Functional Analysis – Final test

Funktionalanalysis – Endklausur

Mathematisches Institut der LMU – SS2010
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Name: / *Name:* ______________________________

Matriculation number: / *Matrikelnr.:* ______________________________

Degree programme: / *Studiengang:* ☐ Bachelor ☐ Master ☐ Diplom ☐ Lehramt ☐ TMP ☐ ____________

Discipline: / *Studienfach:* ☐ Mathematics ☐ Physics ☐ Informatics ☐ ________

Extra solution sheets submitted: ☐ **NO** / *NEIN*
Zusätzlich abgegebene Lösungsblätter: ☐ **YES** / *JA* ________

Marks: / *Punkte:*

problem	1	2	3	4	5	6	7	8	$\sum$
total points	15	15	15	20	20	15	20	15	135
scored points									

homework performance		final test performance		total performance	

INSTRUCTIONS:

- This booklet is made of eighteen pages, including the cover, numbered from 1 to 18. The test consists of eight problems. Each problem is worth the number of points specified in the table above. 100 points are counted as the full mark in this test. You are free to attempt any problem and collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one two-sided A4-paper "cheat sheet" (Spickzettel). You cannot use your own paper: should you need more paper, raise your hand and you will be given extra sheets.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in English or in German. Put your name on every sheet you hand in.
- You have 150 minutes.

GOOD LUCK!

Name

PROBLEM 1. (15 points) Which of these sets are dense subspaces of ℓ^2?

(i) $\{a = (a_1, a_2, \dots) \in \ell^2 \mid |a_{2010}| \leqslant |a_{2011}|\}$

(ii) $\{a = (a_1, a_2, \dots) \in \ell^2 \mid a_{2010} = a_{2011}\}$

(iii) $\left\{a = (a_1, a_2, \dots) \in \ell^2 \mid \sum_{n=1}^{\infty} \frac{a_n}{n} = 0\right\}$

SOLUTION:

Name

PROBLEM 2. (15 points) In $L^2[\frac{1}{2}, 2]$ consider the subspace

$$\mathcal{U} := \left\{ f \in L^2[\tfrac{1}{2}, 2] \mid f(x) = f(\tfrac{1}{x}) \text{ a.e. for } x \in [\tfrac{1}{2}, 2] \right\}.$$

(i) Prove that $\mathcal{U}^\perp = \{ f \in L^2[\frac{1}{2}, 2] \mid f(\frac{1}{x}) = -x^2 f(x) \text{ a.e. for } x \in [\frac{1}{2}, 2] \}$.

(ii) Find the orthogonal projection of the function $f_0(x) = x$ to the subspace $\mathcal{U}$.

SOLUTION:

Name

PROBLEM 3. (15 points)

(i) Show that the partial differential equation

$$(1-\Delta)\, f(x,y) \;=\; 3\cos(x+y)$$

has a unique solution in $H^2(S^1_x \times S^1_y)$ and compute it.

(ii) Exhibit a number $\alpha \in \mathbb{R}$ such that

$$(\alpha-\Delta)\, f(x,y) \;=\; 3\cos(x+y)$$

has no solution.

SOLUTION:

Name

PROBLEM 4 (20 points). Let $(X, \| \|_X)$ and $(Y, \| \|_Y)$ be Banach spaces and let $T : X \to Y$ be a linear map. Decide if the following statements are true or false (and give a proof or provide a counterexample):

(i) T is bounded $\Rightarrow \ker T$ is closed

(ii) $\left.\begin{array}{c} T \text{ is bounded} \\ \ker T = \{0\} \\ \operatorname{Im} T \text{ is closed} \end{array}\right\} \Rightarrow T$ is invertible on $\operatorname{Im} T$ and $T^{-1} : \operatorname{Im} T \to X$ is bounded

(iii) $\ker T$ is closed $\Rightarrow T$ is bounded

(iv) $\left.\begin{array}{c} T \text{ is bounded} \\ \ker T = \{0\} \end{array}\right\} \Rightarrow \operatorname{Im} T$ is closed

Here $\ker T$ is the kernel of T and $\operatorname{Im} T$ is the range of T.

SOLUTION:

Name

PROBLEM 5. (20 points) Decide if the following formulæ give a well-defined and bounded linear functional and, when they do, compute the norm of the functional:

(i) $\phi : L^2[-1,1] \to \mathbb{C}$ with $\phi(f) := \int_{-1}^{1} x f(x)\,\mathrm{d}x$

(ii) $\phi : \ell^1 \to \mathbb{C}$ with $\phi(x) := \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{n}}$ $(x = \{x_n\}_{n=1}^{\infty})$

(iii) $\phi : \ell^2 \to \mathbb{C}$ with $\phi(x) := \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{n}}$ $(x = \{x_n\}_{n=1}^{\infty})$

(iv) $\phi : \ell^\infty \to \mathbb{C}$ with $\phi(x) := \sum_{n=1}^{\infty} \frac{x_n}{n^2}$ $(x = \{x_n\}_{n=1}^{\infty})$.

SOLUTION:

Name

PROBLEM 6. (15 points)

(i) Let $(X, \| \ \|_X)$ be a normed vector space and let $x \in X$ such that $|\phi(x)| \leqslant 1 \ \forall \phi \in X^*$ with $\|\phi\|_{X^*} \leqslant 1$. Prove that $\|x\|_X \leqslant 1$.

(ii) Show that the unit ball in $L^2_{\mathbb{R}}[0,1]$ (the real vector space of square-summable functions $[0,1] \to \mathbb{R}$) can be represented as the intersection of countably many half-spaces. (Recall that a half-space in a real normed space X is the set $\{x \in X \mid \phi(x) \leqslant 1\}$ for some $\phi \in X^*$.)

SOLUTION:

Name

PROBLEM 7. (20 points) For every positive integer n, let

$$M_n \ := \ \left\{ f \in L^2[0,1] \;\middle|\; \int_0^1 |f(x)|^2 \mathrm{d}x \leqslant n \right\}.$$

(i) Show that $L^2[0,1] = \bigcup_{n=1}^{\infty} M_n$.

(ii) Show that each M_n is a closed subset in $L^1[0,1]$. (*Hint:* apply Fatou's lemma on convergent sequences in M_n.)

(iii) Show that the interior of each M_n, in the topology of $L^1[0,1]$, is empty. (*Hint:* you can use the result of a recent homework stating that a proper subspace in a normed space has empty interior.)

(iv) From (i)–(iii) it appears that $L^2[0,1]$ is the countable union of nowhere dense sets. Why does not this contradict Baire's theorem?

SOLUTION:

Name

PROBLEM 8. (15 points) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that

$$\sum_{n=1}^{\infty} a_n b_n < \infty \qquad \text{for all sequences } \{b_n\}_{n=1}^{\infty} \text{ in } c_0.$$

Prove that $\{a_n\}_{n=1}^{\infty}$ is in ℓ^1.

SOLUTION: