilde{r}n} f \cdot 1_{A_i}|| \to ||\lambda_i(f) g_i|| \quad \text{as } n \to \infty, \tag{3.3.147}$$

where $\tilde{r} = \min\{r : \sigma^r = id\}$. Since P is a contraction and $||g_i|| = 1$, for every $f \in L^1$

$$\frac{|\lambda_i(f)|}{||f||} \leqslant \frac{|\lambda_i(f)|}{||P^{\tilde{r}n} f||} = \frac{||\lambda_i(f) g_i||}{||P^{\tilde{r}n} f||} \leqslant \frac{||\lambda_i(f) g_i||}{||P^{\tilde{r}n} f \cdot 1_{A_i}||} \to 1 \tag{3.3.148}$$

as $n \to \infty$, which implies that λ_i is a bounded operator.

Further, $\epsilon_n \to 0$ implies that $\lambda : L^1(X) \to \mathbb{R}$ is unique. Indeed, for another $\tilde{\lambda} : L^1(X) \to \mathbb{R}$ with the same property, we get

$$\begin{aligned} |\lambda_i(f) - \tilde{\lambda}_i(f)| &= ||\lambda_i(f) g_i - \tilde{\lambda}_i(f) g_i|| \\ &\leqslant ||\lambda_i(f) g_i - P^{\tilde{r}n} f \cdot 1_{A_i}|| + ||P^{\tilde{r}n} f \cdot 1_{A_i} - \tilde{\lambda}_i(f) g_i|| \to 0 \end{aligned} \tag{3.3.149}$$

as $n \to \infty$. Since

$$\left\| P^n \left(f_1 - \sum_{i=1}^{r} \lambda(f_1) g_i \right) \right\| \to 0 \quad \text{and} \quad \left\| P^n \left(f_2 - \sum_{i=1}^{r} \lambda(f_2) g_i \right) \right\| \to 0 \tag{3.3.150}$$

imply

$$\left\| P^n \left(f_1 + f_2 - \sum_{i=1}^{r} (\lambda(f_1) + \lambda(f_2) g_i \right) \right\| \to 0, \tag{3.3.151}$$

it holds $\lambda_i(f_1) + \lambda_i(f_2) = \lambda_i(f_1 + f_2)$, so the operator λ_i is linear. □

3.4 Asymptotic periodicity for the driven Rényi transformation

Finally, we want to summarize results of Sections 3.2 and 3.3 to make a precise statement on the dynamics of the density under the driven Rényi transformation.

Theorem 3.3.1 states that the Perron-Frobenius operator corresponding to a transformation is asymptotically periodic, if it is constrictive. By corollary 3.2.9, the sequence of Perron-Frobenius operators corresponding to the sequnce of driven Rényi transformations

$$S_t(x) = a(x + k(y_t - x)) \bmod 1 \tag{3.4.1}$$

is constrictive if $\{y_t\}$ is periodic or if $a \geqslant 2$ and $k < 1 - \frac{\lfloor a \rfloor}{a}$. Unfortunately, the second condition of constrictivness is not enough. If $\{y_t\}$ is not periodic, we have an infinite sequence of different Perron-Frobenius operators and cannot follow asymptotic periodicity.

Suppose that $\{y_t\}$ is periodic with period k. Let $\{P_t\}$ be the sequence of Perron-Frobenius operators corresponding to $\{S_t\}$. Define

$$P = P_k P_{k-1} \cdots P_1. \tag{3.4.2}$$

Clearly, P is constrictive, since $\{P_t\}$ is a constrictive sequence. By theorem 3.3.1, P is asymptotically periodic with some period $\tilde{r} < \infty$. It follows, that the sequence $\{P_t\}$ is asymptotically periodic with period $\tilde{r}k < \infty$.

So the condition for the asymptotic periodicity of $\{P_t\}$ is the periodicity of $\{y_t\}$. When is $\{y_t\}$ periodic? Recall that $\{y_t\}$ is the trajectory of a system determined by a Rényi transformation:

$$y_{t+1} = R(y_t) = by_t \bmod 1 \tag{3.4.3}$$

with $b > 1$ and $y_0 \in [0,1]$. Whether $\{y_t\}$ is periodic, depends both on b and the initial state y_0. In fact, one can consider the transformation R^t as a shift operator applied t times to the initial state y_0 in b-adic representation. Periodicity of $\{y_t\}$ is then equivalent to the periodicity of y_0 in b-adic representation.

If $b \in \mathbb{N}$, then all and only $y_0 \in \mathbb{Q} \cap [0,1]$ have periodic b-adic representations. This can be easily shown using direct computation in the "only" direction and the division algorithm for "all".

For $b \notin \mathbb{N}$, to determine y_0 which have periodic representations is a much more complex question. However, [Schmidt] managed to give a comprehensive classification. We first need some definitions.

Definition 3.4.1. *A* **Pisot-Vijayaraghavan number** *(also a* **Pisot number***) is a real root of a monic polynomial, i.e., a polynomial with coefficient* 1 *by the term with highest power, which is greater than* 1 *and such that all other roots are less than* 1 *in absolute value.*

Definition 3.4.2. *A* **Salem number** *is a real root of a monic polynomial, which is greater than* 1 *and such that all other roots are not greater than* 1 *in absoulte value, whereas at least one of them has absolute value exactly* 1*.*

Golden ratio $(1+\sqrt{5})/2$ is an example of a Pisot number. The smallest Salem number is the largest root of the polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1. \tag{3.4.4}$$

All integers greater than 1 are Pisot numbers and, conversely, every rational Pisot number is an integer. Moreover, Pisot numbers are nowhere dense in $[1,\infty)$ (cf. [Salem, Chapter II][4]) and the same was conjectured, but not yet proved, for Salem numbers.

Finally, define the field extension

$$\mathbb{Q}(b) = \{\lambda_1 + \lambda_2 b : \lambda_1, \lambda_2 \in \mathbb{Q}\}. \tag{3.4.5}$$

Now, it follows from [Schmidt] that

(i) If b is a Pisot number, then all and only $y_0 \in \mathbb{Q}(b) \cap [0,1)$ have periodic b-adic representations.

(ii) If b is not a Pisot number, then only some $y_0 \in \mathbb{Q}(b) \cap [0,1)$ have periodic b-adic representations.

(iii) If b is neither Pisot nor Salem, then those y_0 that have periodic b-adic representations are nowhere dense in $[0,1)$.

In numerical computations one can only simulate rational numbers. It follows that accurate numerical computations can yield periodic $\{y_t\}$ orbits for all (rational) y_0 only if b is a rational Pisot number, i.e., an integer greater than 1.

[4] There closedness of the set of Pisot numbers is proved. An arbitrary interval in $[0,1)$ contains a dense subset of rational numbers, which are not Pisot. By closedness, any convergent sequence of Pisot numbers converges to a Pisot number, so Pisot numbers cannot be dense in the interval.

Chapter 4

Cross-transitivity estimators for two coupled systems

In this chapter we study the relation of the cross-transitivity estimators for two coupled systems.

In the first section we numerically compute the transitivity estimator (2.0.2) and the cross-transitivity (1.1.6) (which is also an estimator in the sense that it is computed from a finite data set and not using the invariant measure) in the same way as [Feldhoff et al.] do. As explained in the introduction, [Feldhoff et al.] study only one system — the Rössler system — and find out that if the systems $\mathcal{X}$ and $\mathcal{Y}$ are governed by the Rössler equations and are coupled via diffusive coupling so that system $\mathcal{Y}$ is independent and system $\mathcal{X}$ is driven by system $\mathcal{Y}$, then $\hat{\mathcal{T}}^{\mathcal{XY}} > \hat{\mathcal{T}}^{\mathcal{YX}}$ for all values of the coupling strength which yield significantly different $\hat{\mathcal{T}}^{\mathcal{XY}}$ and $\hat{\mathcal{T}}^{\mathcal{YX}}$. This illustrated the hypothesis that such a relation of cross-transitivities is typical for the unidirectional coupling.

We reproduce the result of [Feldhoff et al.] and compute estimators for different other systems both in continuous and discrete time. We find out that in general both relations $\hat{\mathcal{T}}^{\mathcal{YX}} > \hat{\mathcal{T}}^{\mathcal{XY}}$ and $\hat{\mathcal{T}}^{\mathcal{XY}} > \hat{\mathcal{T}}^{\mathcal{YX}}$ are possible. The results are discussed further at the end of the section.

In the second section we propose four simple models for attractors of two coupled systems on the plane, illustrating how different geometry of the attractors could lead to different relations between cross-transitivities.

4.1 Estimation of (cross-)transitivities for different systems

In the following we plot transitivity dimensions for different coupled systems of the same type. We always consider unidirectional coupling, where system $\mathcal{Y}$ is independent and drives system $\mathcal{X}$. Following [Feldhoff et al., Sec.3], for different values of the coupling strength k, ensembles of 200 realisations are considered. Values of k are chosen from the interval $[0, 1]$ or a smaller one in case cross-transitivities start to have similar behaviour long before $k = 1$, with a step of at most 0.02. Only for the Lorenz system (equation (4.1.5)) we considered values of k also beyond 1, since the transitivities demonstrated "smooth" behaviour on the interval $[0, 5]$. In this case the step is equal to 0.1.

For continuous-time systems, the first-order differential equations are integrated with a step size $h = 0.01$ for a total time $T = 5000$, leading to $500,000$ points on each simulated trajectory. After discarding $100,000$ first points, which probably correspond to the transient phase, $N = 1500$ points are chosen randomly to construct the recurrence network.[1] For discrete-time systems, the maps are iterated $500,000$ times and the same procedure is used afterwards.

The networks are constructed with fixed recurrence rates $RR^{\mathcal{X}} = RR^{\mathcal{Y}} = 0.02$ and $RR^{\mathcal{XY}} = RR^{\mathcal{YX}} = 0.03$, where for the inter-system adjacency matrix A defined by (1.1.3),

$$RR^{\mathcal{X}} = \frac{1}{N(N-1)} \sum_{i,j \in \{1,\ldots,N\}} A_{ij}^{\mathcal{X}} \tag{4.1.1}$$

and

$$RR^{\mathcal{XY}} = \frac{1}{NM} \sum_{i \in \{1,\ldots,N\}, j \in \{1,\ldots,M\}} A_{ij}^{\mathcal{XY}} \tag{4.1.2}$$

with $N \in \mathbb{N}$ — the number of vertices in the network of the system $\mathcal{X}$ and $M \in \mathbb{N}$ — of the system $\mathcal{Y}$ (cf. [Donner et al., 2010, Sec.3.1.2]). Fixed recurrence rates determine thresholds, which thus slightly vary, but normally satisfy $\epsilon^{\mathcal{X}} \approx \epsilon^{\mathcal{Y}}$ and $\epsilon^{\mathcal{XY}} = \epsilon^{\mathcal{YX}} > \epsilon^{\mathcal{X}}, \epsilon^{\mathcal{Y}}$. This approach allows to preserve the density of the network and thus to compare networks of different systems without normalising their time series beforehand. The exact values are chosen following [Feldhoff et al., Sec.3].

We normally consider some frequency mismatch, i.e., systems $\mathcal{X}$ and $\mathcal{Y}$ are described by the same equations (up to coupling), but with slightly different coefficients. This is done to prevent the systems from completely synchronising at very low coupling strengths.

Since much of this thesis is dedicated to dimensions, we estimate here not the (cross)-transitivities, but the (cross)-transitivity dimensions, using the following estimators for some choice of ϵ:

$$\hat{D}_{\hat{\mathcal{T}}^{\mathcal{X}}} = \frac{\log \hat{\mathcal{T}}^{\mathcal{X}}}{\log(3/4)} \quad \text{and} \quad \hat{D}_{\hat{\mathcal{T}}^{\mathcal{XY}}} = \frac{\log \hat{\mathcal{T}}^{\mathcal{XY}}}{\log(3/4)}. \tag{4.1.3}$$

This does not change the analysis, since the transformation is monotonic. Ensemble means and standard deviations (error bars) for $\hat{D}_{\hat{\mathcal{T}}^{\mathcal{X}}}$, $\hat{D}_{\hat{\mathcal{T}}^{\mathcal{Y}}}$ as well as for $\hat{D}_{\hat{\mathcal{T}}^{\mathcal{XY}}}$ and $\hat{D}_{\hat{\mathcal{T}}^{\mathcal{YX}}}$ are given. Note that in our case, where the coupling direction is from $\mathcal{Y}$ to $\mathcal{X}$ ($\mathcal{Y} \to \mathcal{X}$), we expect $\hat{\mathcal{T}}^{\mathcal{XY}} > \hat{\mathcal{T}}^{\mathcal{YX}}$ according to [Feldhoff et al.]. Since $\log(3/4) < 0$, this corresponds to $\hat{D}_{\hat{\mathcal{T}}^{\mathcal{YX}}} > \hat{D}_{\hat{\mathcal{T}}^{\mathcal{XY}}}$.

For calculations the python language and the software package pyunicorn have been used on the IBM iDataPlex Cluster of the Potsdam Institute of Climate Impact Research.

The choice of the systems, parameters and variables for coupling is rather subjective. In view of the limited time and the goals of the thesis it was impossible to conduct a comprehensive analysis. It is in line with the tradition of the current research, since the zoo of chaotic systems is already very big and the chaotic systems can demonstrate very different behaviour for different parameters, making researchers concentrate on specific cases.

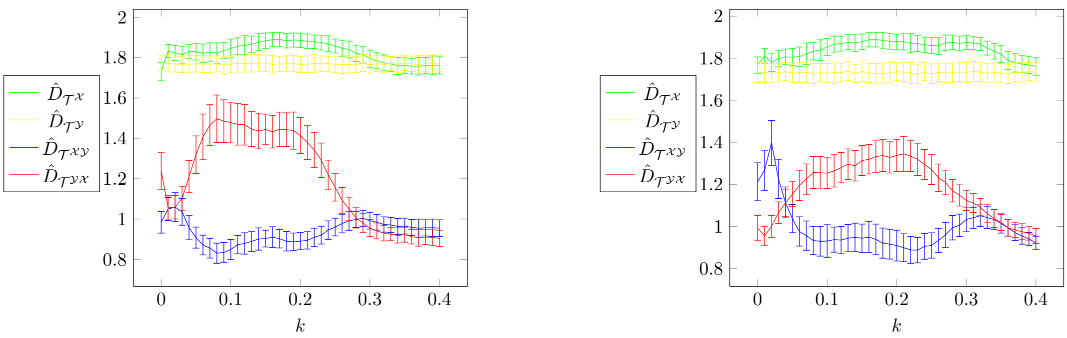
[1] This is the widely used bootstrapping technique, see [Efron and Tibshirani] for the explanations.

4.1.1 Continuous time systems

1. Rössler system

$$\begin{cases} \dot{x_1} = -(1+n)x_2 - x_3 \\ \dot{x_2} = (1+n)x_1 + ax_2 + k(y_2 - x_2) \\ \dot{x_3} = b + x_3(x_1 - c) \end{cases} \qquad \begin{cases} \dot{y_1} = -(1-n)y_2 - y_3 \\ \dot{y_2} = (1-n)y_1 + ay_2 \\ \dot{y_3} = b + y_3(y_1 - c) \end{cases} \tag{4.1.4}$$

In order to reproduce the results of [Feldhoff et al.], we consider the so-called funnel regime with $a = 0.2925$, $b = 0.1$ and $c = 8.5$ and choose the ensemble of initial conditions randomly from the set $[0,1)^3$. [Rössler] initially proposed different parameters and the funnel regime was studied in [Osipov et al., 1997] and [Osipov et al., 2003].



(a) frequency mismatch $n = -0.02$

(b) frequency mismatch $n = 0.02$

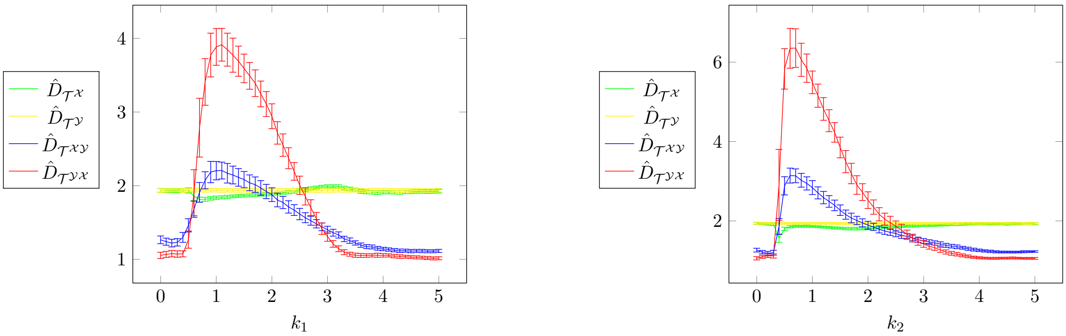
Figure 4.1: Coupling analysis for two Rössler systems (eq. (4.1.4)) with mismatch (a) $n = -0.02$ and (b) $n = 0.02$

As also for all following systems, we discuss the results in the end of this section.

2. Lorenz system

$$\begin{cases} \dot{x_1} = (1+n)\sigma(x_2 - x_1) \\ \dot{x_2} = -x_1x_3 + (1+n)\rho x_1 - x_2 + k_1(y_2 - x_2) \\ \dot{x_3} = x_1x_2 - (1+n)\beta x_3 + k_2(y_3 - x_3) \end{cases} \qquad \begin{cases} \dot{y_1} = (1-n)\sigma(y_2 - y_1) \\ \dot{y_2} = -y_1y_3 + (1-n)\rho y_1 - y_2 \\ \dot{y_3} = y_1y_2 - (1-n)\beta y_3 \end{cases} \tag{4.1.5}$$

Here different variables are coupled: in case (a), the systems are coupled via the second variable ($k_2 = 0$), and in case (b) — via the third variable ($k_1 = 0$). We take the canonical parameters $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$ proposed by [Lorenz] and approximate β by 2.6667. The ensemble of initial conditions is chosen for the third variable randomly from the interval $[20, 24)$, the two first variables are fixed at 1.



(a) second variable coupling, frequency mismatch $n = 0.02$

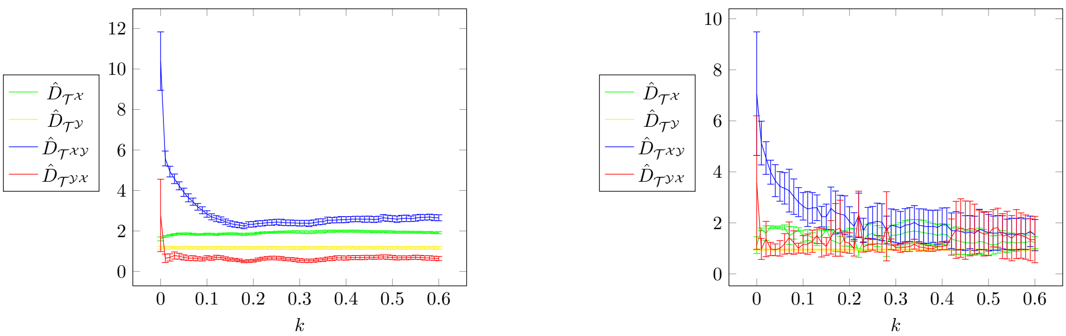
(b) third variable coupling, frequency mismatch $n = 0.02$

Figure 4.2: Coupling analysis for two Lorenz systems (eq. (4.1.5)) with coupling via (a) the second or (b) the third variable, with frequency mismatch $n = 0.02$ in both cases

3. Thomas' cyclically symmetric operator

$$\begin{cases} \dot{x_1} = -(1+n)bx_1 + \sin x_2 \\ \dot{x_2} = -(1+n)bx_2 + \sin x_3 \\ \dot{x_3} = -(1+n)bx_3 + \sin x_1 + k(y_3 - x_3) \end{cases} \qquad \begin{cases} \dot{y_1} = -(1-n)by_1 + \sin y_2 \\ \dot{y_2} = -(1-n)by_2 + \sin y_3 \\ \dot{y_3} = -(1-n)by_3 + \sin y_1 \end{cases} \tag{4.1.6}$$

We consider $b = 0.18$ as proposed by [Thomas]. The ensemble of initial conditions is chosen randomly from the set $[0, 1)^3$.



(a) frequency mismatch $n = -0.02$

(b) frequency mismatch $n = 0.02$

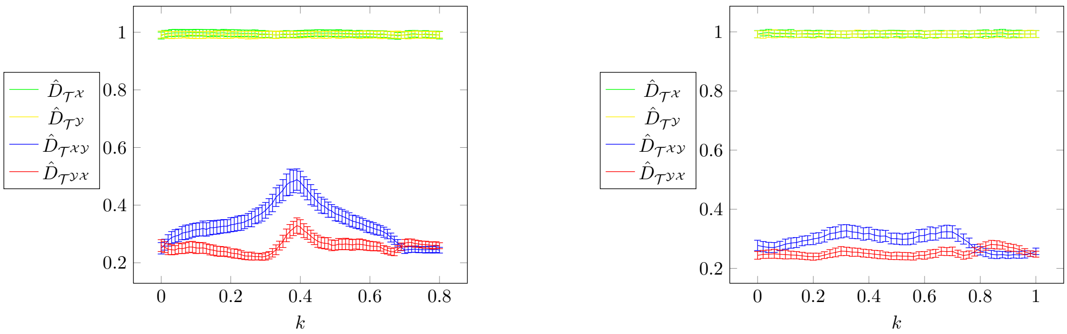
Figure 4.3: Coupling analysis for two Thomas systems (eq. (4.1.6)) with mismatch (a) $n = -0.02$ and (b) $n = 0.02$. Note that part of the difference in the variability of dimensions in figures (a) and (b) is due to the different scaling of the y-axes

4.1.2 Discrete time systems

1. Rényi transformation

$$\begin{aligned} x_{t+1} &= a(x_t + k(y_t - x_t)) \mod 1 \\ y_{t+1} &= (1-n)ay_t \mod 1 \end{aligned} \tag{4.1.7}$$

The Rényi transformation is a generalization of the Bernoulli shift map for which $a = 2$. [Rényi] began the thorough study of this class of transformations and did not propose any specific value for a. Since then there exists no canonical value of a in the literature. We take two examples: $a = 1.5$ and $a = 2.7$. The ensemble of initial conditions is chosen randomly from the set $[0, 1) \times [0, 1)$.



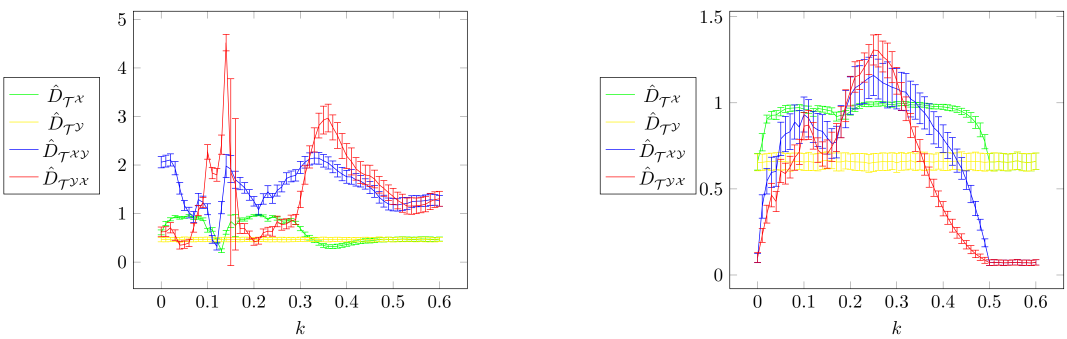
(a) $a = 1.5$, frequency mismatch $n = 0.03$ (b) $a = 2.7$, frequency mismatch $n = 0.03$

Figure 4.4: Coupling analysis for two Rényi systems (eq. (4.1.7)) for parameters (a) $a = 1.5$ and (b) $b = 2.7$, both with frequency mismatch $n = 0.03$

2. Logistic map

$$
\begin{aligned}
x_{t+1} &= (1+n)a(x_t + k(y_t - x_t))(1 - (x_t + k(y_t - x_t))) \\
y_{t+1} &= (1-n)ay_t(1 - y_t)
\end{aligned} \tag{4.1.8}
$$

The logistic map (popularized by [May]) demonstrates very different behaviour depending on a. We want to consider $a = 3.7$ and the often used $a = 4$, both of which correspond to the chaos regime (see [Sprott, Sec.2.3]). $a = 4$ is approximated by 3.9999999, since multiplication of rational numbers in the binary machine representation with a power of 2 soon leads to degenerate values. The ensemble of initial conditions is chosen randomly from the set $[0, 1) \times [0, 1)$ in accordance with [Sprott, A.1.1].



(a) $a = 3.7$, frequency mismatch $n = 0.02$ (b) $a = 3.9999999$, no frequency mismatch

Figure 4.5: Coupling analysis for two Logistic systems (eq. (4.1.8)) for (a) $a = 3.7$ and frequency mismatch $n = 0.02$ and (b) $a = 3.9999999$ and no frequency mismatch

3. Arnold's cat map

$$\begin{cases} x^1_{t+1} = (x^1_t + (x^2_t + k(y^2_t - x^2_t))) \mod 1 \\ x^2_{t+1} = (x^1_t + 2(x^2_t + k(y^2_t - x^2_t))) \mod 1 \end{cases} \quad \begin{cases} y^1_{t+1} = (y^1_t + y^2_t) \mod 1 \\ y^2_{t+1} = (y^1_t + 2y^2_t) \mod 1 \end{cases}, \tag{4.1.9}$$

where z^i_t denotes the i-th component of z, $i = 1, 2$, $z \in \{x, y\}$.

Arnold's cat map, the best known example of the so-called Anosov isomorphism was proposed in the given form by [Arnold and Avez, Sec.3.13]. The ensemble of initial conditions is chosen randomly from the set $[0, 1) \times [0, 1)$.

4. Baker's map

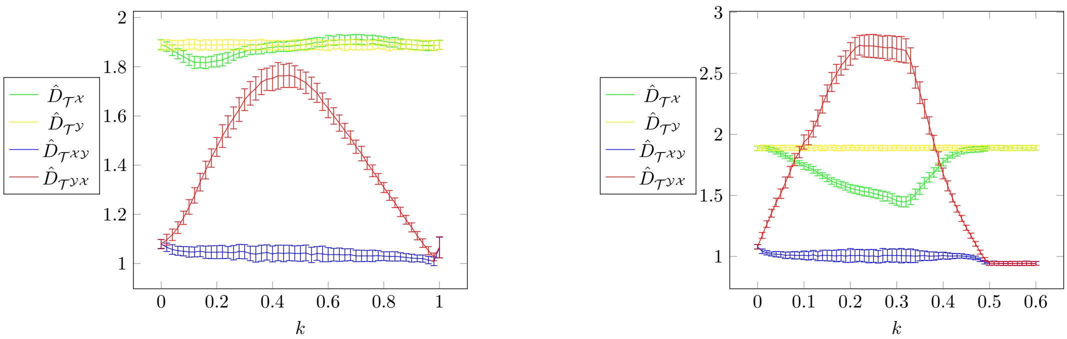
$$(x^1_{t+1}, x^2_{t+1}) = \begin{cases} (2(x^1_t + k(y^1_t - x^1_t)), x^2_t/2), & \text{if } 0 \leqslant (x^1_t + k(y^1_t - x^1_t)) < 1/2 \\ (2 - 2(x^1_t + k(y^1_t - x^1_t)), 1 - x^2_t/2), & \text{if } 1/2 \leqslant (x^1_t + k(y^1_t - x^1_t)) < 1 \end{cases}$$

$$(y^1_{t+1}, y^2_{t+1}) = \begin{cases} (2y^1_t, y^2_t/2), & \text{if } 0 \leqslant y^1_t < 1/2 \\ (2 - 2y^1_t, 1 - y^2_t/2), & \text{if } 1/2 \leqslant y^1_t < 1 \end{cases} \tag{4.1.10}$$

where z^i_t denotes the i-th component of z, $i = 1, 2$, $z \in \{x, y\}$.

This is the standard Baker's map, cf. [Driebe, Sec.5.1], with coupling via the first variable. Since for any rational initial conditions one gets $y_t = 0$ after a low number of iterations, we substitute all 2's in the equations with 1.9999999's, as proposed in [Sprott, Sec.2.5.4].

The map acts on $[0, 1)^2$, so the ensemble of initial conditions is chosen randomly from the set $[0, 1)^2$.



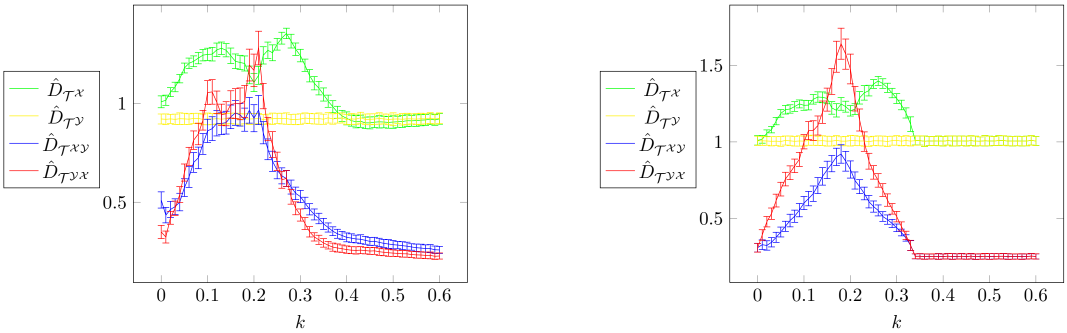
(a) canonical Arnold's cat map (b) Baker's map

Figure 4.6: Coupling analysis for (a) two Arnold's cat maps (eq. (4.1.9)) and (b) two Baker's maps (eq. (4.1.10)) with no mismatch

5. Henon map

$$\begin{cases} x^1_{t+1} = x^2_t + 1 - a(x^1_t + k(y^1_t - x^1_t))^2 \\ x^2_t = bx^1_t \end{cases} \quad \begin{cases} y^1_{t+1} = y^2_t + 1 - (1 - n)a(y^1_t)^2 \\ y^2_{t+1} = by^1_t \end{cases} \tag{4.1.11}$$

We consider $a = 1.4$ and $b = 0.3$ as proposed in the original article by [Henon]. The ensemble of initial conditions is chosen randomly from the interval $[0, 1)$ for the first variable of each system and from the interval $[0, 0.1)$ for the second variable. This was chosen by trial and is in line with the initial conditions $(0, 0)$ proposed in [Sprott, Sec.5.2.2].



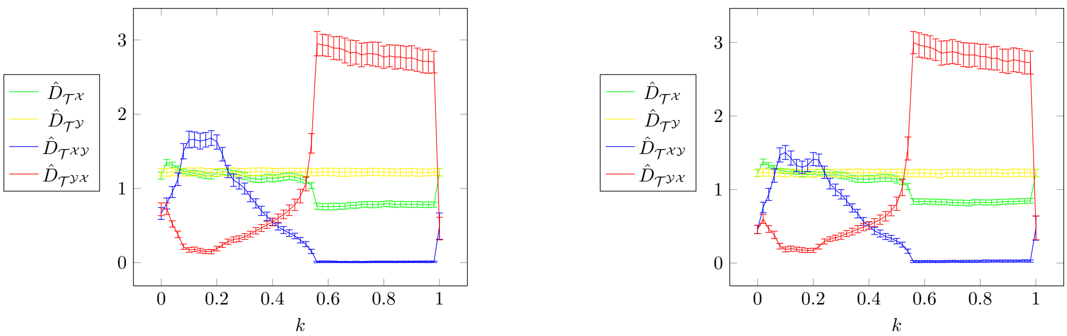
(a) frequency mismatch $n = 0.03$ (b) no mismatch

Figure 4.7: Coupling analysis for two Henon systems (eq. (4.1.11)) with $a = 1.4$, $b = 0.3$ and (a) frequency mismatch $n = 0.03$ or (b) no mismatch ($n = 0$)

6. Burger's map

$$\begin{cases} x^1_{t+1} = a(x^1_t + k(x^2_t - x^1_t)) - (x^2_t)^2 \\ x^2_t = (1+n)bx^2_t + (x^1_t + k(x^2_t - x^1_t))x^2_t \end{cases} \qquad \begin{cases} y^1_{t+1} = ay^1_t - (y^2_t)^2 \\ y^2_t = (1-n)by^2_t + y^1_t y^2_t \end{cases} \tag{4.1.12}$$

We consider $a = 0.75$ and $b = 1.75$ as proposed in [Sprott, A.2.5]. The ensemble of initial conditions is chosen randomly from the interval $[-0.15, -0.05)$ for the first variable of each system and from the interval $[0.05, 0.15)$ for the second variable in line with the initial conditions $(-0.1, 0.1)$ proposed in [Sprott, ebd.].



(a) frequency mismatch $n = 0.03$ (b) no mismatch

Figure 4.8: Coupling analysis for two Burger systems (eq. (4.1.12)) with $a = 0.75$, $b = 1.75$ and (a) no frequency mismatch or (b) frequncy mismatch ($n = 0.02$). Note that both figures differ

7. Kaplan-Yorke map

$$\begin{cases} x^1_{t+1} = (2(x^1_t + k(y^1_t - x^1_t)) \mod 1 \\ x^2_{t+1} = (1+n)ax^2_t + \cos(4\pi x^1_t) \end{cases} \quad \begin{cases} y^1_{t+1} = (2y^1_t) \mod 1 \\ y^2_{t+1} = (1-n)ay^2_t + \cos(4\pi y^1_t) \end{cases}, \tag{4.1.13}$$

where z^i_t denotes the i-th component of z, $i = 1, 2$, $z \in \{x, y\}$.

We consider $a = 0.2$ as proposed by [Grassberger and Procaccia, Sec.2.2].

Since direct numerical simulation with a rational initial condition will lead to $y^1_t = 0$ after a low number of iterations, we compute the trajectory using a different, but equivalent algorithm. E.g., for y^1_t, we compute

$$\begin{aligned} \bar{y}^1_{t+1} &= (2\bar{y}^1_t) \mod 514229 \\ y^1_{t+1} &= \bar{y}^1_{t+1}/514229. \end{aligned}$$

Here, 514229 is an arbitrarily chosen prime number. Any other large prime number would work as well.

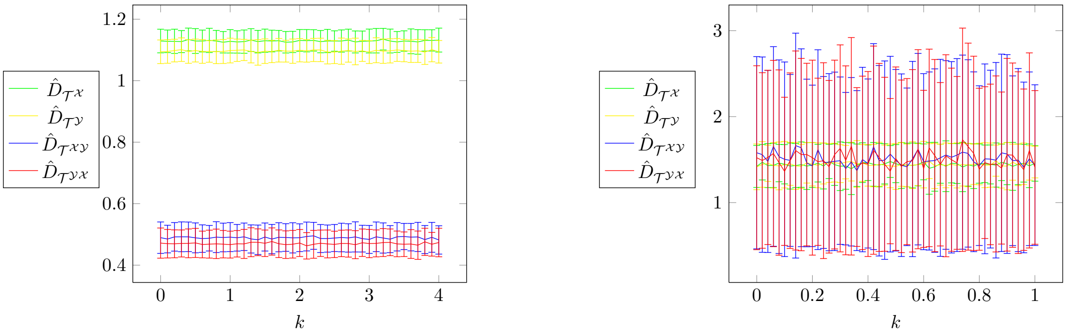
Initial conditions are random from the interval $[0, 1)$ for each coordinate. For the implementation, $\bar{x}_0$ and $\bar{y}_0$ are random from the interval $[0, 514229)$.

8. Chirikov standard map

$$\begin{cases} x^1_{t+1} = (x^1_t + k(y^1_t - x^1_t) + x^2_{t+1}) \mod 2\pi \\ x^2_{t+1} = (x^2_t + a\sin(x^1_t + k(y^1_t - x^1_t))) \mod 2\pi \end{cases} \quad \begin{cases} y^1_{t+1} = (y^1_t + y^2_{t+1}) \mod 2\pi \\ y^2_{t+1} = (y^2_t + a\sin(y^1_t)) \mod 2\pi \end{cases}, \tag{4.1.14}$$

where z^i_t denotes the i-th component of z, $i = 1, 2$, $z \in \{x, y\}$.

We consider $a = 1$ as proposed by [Sprott, Sec.A.3.1]. Initial conditions are random from the interval $[0, 1)$ for each coordinate.



(a) $a = 0.02$, frequency mismatch $n = 0.02$ (b) Y→ X coupling

Figure 4.9: Coupling analysis for (a) two Kaplan-Yorke systems (eq. (4.1.13)) with mismatch $n = 0.02$ and (b) two Chirikov systems (eq. (4.1.14)) with no mismatch

4.1.3 Discussion of results

We start with the observation that the estimated values normally make sense as estimations for the attractors' dimensions. Attractors that look like curves and are embedded in the three-dimensional space have estimated values between 1 and 2 (Rössler, many values for the Lorenz systems); Rényi system, whose attractor is visually represented by a combination of isolated points and intervals, yields values between 0 and 1. Two-dimensional maps (Arnold's cat, baker's, Chirikov) yield values between 1 and 2. However, there are pathological cases such as Lorenz for coupling strength between 1 and 2, Thomas for small coupling strengths, logistic map for $a = 3.7$, and Burger's for larger coupling strengths.

For different systems, both $\hat{D}_{\hat{\mathcal{T}}\mathcal{X}\mathcal{Y}}$ and $\hat{D}_{\hat{\mathcal{T}}\mathcal{Y}\mathcal{X}}$ can demonstrate very different behaviour. The range of the variability of the mean values of these estimators depending on k is normally large (especially Lorenz, Thomas, Burger systems), but can also be much lower (Rössler and, for the absolute values, Rényi systems). For low coupling strengths, the values of both estimators are similar, then they start to change significantly, getting similar again for higher values of k. From some value of k both estimators do not change much, however, this value differs significantly for different systems. Kaplan-Yorke and Chirikov systems do not yield any clear change in the estimators in dependence on k.

In most cases, one of the $\hat{D}_{\hat{\mathcal{T}}\mathcal{X}\mathcal{Y}}$, $\hat{D}_{\hat{\mathcal{T}}\mathcal{Y}\mathcal{X}}$ has higher variability than the other. Normally, both $\hat{D}_{\hat{\mathcal{T}}\mathcal{X}\mathcal{Y}}$ and $\hat{D}_{\hat{\mathcal{T}}\mathcal{Y}\mathcal{X}}$ deviate in the same direction, but different directions are also possible (Rössler, logistic map for $a = 3.7$ and some values of k, Burger's map).

Both relations $\hat{D}_{\hat{\mathcal{T}}\mathcal{X}\mathcal{Y}} > \hat{D}_{\hat{\mathcal{T}}\mathcal{Y}\mathcal{X}}$ and $\hat{D}_{\hat{\mathcal{T}}\mathcal{Y}\mathcal{X}} > \hat{D}_{\hat{\mathcal{T}}\mathcal{X}\mathcal{Y}}$ are possible for different, both continuous (cf. Rössler and Thomas) and discrete (cf. Rényi and Arnold's cat or baker's) time systems. For any specific system, there is only one interval of significant difference and there one specific estimator is always higher than the other one. Burger's map yields two intervals with different relations.

Finally, most systems demonstrate a smooth change of estimators in dependence on k. The logistic map for $a = 3.7$ is an exception. Most systems have one extremum for the estimators, at one value of k for both estimators. The logistic and Henon maps may have several extrema.

The transitivity dimension estimator $\hat{D}_{\hat{\mathcal{T}}\mathcal{X}}$ also demonstrates interesting behaviour, though its variability in specific systems and across systems is not as high as for cross-transitivity dimension estimators (clearly, some tiny variability of $\hat{D}_{\hat{\mathcal{T}}\mathcal{Y}}$ is solely due to a finite number of realisations, since system $\mathcal{Y}$ does not depend on k). We do not discuss $\hat{D}_{\hat{\mathcal{T}}\mathcal{X}}$ thoroughly, one can study the figures above to see the range and direction of the variability, possible positive or negative correlation with $\hat{D}_{\hat{\mathcal{T}}\mathcal{X}\mathcal{Y}}$ and $\hat{D}_{\hat{\mathcal{T}}\mathcal{Y}\mathcal{X}}$, the intervals of stabilisation and extrema.

Clearly, the whole range of phenomena on these figures remains mainly unexplained. However, most results seem to be plausible and only several issues raise specific questions. These include the very high values of $\hat{D}_{\hat{\mathcal{T}}\mathcal{X}\mathcal{Y}}$ for the Thomas operator and low coupling strength and extremely low values of $\hat{D}_{\hat{\mathcal{T}}\mathcal{X}\mathcal{Y}}$ for the Burger's mapping and $k \in [0.6, 0.9]$. In any case, the behaviour of the estimators is not random and deserves a deeper study.

In the end of this subsection, we want to emphasize that unfortunately our analysis does not allow us to test the conjecture of Section 2.3, i.e., whether the transitivity dimension of the driven system has integer value for small coupling strengths. To compute good estimators of dimension, we need to consider very small thresholds ϵ and thus to construct very large networks (otherwise small thresholds would imply zero density of the network). However, the algorithms and the memory of the computer used do not allow computations with a significantly higher number of vertices.

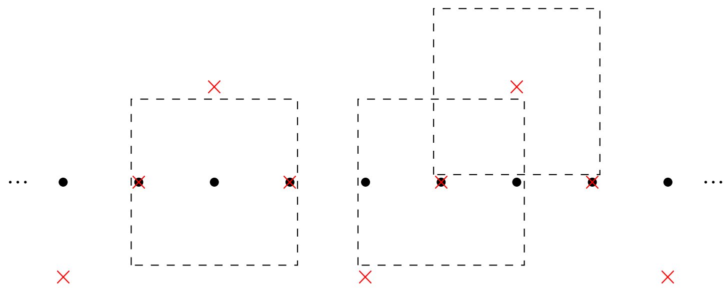
4.2 Sample models for two attractors of coupled systems with different relations of cross-transitivities

Here we demonstrate what kind of geometrical relation between the two attractors could lead to different relations of $\hat{\mathcal{T}}^{\mathcal{XY}}$ and $\hat{\mathcal{T}}^{\mathcal{XY}}$. We consider four simple models with attractors on the plane. In each case, system $\mathcal{Y}$'s trajectory is just a regular one-dimensional grid. System $\mathcal{X}$ has more complex behaviour. We believe that system $\mathcal{X}$ would rather correspond to the driven system: it shows more variability, since it is being constantly disturbed. Points of $\mathcal{Y}$ are depicted as black balls and points of $\mathcal{X}$ - as red crosses. We recall that

$$\hat{\mathcal{T}}^{\mathcal{XY}}(\epsilon) = \frac{\sum_{v\in V_{\mathcal{X}},\ p,q\in V_{\mathcal{Y}}} A_{vp}(\epsilon)A_{vq}(\epsilon)A_{pq}(\epsilon)}{\sum_{v\in V_{\mathcal{X}},\ p,q\in V_{\mathcal{Y}},\ p\neq q} A_{vp}(\epsilon)A_{vq}(\epsilon)}. \tag{4.2.1}$$

1. Oscillation, $\hat{\mathcal{T}}^{\mathcal{XY}} > \hat{\mathcal{T}}^{\mathcal{YX}}$

Suppose that after the transient time, system $\mathcal{X}$ oscillates around system $\mathcal{Y}$ as a sine wave with the peak-to-peak amplitude more than twice the distance between two subsequent points of $\mathcal{Y}$. We observe only the "peak" and "zero" points and take ϵ less than the half of the peak-to-peak amplitude, but more than the distance between two subsequent points of $\mathcal{Y}$. See the figure below with three ϵ-balls:



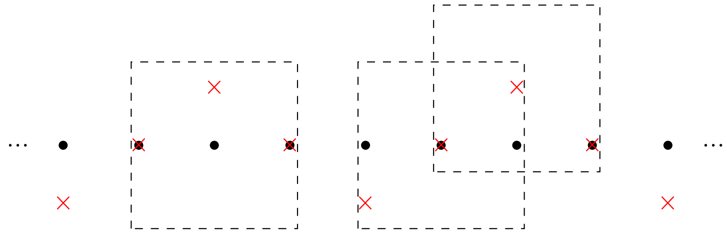
Half of the red points do not have any black points inside the ϵ-balls around them. The other half has three points in each ball and thus $3 \cdot 2 = 6$ triples and 4 triangles. It follows that

$$\hat{\mathcal{T}}^{\mathcal{XY}}(\epsilon) = \frac{\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 4}{\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 6} = \frac{2}{3}.$$

Half of the black points have one red point inside their ϵ-balls, which corresponds to one triple and zero triangles. The other half has two red points, but more than ϵ far away from each other, which corresponds to four triples and zero triangles. Thus, $\hat{\mathcal{T}}^{\mathcal{YX}}(\epsilon) = 0$.

2. Oscillation, $\hat{\mathcal{T}}^{\mathcal{XY}} = \hat{\mathcal{T}}^{\mathcal{YX}}$

We consider the same model as above, but with the sine amplitude less than twice the distance between two subsequent points of $\mathcal{Y}$.



In terms of the supremum norm we use here, this model is equivalent to just two identical regular one-dimensional grids. Every point has three points of the other system inside the ϵ-ball around it, which corresponds to 6 triples and 4 triangles. We get

$$\hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}}(\epsilon) = \hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}}(\epsilon) = \frac{2}{3}.$$

3. Drag and push, $\hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}} > \hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}}$, $\hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}} = 0$

Here both systems are on the same one-dimensional submanifold of the plane. The system $\mathcal{X}$ evolves in the same direction as system $\mathcal{Y}$, but its motion is not steady: sometimes it is slower, sometimes faster. Below you can see one example of this "drag and push" trajectory with three ϵ-balls:

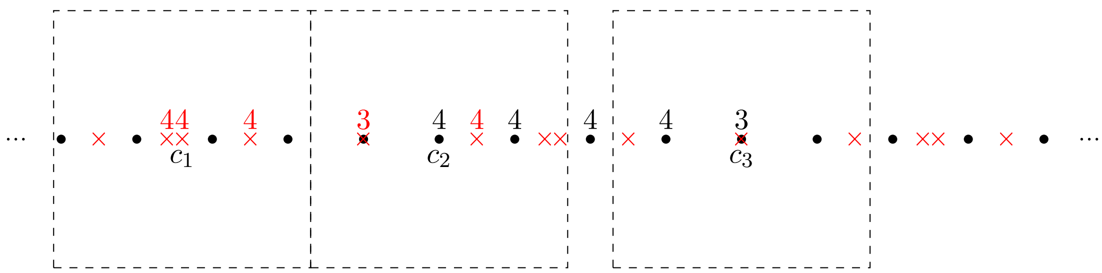
Red points have at most one black point inside the ϵ-ball around them, so there are no triangles and $\hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}} = 0$.

Half of the black points have two red points inside their ϵ-balls, which means that there are 2 triples and 2 triangles. The other half has no red points. It follows that

$$\hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}}(\epsilon) = \frac{\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2}{\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2} = 1.$$

4. Drag and push, $\hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}} > \hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}}$, $\hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}} \neq 0$

This is a more complex "drag and push" model. One section of the repeating pattern starts with two red points, which are very close to each other, and ends before the next such two points, thus containing five points. We take ϵ so that balls contain more points. In the figure below, letters indicate centres of the three given ϵ-balls and numbers indicate how many points of another system are in the ϵ-ball around the given point.



For the one black point (out of five in a single section of the repeating pattern) there are 3 red points inside its ϵ-ball, which corresponds to 6 triples and 4 triangles. Out of other four points in a single section, for two (such as c_2) the 4 red points inside the ϵ-ball correspond to 8 triangles, and for the other two - to 10 triangles. The 4 points always correspond to $4 \cdot 3 = 12$ triples. It follows that

$$\hat{\mathcal{T}}^{\mathcal{Y}\mathcal{X}}(\epsilon) = \frac{\frac{1}{5} \cdot 4 + \frac{2}{5} \cdot 8 + \frac{2}{5} \cdot 10}{\frac{1}{5} \cdot 6 + \frac{4}{5} \cdot 12} = \frac{20}{27}.$$

6 triples and 4 triangles correspond to every red point with 3 black points inside the ϵ-ball. 12 triples and 6 triangles correspond to every red point with 4 black points. Consequently,

$$\hat{\mathcal{T}}^{\mathcal{X}\mathcal{Y}}(\epsilon) = \frac{\frac{1}{5} \cdot 4 + \frac{4}{5} \cdot 6}{\frac{1}{5} \cdot 6 + \frac{4}{5} \cdot 12} = \frac{14}{27}.$$

Finally, we want to note that the first model might be attributed to continuous time systems such as the Rössler or the Lorenz systems, whereas the third and the first models might be some approximations of one-dimensional maps such as the Rényi map. We see that the results for cross-transitivity estimators coincide qualitatively. A detailed study of the trajectories is needed to be able to formulate better approximating models.

Summary and outlook

In this thesis we discussed several aspects of dimensions and clustering in recurrence networks. We proved that the transitivity and the Rényi entropy dimensions have integer values of the phase space dimension in case the invariant measure is absolutely continuous w.r.t. Lebesgue and the corresponding Radon-Nikodym derivative is bounded and has at least one point of continuity in its support. We studied how this result can be generalized for measures, that are absolutely continuous with respect to some smooth submanifold of the phase space. Further, we showed how the transitivity estimator converges to the value expected with a rising number of observations.

We demonstrated the approach of approximating a weak coupling of two systems with stochastic noise. For the simple case of a system with fixed point and Gaussian white noise we showed that noise will raise the dimension of the attractor to some integer up to the phase space dimension.

Turning to specific complex system, the theory of the invariant measure of the Rényi transformation, developed by Prof. Lasota and colleagues, was presented and applied to the driven Rényi transformation.

Finally, we gave numerical estimations for (cross-)transitivity dimensions of several coupled systems and presented four simple models, aiming to explain how different geometry of the attractors in a two-dimensional space can result in different relation between the two cross-transitivities.

Several further steps follow directly from the work presented here. The generalization of the proof of the integer dimension to the case of absoulute continuity w.r.t. a submanifold of the phase space should be elaborated. The sufficient conditions on the submanifold for the generalization to work should be specified. An example of an attractor with invariant measure, which is absolutely continuous w.r.t. Lebesgue, but with a nowhere continuous Radon-Nikodym derivative and non-integer dimension, would make the Proposition sharp.

A comprehensive study of a wide range of chaotic systems with respect to the cross-transitivities could result in a new method of detecting coupling. We have seen that there is no straightforward relationship between coupling direction and the relation of both cross-transitivities. However, one could probably classify systems according to this relation. We suggest conducting the same estimations as in the last chapter with all the systems given, e.g., in [Sprott, Appendix]. The estimations can be supplemented by plots of the systems' trajectories for different coupling strengths, allowing better understanding of the attractors' geometry leading to different relation of cross-transitivities. Consequently, more models, similar to those in the last section, can be created.

A more far-reaching theoretical task is to understand the behaviour of dimensions for invariant measures, which are not absolutely continuous w.r.t. Lebesgue. We have seen that dimensions defined as limits should not necessarily exist, so one task is to specify the properties of measures which guarantee the existence of limits. Since all estimators of dimensions involve two limits — in the number of observations and in the threshold ϵ, it is also important to understand the properties of the estimators' convergence and the interdependence of both limits.

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