EXERCISE SHEET 10, issued on Tuesday 18 January 2011
Due: Tuesday 25 January 2011 by 8,15 a.m. in the designated "MQM" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS10_MQM.html`

Each exercise is worth a full mark as specified below. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English. The exercise marked with ★ is for extra credit.

Exercise 37 [13 points].

(i) Let $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Use the notation $k(x,y)$ with $x,y \in \mathbb{R}^d$. Show that

$$(Af)(x) := \int_{\mathbb{R}^d} k(x,y)f(y)\mathrm{d}y\,, \qquad f \in L^2(\mathbb{R}^d) \tag{$*$}$$

defines a bounded operator $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with operator norm at most $\|k\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}$.

(ii) Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a measurable function such that the functions k_1 and k_2 defined by $k_1(x) := \int_{\mathbb{R}^3} |k(x,y)|\mathrm{d}y$ and $k_2(y) := \int_{\mathbb{R}^3} |k(x,y)|\mathrm{d}x$ belong both to $L^\infty(\mathbb{R}^d)$. Show that $(*)$ defines a bounded operator $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with operator norm at most $\|k_1\|_\infty^{1/2}\|k_2\|_\infty^{1/2}$.

★Exercise 38 [15 points].

(i) Let $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Use the notation $k(x,y)$ with $x,y \in \mathbb{R}^d$. Consider the bounded $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by

$$(Af)(x) := \int_{\mathbb{R}^d} k(x,y)f(y)\mathrm{d}y\,, \qquad f \in L^2(\mathbb{R}^d)$$

(the fact that A is a bounded operator is proved in Exercise 37). Exhibit a sequence $\{A_N\}_{N=1}^\infty$ of finite rank $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ operators such that $A_N \xrightarrow{N\to\infty} A$ in the operator norm, thus concluding that A is a compact operator.

(ii) Show that A defined in (i) is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^d)$ with Hilbert-Schmidt norm equal to $\|k\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}$.

(iii) Assume that $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a Hilbert-Schmidt operator. Show that there exists $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ such that $Af = \int_{\mathbb{R}^d} k(\cdot,y)f(y)\mathrm{d}y\ \forall f \in L^2(\mathbb{R}^d)$. (*Hint 1:* use the singular value decomposition for compact operators stated in the handout "*Many-body quantum systems*". *Hint 2:* as an alternative strategy, show that the correspondence $k \mapsto A$ defined in (i), which by (ii) is an isometry between $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and the space of Hilbert-Schmidt operators, is actually surjective owing to the density of finite rank operators in the Hilbert-Schmidt operators.)

Exercise 39 [15 points]. Let $\mathcal{H}$ be a complex Hilbert space. Recall that when a bounded self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$ is such that $\langle \psi, A\psi \rangle \geqslant 0 \ \forall \psi \in \mathcal{H}$ one uses the short-hand notation $A \geqslant \mathbf{0}$, $\mathbf{0}$ being the zero operator on $\mathcal{H}$, and says that A is positive semi-definite. Similarly, $B - A \geqslant \mathbf{0}$ means $\mathbf{0} \leqslant A \leqslant B$ where $A, B : \mathcal{H} \to \mathcal{H}$ are bounded.

(i) Assume that $A \geqslant \mathbb{1}$ ($\mathbb{1}$ being the unit operator on $\mathcal{H}$). Prove that A is invertible and $\mathbf{0} \leqslant A^{-1} \leqslant \mathbb{1}$.

(ii) Assume that $\mathbf{0} \leqslant A \leqslant B$. Show that both $A + \lambda\mathbb{1}$ and $B + \lambda\mathbb{1}$ are invertible for any $\lambda > 0$ and that $(B + \lambda\mathbb{1})^{-1} \leqslant (A + \lambda\mathbb{1})^{-1}$. (*Hint:* use (ii).)

(iii) Assume that $\mathbf{0} \leqslant A \leqslant B$. Show that $\sqrt{A} \leqslant \sqrt{B}$. (*Hint:* use (iii) and the representation

$$\sqrt{A} \ = \ \int_0^\infty \frac{\mathrm{d}\lambda}{\sqrt{\lambda}} \big(\mathbb{1} - \lambda(A + \lambda\mathbb{1})^{-1}\big)$$

where the integral in the r.h.s. is in the Riemann sense for an operator-valued function.)

(iv) Assume that $\mathbf{0} \leqslant A \leqslant B$. Show that this neither implies $A^2 \leqslant B^2$ nor $AB \geqslant \mathbf{0}$ (i.e., disprove both inequalities with counterexamples, you can think of 2×2 matrices).

Exercise 40 [12 points].

(i) Let Γ be a N-body bosonic or fermionic density matrix on $L^2(\mathbb{R}^{Nd})$. Show that for any $k \in \{1, \dots, N\}$ the k-particle reduced density matrix $\gamma^{(k)}$ associated with Γ is positive semi-definite, that is (see also Exercise 39), $\langle \varphi, \gamma^{(k)} \varphi \rangle \geqslant 0 \ \forall \varphi \in L^2(\mathbb{R}^{kd})$. (*Hint:* use the spectral decomposition of Γ.)

(ii) Let $\Psi \in L^2(\mathbb{R}^{Nd})$ be a bosonic or fermionic wave-function with unit norm. Let $k \in \{1, \dots, N\}$ and let $\gamma^{(k)}$ be the k-particle reduced density matrix associated with Ψ. Consider the kernel $\mathcal{O}(\mathbf{x}_k; \mathbf{x}'_k)$ with $\mathcal{O} \in C_0^\infty(\mathbb{R}^{kd} \times \mathbb{R}^{kd})$ and the notation $\mathbf{x}_k = (x_1, \dots, x_k)$, $\mathbf{x}_{N-k} = (x_{k+1}, \dots, x_N)$, etc., where each $x_j \in \mathbb{R}^d$. Show that

$$\begin{aligned} \frac{N!}{(N-k)!} \int \mathrm{d}\mathbf{x}_k \, \mathrm{d}\mathbf{x}'_k \, \mathrm{d}\mathbf{x}_{N-k} \, \overline{\Psi(\mathbf{x}_k, \mathbf{x}_{N-k})} \, \mathcal{O}(\mathbf{x}_k; \mathbf{x}'_k) \, \Psi(\mathbf{x}'_k, \mathbf{x}_{N-k}) \ &= \\ = \int \mathrm{d}\mathbf{x}_k \, \mathrm{d}\mathbf{x}'_k \, \mathcal{O}(\mathbf{x}_k; \mathbf{x}'_k) \, \gamma^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) \, . \end{aligned}$$