Institute of Mathematics, LMU Munich – Spring Term 2012
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PROBLEM IN CLASS – WEEK 11

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 41. (Regularity of Fourier series)

Consider the Hilbert space $L^2[0,1]$, the canonical ONB $\{e_n\}_{n\in\mathbb{Z}}$ where $e_m(x) := \dfrac{e^{inx}}{\sqrt{2\pi}}$, $n\in\mathbb{Z}$, the subspaces $C^k(\mathbb{S}^1) := \{f \in C^k([0,2\pi]) \,|\, f^{(m)}(0) = f^{(m)}(2\pi),\ m = 0,\dots,k\}$, and, for every $f\in L^2[0,2\pi]$, the N-th approximate Fourier sum $S_N(f) := \sum_{n=-N}^{N} \langle e_n, f\rangle\, e_n$ (see Exercise 38).

(i) Show that if $f\in C^1(\mathbb{S}^1)$ then $S_N(f) \xrightarrow{N\to\infty} f$ uniformly (and not just in the L^2-sense as for general L^2-functions – see Exercise 28(ii)).

(ii) Show that if $f\in L^2[0,2\pi]$, $k\in\mathbb{N}_0$, and $\sum_{n\in\mathbb{Z}} |\langle e_n, f\rangle|\, |n|^k < \infty$ then $f\in C^k(\mathbb{S}^1)$.

(iii) Show that if $f\in L^2[0,2\pi]$, $k\in\mathbb{N}_0$, and $f\in C^k(\mathbb{S}^1)$ then $\sum_{n\in\mathbb{Z}} |\langle e_n, f\rangle|^2\, |n|^{2k} < \infty$.

Problem 42. (Sobolev spaces on $[0,2\pi]$)

Same setting as in Problem 41. Let $k\geqslant 0$. Define:

$$
\begin{aligned}
H^k(\mathbb{S}^1) &:= \Big\{ f\in L^2[0,2\pi] \,\Big|\, \sum_{n\in\mathbb{Z}} |n|^{2k}\, |\langle e_n, f\rangle|^2 \Big\} < \infty \\
\langle f,g\rangle_{H^k} &:= \sum_{n\in\mathbb{Z}} (1+|n|^{2k})\, \overline{\langle e_n, f\rangle}\, \langle e_n, g\rangle \qquad && f,g\in H^k(\mathbb{S}^1) \\
\|f\|_{H^k} &:= \sqrt{\langle f,g\rangle_{H^k}} && f\in H^k(\mathbb{S}^1) \\
f^{(k)} &:= \sum_{n\in\mathbb{Z}} (\mathrm{i}n)^k \langle e_n, f\rangle e_n && k\in\mathbb{N}_0,\ f\in H^k(\mathbb{S}^1)\,.
\end{aligned}
$$

The notation for $f^{(1)}, f^{(2)}, \dots$ is also $f', f'', \dots$

(i) Show that for each $k\geqslant 0$ $H^k(\mathbb{S}^1)$ is a Hilbert space with scalar product $\langle\ ,\ \rangle_{H^k}$ and that $H^0(\mathbb{S}^1) = L^2[0,\pi]$.

(ii) Show that if $k\in\mathbb{N}$ and $f\in C^k(\mathbb{S}^1)$ then $f\in H^k(\mathbb{S}^1)$ and $f^{(k)}$ is precisely the classical derivative of f (whereas $f^{(k)}$ still makes sense as a $L^2[0,2\pi]$-function even when f is not k times differentiable in the classical sense).

(iii) Let $f \in H^1(\mathbb{S}^1)$. Show that the limit

$$\lim_{\epsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

exists in the L^2-sense and is precisely f' defined above.

Problem 43. (Computation of weak derivatives. Weak differentiation and integration commute.)

(i) Compute the weak derivative of the function $u : (-1,1) \to \mathbb{R}$, $u(x) = |x|$.

(ii) Let $d \in \mathbb{N}$ and $\alpha > -(d-1)$. Compute the weak derivative of the function $u : \mathbb{R}^d \to \mathbb{R}$, $u(x) = |x|^\alpha$.

(iii) Consider the function $u : (0,1) \times (0,1) \to \mathbb{R}$, $u(x) = \begin{cases} 0 & \text{if } y = 0 \\ \frac{x^3}{y^2} e^{-x^2/y} & \text{otherwise} \end{cases}$. Disprove

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^1 u(x,y)\,\mathrm{d}y \;=\; \int_0^1 \partial_x u(x,y)\,\mathrm{d}y$$

where the derivatives here are meant in the classical sense.

(iv) Let $\Omega := (a,b) \times (c,d) \subset \mathbb{R}^2$ and $u \in L^1_{\mathrm{loc}}(\Omega)$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_c^d u(x,y)\,\mathrm{d}y \;=\; \int_c^d \partial_x u(x,y)\,\mathrm{d}y$$

whenever both sides make sense, where the derivatives here are meant in the weak sense.

Problem 44. (Weak derivatives commute. Multiplication by a C_0^∞-function leaves Sobolev spaces invariant.)

Let $d \in \mathbb{N}$, Ω be an open in $\mathbb{R}^d$, $f \in L^1_{\mathrm{loc}}(\Omega)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.

(i) Let $j, k \in \{1, \dots, d\}$. Assume that the weak derivatives $\partial_{x_j} f$ and $\partial_{x_k} f$ exist. Show that if $\partial_{x_k} f$ is weakly differentiable with respect to x_j, so is $\partial_{x_j} f$ with respect to x_k and

$$\partial_{x_j}(\partial_{x_k} f) \;=\; \partial_{x_k}(\partial_{x_j} f)\,.$$

(ii) Show that thinking of ∂_{x_j} and ∂_{x_k} as classical derivatives, (i) is not true any longer for the function $f_0(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \end{cases}$ and explain why this example does not yields any inconsistency between (i) and the fact (Lemma 3.23 in class) that classical and weak derivative agrees if the former exists.

(iii) Assume further that $f \in H^{m,p}(\Omega)$ for some $m \in \mathbb{N}$ and some $p \in [1,\infty]$ and let $\varphi \in C_0^\infty(\Omega)$. Show that $\varphi f \in H^{m,p}(\Omega)$ and

$$D^\beta(\varphi f) \;=\; \sum_{\alpha \leqslant \beta} \binom{\beta}{\alpha} (D^\alpha \varphi)(D^{\beta-\alpha} f)\,, \qquad \binom{\beta}{\alpha} := \prod_{k=1}^d \binom{\beta_k}{\alpha_k}$$

(α and β are multi-indices).