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Few-body and many-body physics with zero-range interactions: theory and experiments with ultra-cold atomic gases

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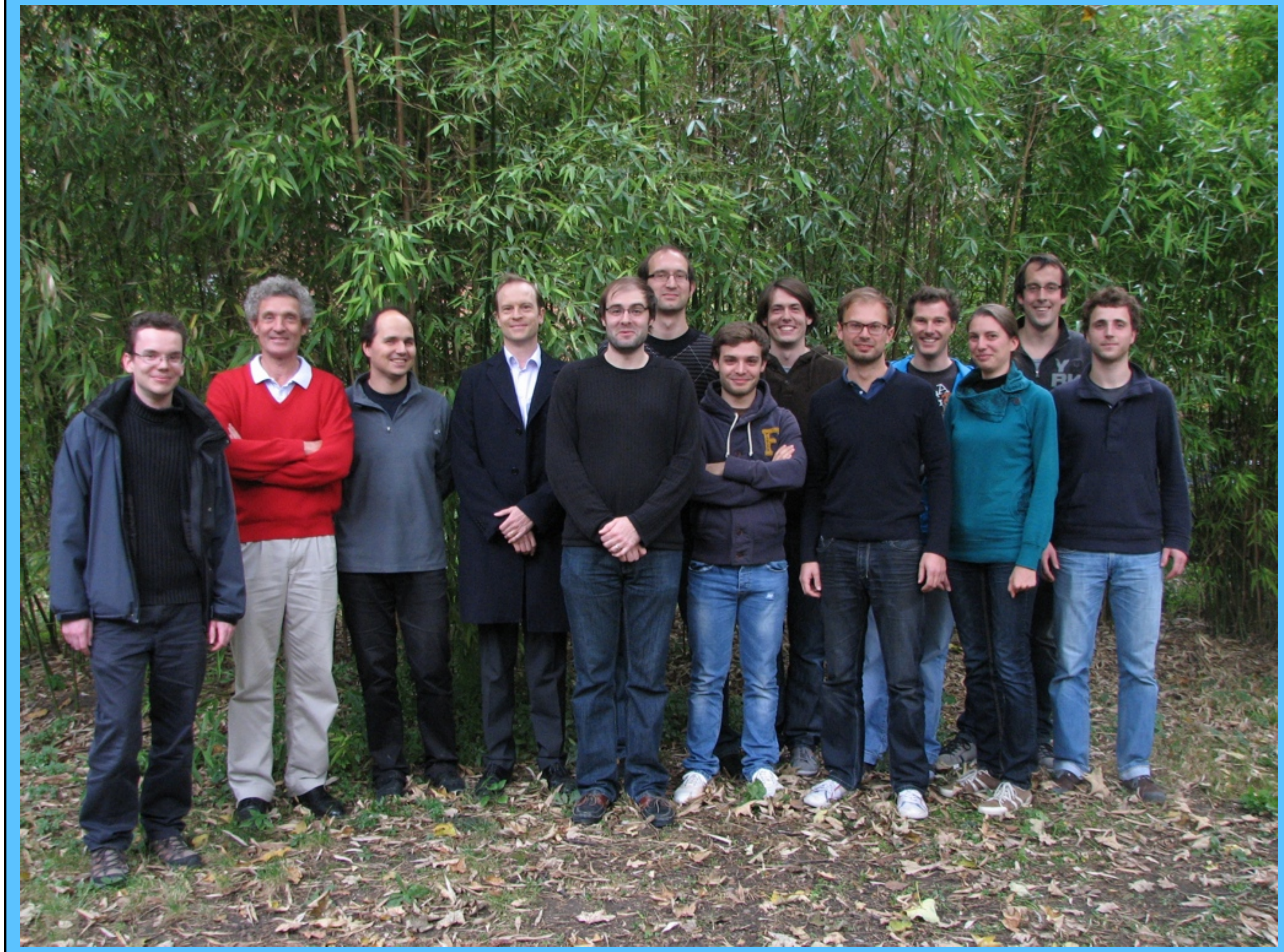

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Some history

nuclear physics

- **Wigner** 1933
- **Bethe and Peierls** 1935
- **...**
- **Efimov** 1970

quantum gases

- **Einstein** 1924
- **cooling and trapping of atomic vapors, BEC** 1995
  key: low-density limit
- **Feshbach resonances** ~2000
- **Strongly correlated degenerate Fermi gases, unitary gas and BEC-BCS crossover** since ~2005
- **Efimov physics (mostly in Bose gases)** since ~2006

Models

3D N particles $\vec{r}_i \quad i \in \{1, 2, \ldots, N\}$

$\psi(\vec{r}_1, \ldots, \vec{r}_N) \qquad H\,\psi = E\,\psi$

lattice model $\vec{r}_i \in (b\mathbb{Z})^3 \qquad \Delta_{\vec{r}}\, e^{i\vec{k}\cdot\vec{r}} = -k^2\, e^{i\vec{k}\cdot\vec{r}}$

$$H = \sum_{i=1}^{N} \left[-\frac{1}{2m_i}\,\Delta_{\vec{r}_i} + U(\vec{r}_i) \right] + g_0 \sum_{1\leq i<j\leq N} \frac{\delta_{\vec{r}_i,\vec{r}_j}}{b^3}$$

choices for U(r)

- free space: $U(\vec{r}) = 0$
- harmonic trap: $U(\vec{r}) = \frac{1}{2}\, m_i\, \omega^2\, r^2$
- homogeneous system in the thermodynamic limit:
  - box : $U(\vec{r}) = 0$ if $\vec{r} \in [0; L]^3$, $+\infty$ otherwise
  - then take $N \to \infty$, $L \to \infty$ with N/L^3 fixed

scattering length $\quad N = 2 \quad$ free space

$$E = \frac{k^2}{2\,m_r} > 0 \qquad \frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}$$

$$\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}) \qquad \vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\psi(\vec{r}) \underset{r\to\infty}{\simeq} e^{i\,\vec{k}\cdot\vec{r}} + f_{\vec{k}}(\hat{r})\,\frac{e^{ikr}}{r}$$

$$f_{\vec{k}}(\hat{r}) \xrightarrow[k\to 0]{} -a$$

$$\frac{1}{g_0} = \left(-K\,\frac{1}{b} + \frac{1}{a}\right)\frac{m_r}{2\pi}\,, \qquad K = 2.442749\ldots$$

continuum limit $\quad b \to 0 \;\text{ with }\; \dfrac{1}{a} \;\text{ fixed}$

remarks:
- $\Leftrightarrow$ zero-range limit
- $g_0 \to 0^-$

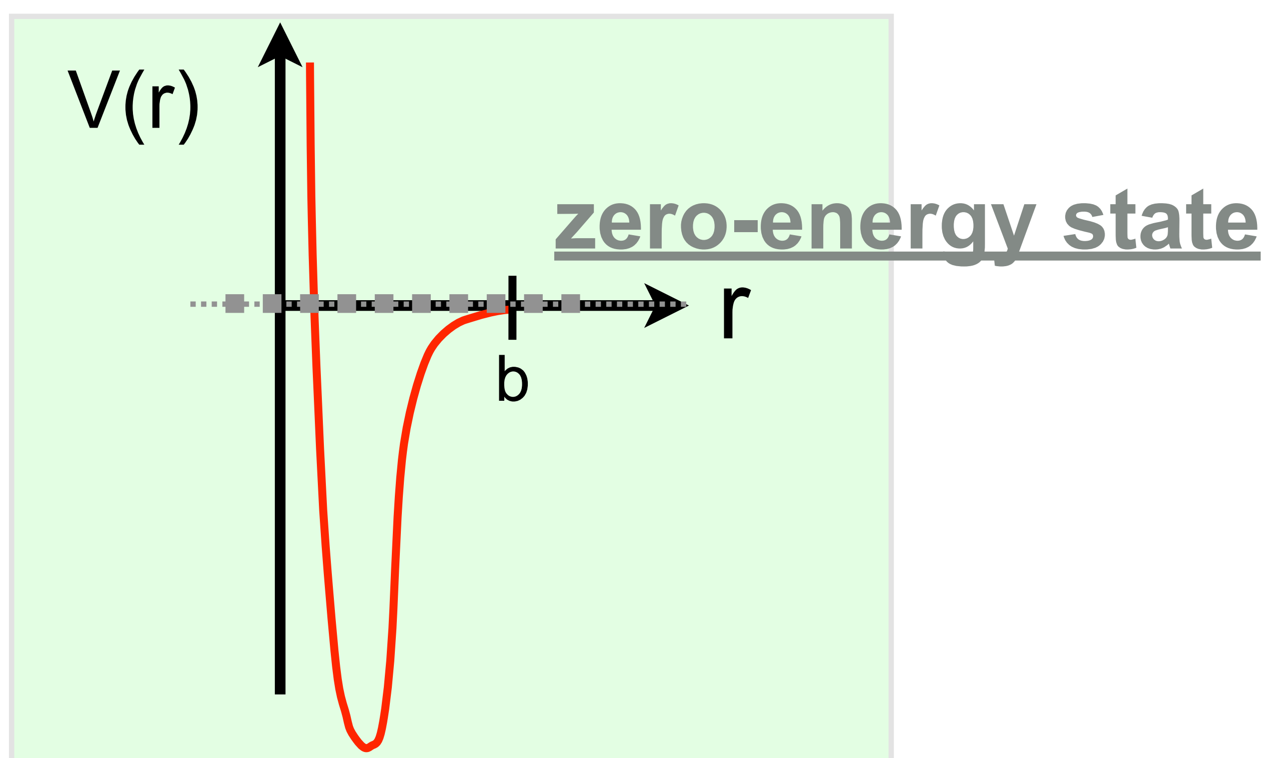
continuous-space finite-range models

$\vec{r}_i \in \mathbb{R}^3$

$$H = \sum_{i=1}^{N} \left[-\frac{1}{2m_i} \Delta_{\vec{r}_i} + U(\vec{r}_i) \right] + \sum_{1 \le i < j \le N} V(r_{ij})$$

$$r_{ij} = \| \vec{r}_i - \vec{r}_j \|$$

For $a = \infty$: $\quad V(r) = \dfrac{1}{mb^2} \, f\left(\dfrac{r}{b}\right)$



Bosons: $\psi(\vec{r}_1, \ldots, \vec{r}_N)$ symmetric

$$m_i = m$$

Fermions: 2 components $\uparrow$ and $\downarrow$

$$\psi(\underbrace{\vec{r}_1, \ldots, \vec{r}_{N_\uparrow}}_{\substack{\uparrow \\ \text{antisymmetric}}}, \underbrace{\vec{r}_{N_\uparrow+1}, \ldots, \vec{r}_{N_\uparrow+N_\downarrow}}_{\substack{\downarrow \\ \text{antisymmetric}}})$$

$$m_\uparrow \qquad\qquad m_\downarrow$$

Universality

(strict sense)
expected for equal mass fermions

- **all eigenvalues and eigenstates of H converge when b -> 0**

- **the limit is the same for “any” finite-range model (lattice models, continuous space models)**

- **this limit is described by the zero-range model**

zero-range model

$$H = \sum_{i=1}^{N} \left[-\frac{1}{2m_i} \Delta_{\vec{r}_i} + U(\vec{r}_i) \right]$$

2-body contact condition:

$$\psi(\vec{\mathbf{r}}_1, \ldots, \vec{\mathbf{r}}_N) \underset{\substack{r_{ij} \to 0 \\ \vec{R}_{ij} \text{ fixed}}}{=} \left(\frac{1}{r_{ij}} - \frac{1}{a} \right) A_{ij}\left(\vec{\mathbf{R}}_{ij}, (\vec{\mathbf{r}}_k)_{k \neq i,j} \right) + O(r_{ij})$$

$$\vec{R}_{ij} = \frac{m_i \vec{r}_i + m_j \vec{r}_j}{m_i + m_j}$$

if Efimov effect: *3-body contact condition* *(see later)*

(no universality in the strict sense)

2-body problem

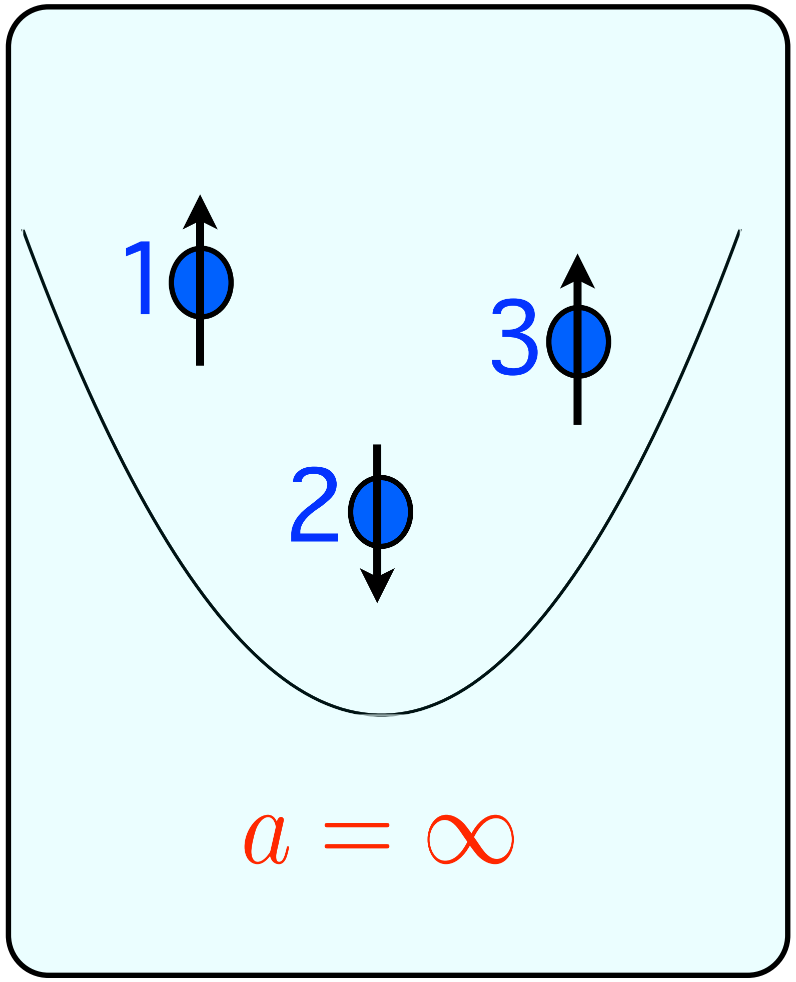
$$f_{\vec{k}} = -\frac{1}{\frac{1}{a} + ik}$$

$$\sigma \propto \frac{1}{\frac{1}{a^2} + k^2}$$

For $a > 0$: 2−body bound state $\quad E = -\frac{\hbar^2}{ma^2}$

3-body problem

Unitary 3-body problem in a trap - equal mass fermions



- $\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = -\psi(\vec{r}_3, \vec{r}_2, \vec{r}_1)$
- Zero-range interactions:

$$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) \underset{r_{ij} \to 0}{=} A \cdot \left(\frac{1}{r_{ij}} - \cancel{\frac{1}{a}} \right) + O(r_{ij})$$

- When all r_{ij} 's are > 0 :

$$\sum_{i=1}^{3} \left[-\frac{\hbar^2}{2m} \Delta_{\mathbf{r}_i} + \frac{1}{2} m \omega^2 r_i^2 \right] \psi = E \psi.$$

Exactly solvable

Center-of-mass is separable: $\vec{C} = (\vec{r}_1 + \vec{r}_2 + \vec{r}_3)/3$

$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \psi_{\rm cm}(\vec{C})\ \psi_{\rm int}$ $\qquad E = E_{\rm cm} + E_{\rm int}$

Hyperradius: $R = \sqrt{\sum_{i<j} {r_{ij}}^2 / 3}$

5 remaining coordinates:
Hyperangles $\vec{\Omega} \equiv \vec{R}/R$

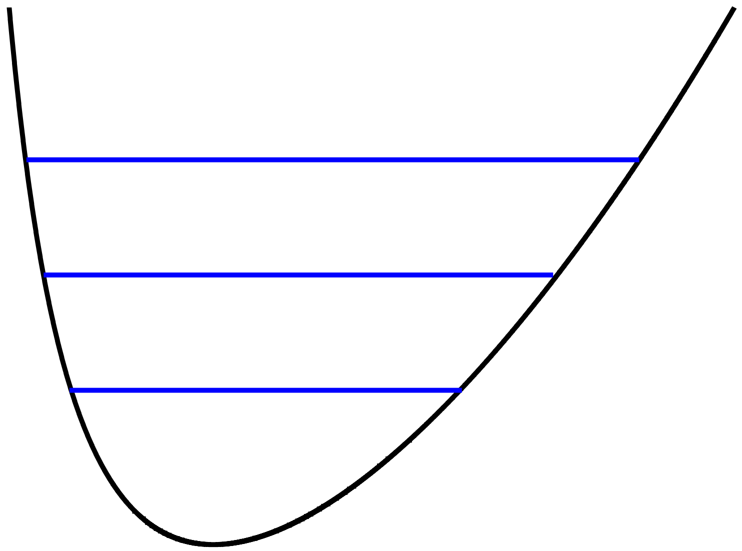
$$\boxed{\psi_{\rm int} = F(R)\, R^{-2}\ \phi(\vec{\Omega})}$$

$F(R)$ = wavefunction of a fictitious particle

in a potential $U_{\rm eff}(R) = \dfrac{\hbar^2}{2m}\dfrac{s^2}{R^2} + \dfrac{1}{2} m\omega^2 R^2$

$E_{\rm int}$ = energy of this fictitious particle

$= (s + 1 + 2q)\hbar\omega, \ \ q \in \mathbb{N}$



$\phi(\vec{\Omega})$ and s are the same than in free space

known since Efimov

For $l = 0$

$$-s\cos\left(s\frac{\pi}{2}\right) + \eta\frac{4}{\sqrt{3}}\sin\left(s\frac{\pi}{6}\right) = 0 \qquad \eta = -1$$

For $l = 1:$

smallest solution $s = 1.772724267\ldots$

Unitary 3-body problem - BOSONS

$$\psi_{\text{int}} = F(R)\, R^{-2}\;\; \phi(\vec{\Omega})$$

$$U_{\text{eff}}(R) = \frac{\hbar^2}{2m}\frac{s^2}{R^2} + \frac{1}{2}m\omega^2 R^2$$

one of the allowed values of s is imaginary $^{(*)}$

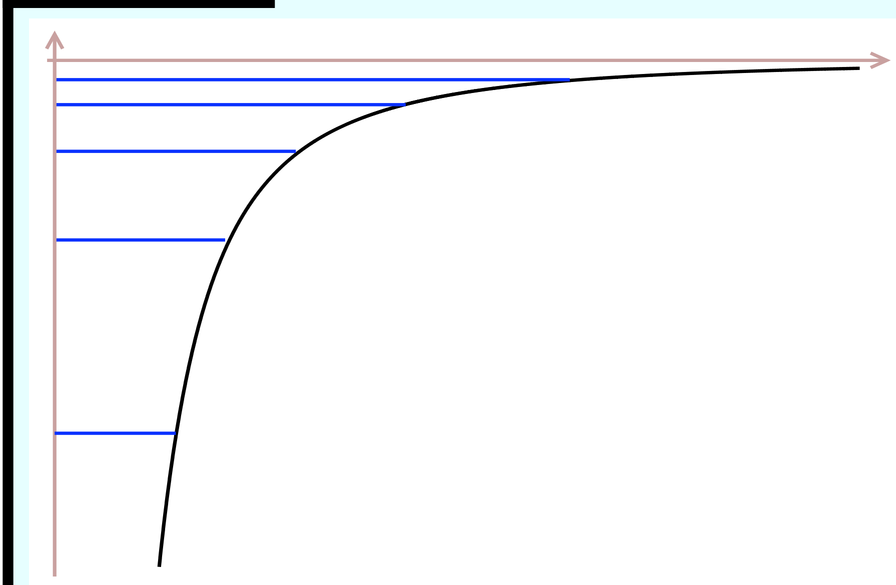
$\Rightarrow$ $U_{\text{eff}}(R)$ is pathologically attractive

$\Rightarrow$ need to add the 3-body contact condition:

$$F(R) \underset{R\to 0}{\propto} \sin\left[|s|\ln(R/R_t)\right]$$

3-body parameter

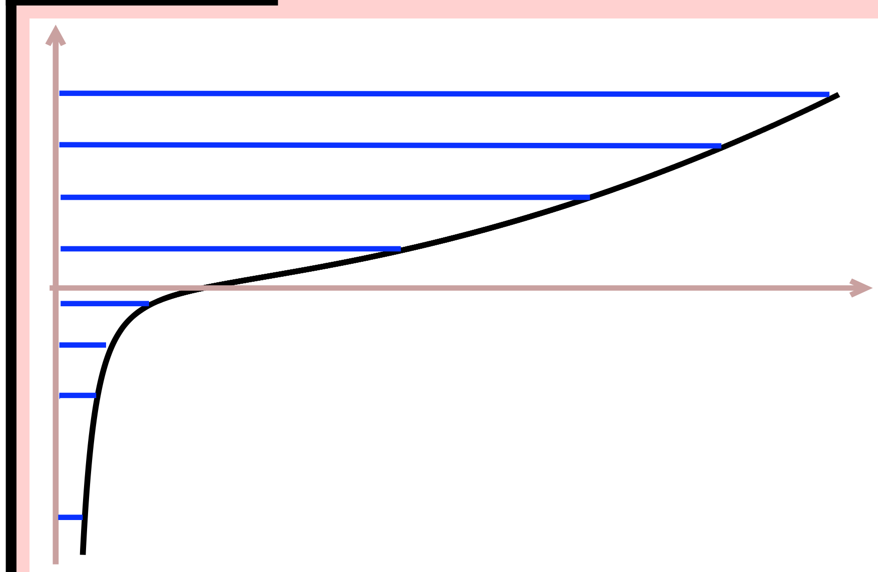
$\omega = 0$



Efimov trimer states

E_{int} =geometric series

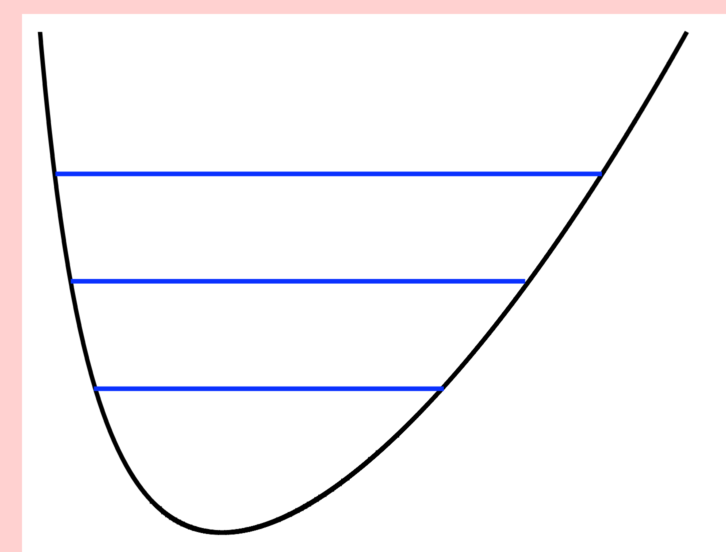
$\omega > 0$



Efimovian states

$E_{\text{int}}(R_t)$

Other s are real



Universal states

$E_{\text{int}} = (s + 1 + 2q)\hbar\omega$

(*)

For $l = 0$

$$-s\cos\left(s\frac{\pi}{2}\right)+\eta\frac{4}{\sqrt{3}}\sin\left(s\frac{\pi}{6}\right)=0 \qquad \eta=+2$$

imaginary solution $s = i\cdot 1.0062378251\ldots$

- **3 equal mass fermions: universal**
- **3 bosons: Efimov effect**
- **3 unequal mass fermions:**

$$N_\uparrow = 2, \quad N_\downarrow = 1$$

$$\text{Efimov effect for } m_\uparrow / m_\downarrow > 13.607\ldots$$

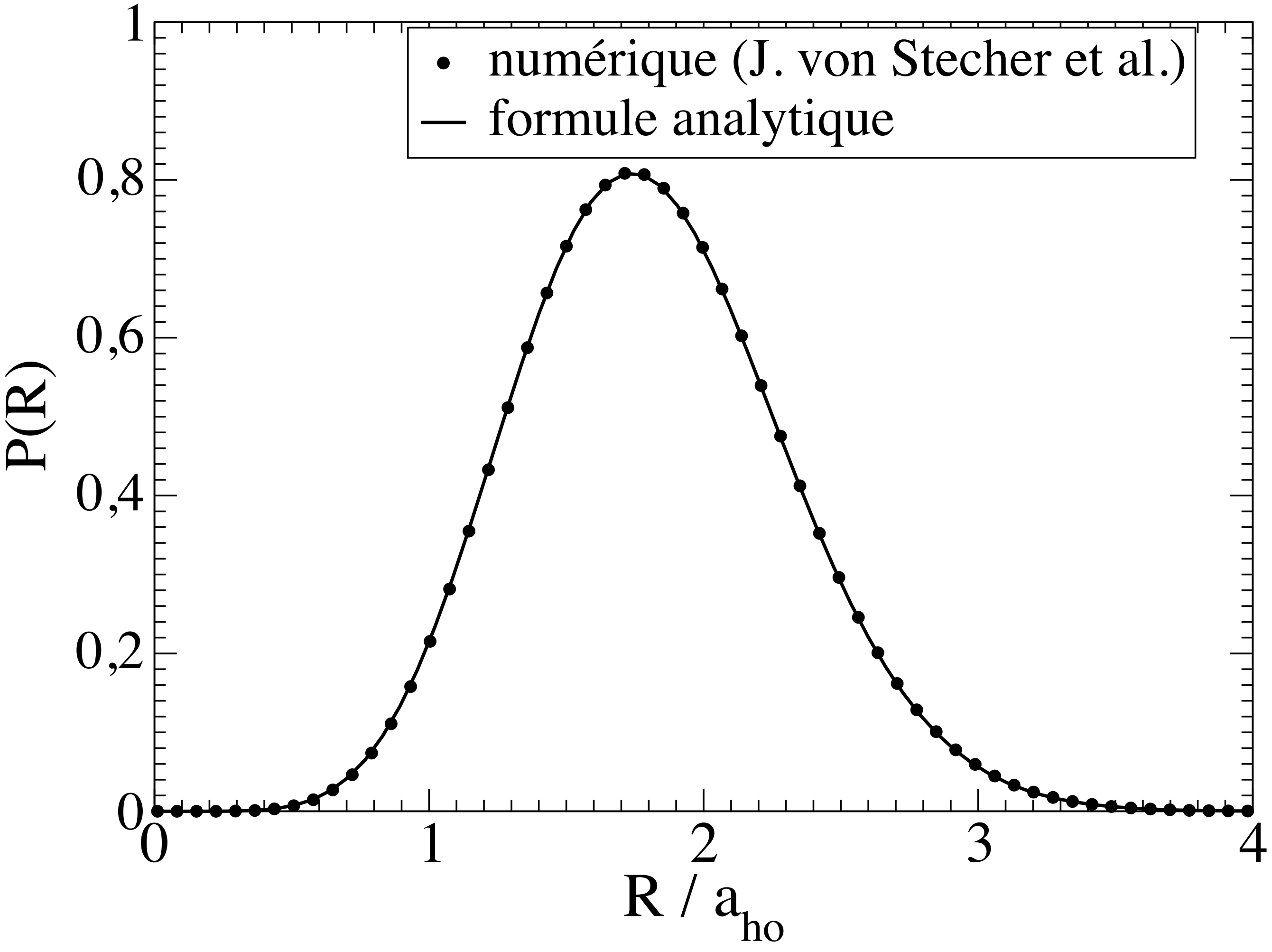
4-body problem

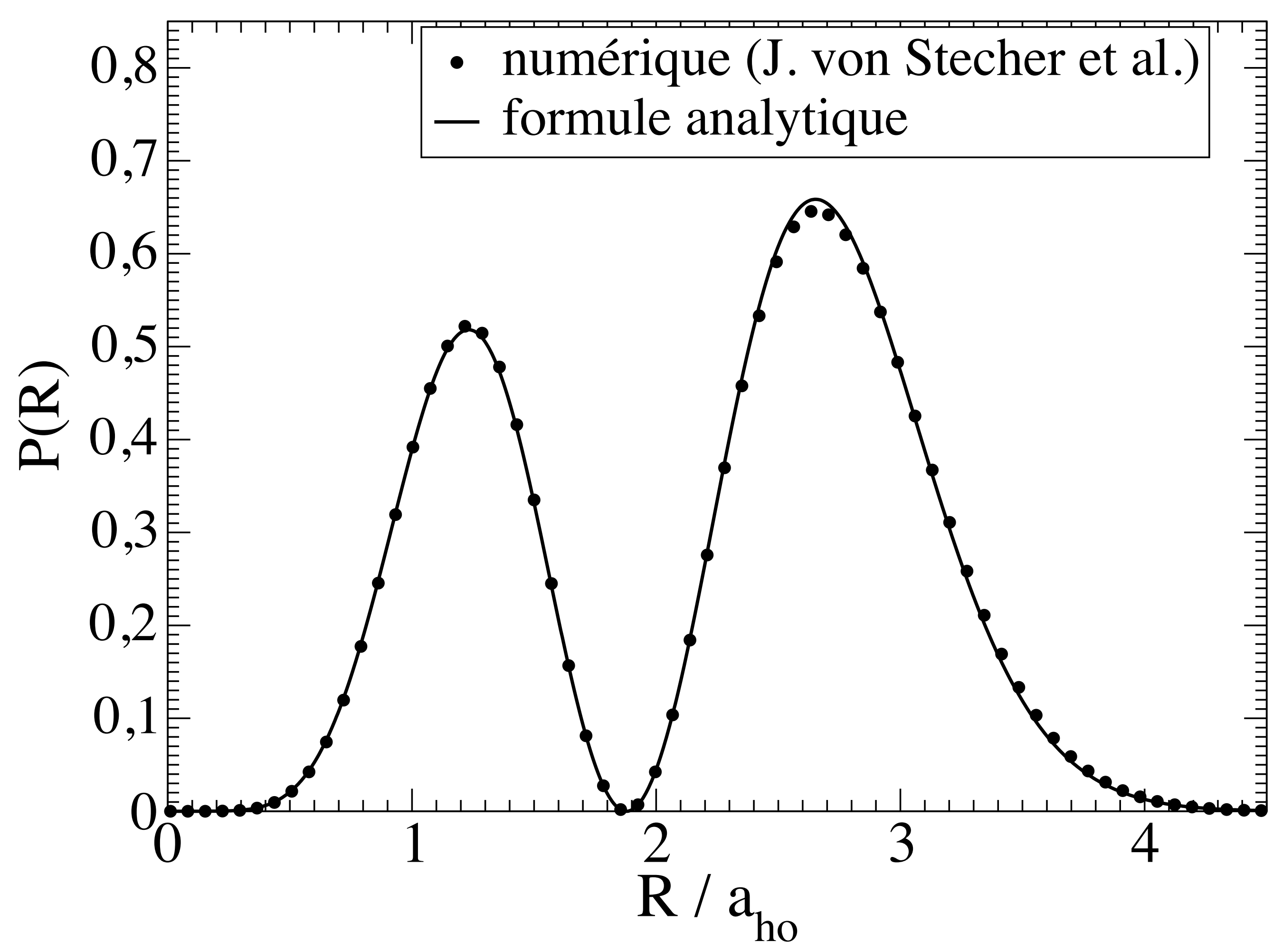
- **Fermions,** $m_\uparrow = m_\downarrow$
- **Fermions,** $m_\uparrow \neq m_\downarrow$
- **Bosons**

4 fermions, $m_\uparrow = m_\downarrow$

universal

$$N_\uparrow = N_\downarrow = 2, \quad \text{harmonic trap}$$





4 fermions, $m_\uparrow \neq m_\downarrow$

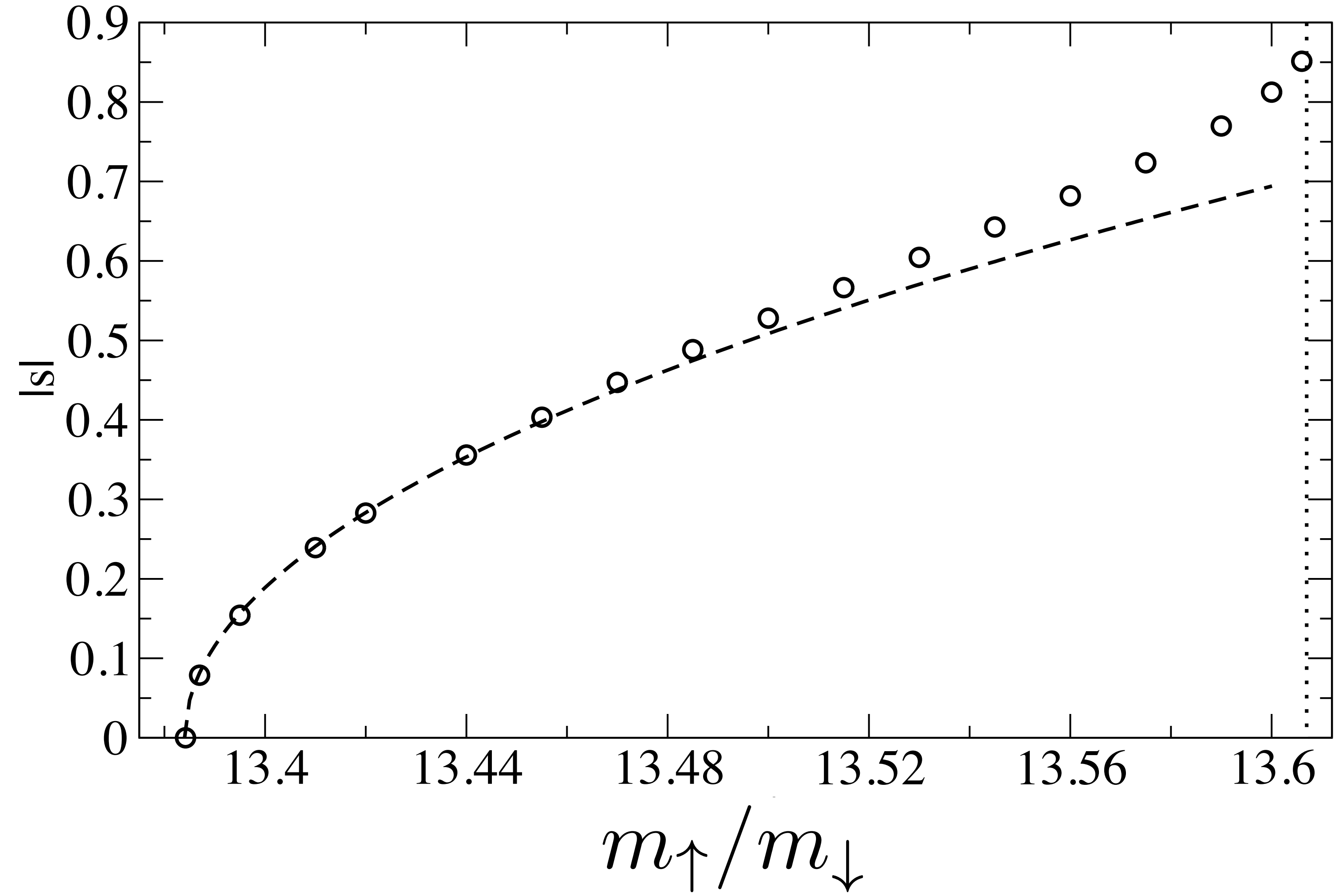
$$N_\uparrow = 3,\ N_\downarrow = 1$$

$$\text{if } 13.384\ldots < m_\uparrow/m_\downarrow < 13.607\ldots \ : \ \ 4-\text{body Efimov effect}$$

[Castin Mora Pricoupenko 2010]

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = R^{-7/2} F(R) f(\Omega) \qquad F(R) \underset{R\to 0}{\sim} \mathrm{Im}\left[\left(\frac{R}{R_f}\right)^s\right]$$

numerical calculation of s



4 bosons

strong numerical + some experimental evidence:
no 4-body parameter
only a, R_t

[Hammer Platter 2007; von Stecher D'Incao Greene 2009; Deltuva 2011]
[Ferlaino et al. 2009]

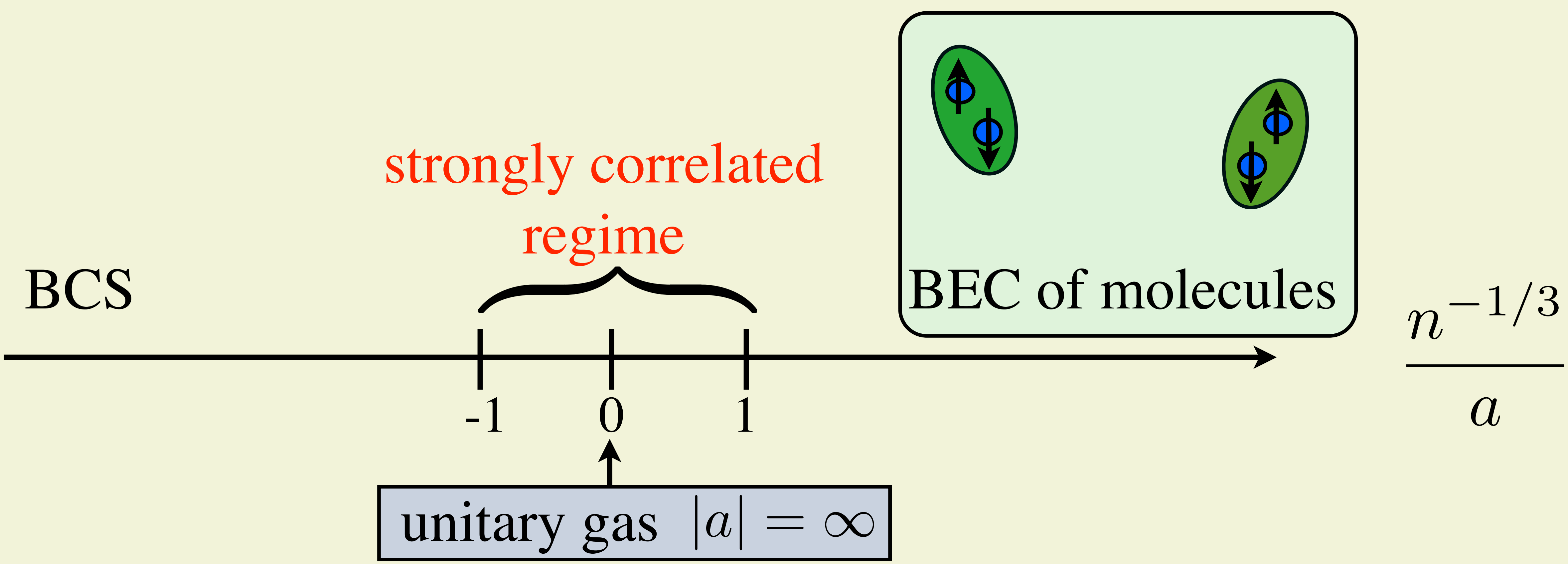
4-body quasi-bound states (i.e. resonances)

Many-body problem

$$\text{Fermions}, \quad N_\uparrow = N_\downarrow \to \infty$$

$$n = \frac{N}{L^3} \quad \text{fixed}$$

BEC-BCS crossover



Equation of state of the unitary gas in the normal unpolarised phase: Cross-validation between Bold Diagrammatic Monte Carlo and ultracold-atoms precision measurements

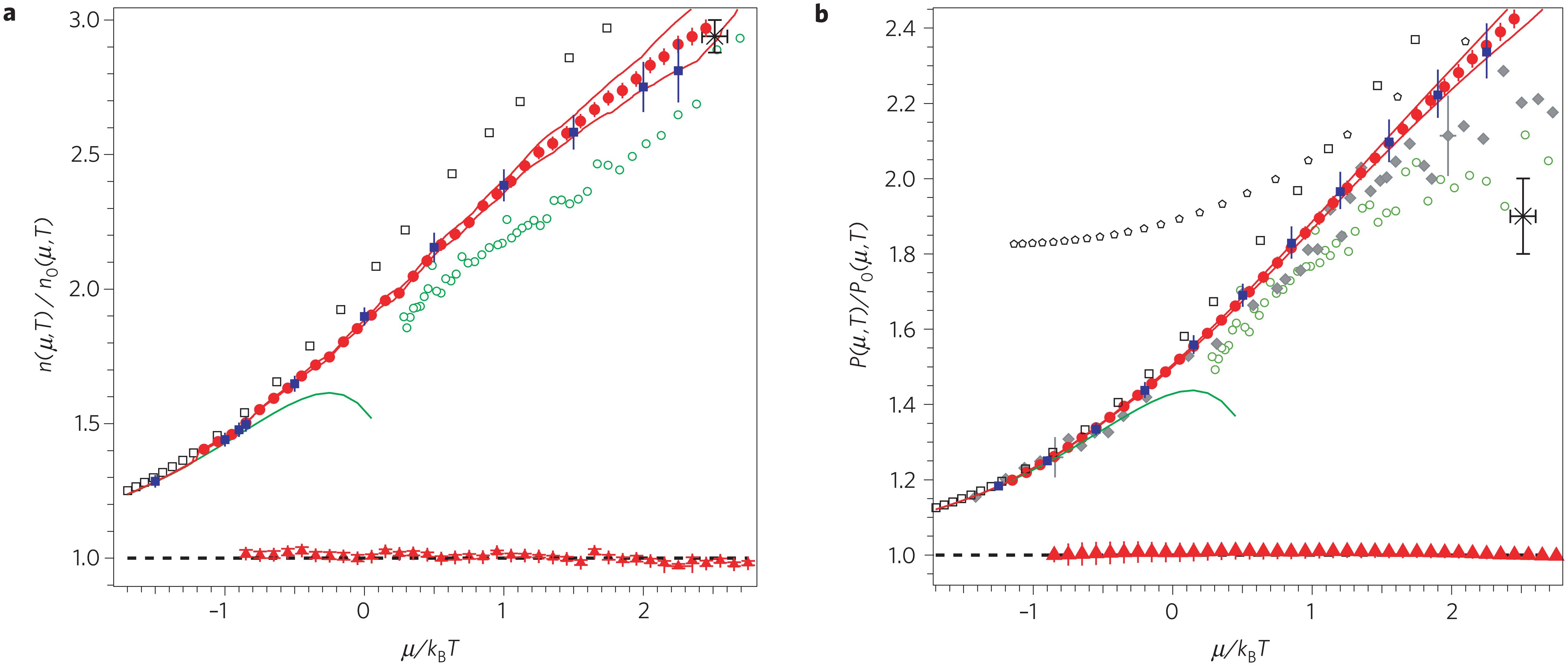
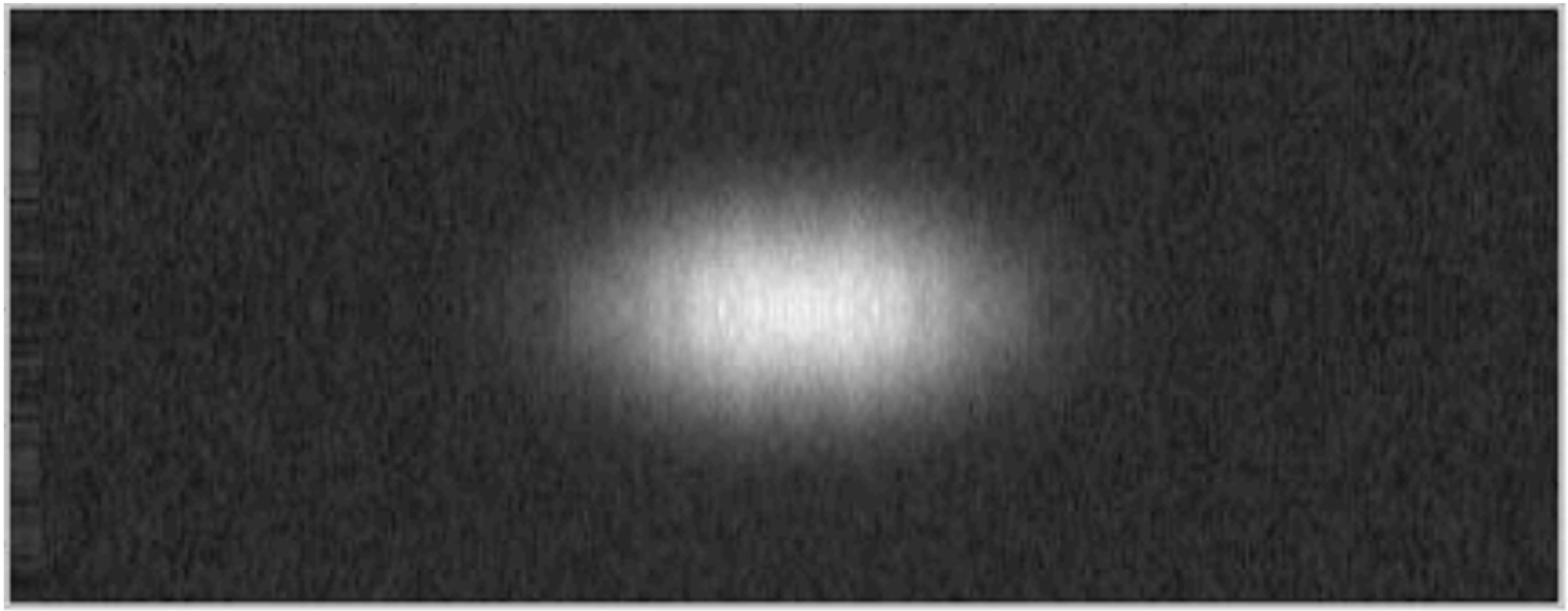


Figure 4 | Equation of state of the unitary Fermi gas in the normal phase. Density n (**a**) and pressure P (**b**) of a unitary Fermi gas, normalized by the density n_0 and the pressure P_0 of a non-interacting Fermi gas, versus the ratio of chemical potential μ to temperature T. Blue filled squares: BDMC (this work), red filled circles: experiment (this work). The BDMC error bars are estimated upper bounds on systematic errors. The error bars are one standard deviation systematic plus statistical errors, with the additional uncertainty from the Feshbach resonance position shown by the upper and lower margins as red solid lines. Black dashed line and red triangles: Theory and experiment (this work) for the ideal Fermi gas, used to assess the experimental systematic error. Green solid line: third order virial expansion. Open squares: first order bold diagram[15,21]. Green open circles: Auxiliary Field QMC (ref. 11). Star: superfluid transition point from Determinental Diagrammatic Monte Carlo[13]. Filled diamonds: experimental pressure EOS (ref. 22). Open pentagons: pressure EOS (ref. 23).



in-situ absorbtion image [MIT]

virial expansion:

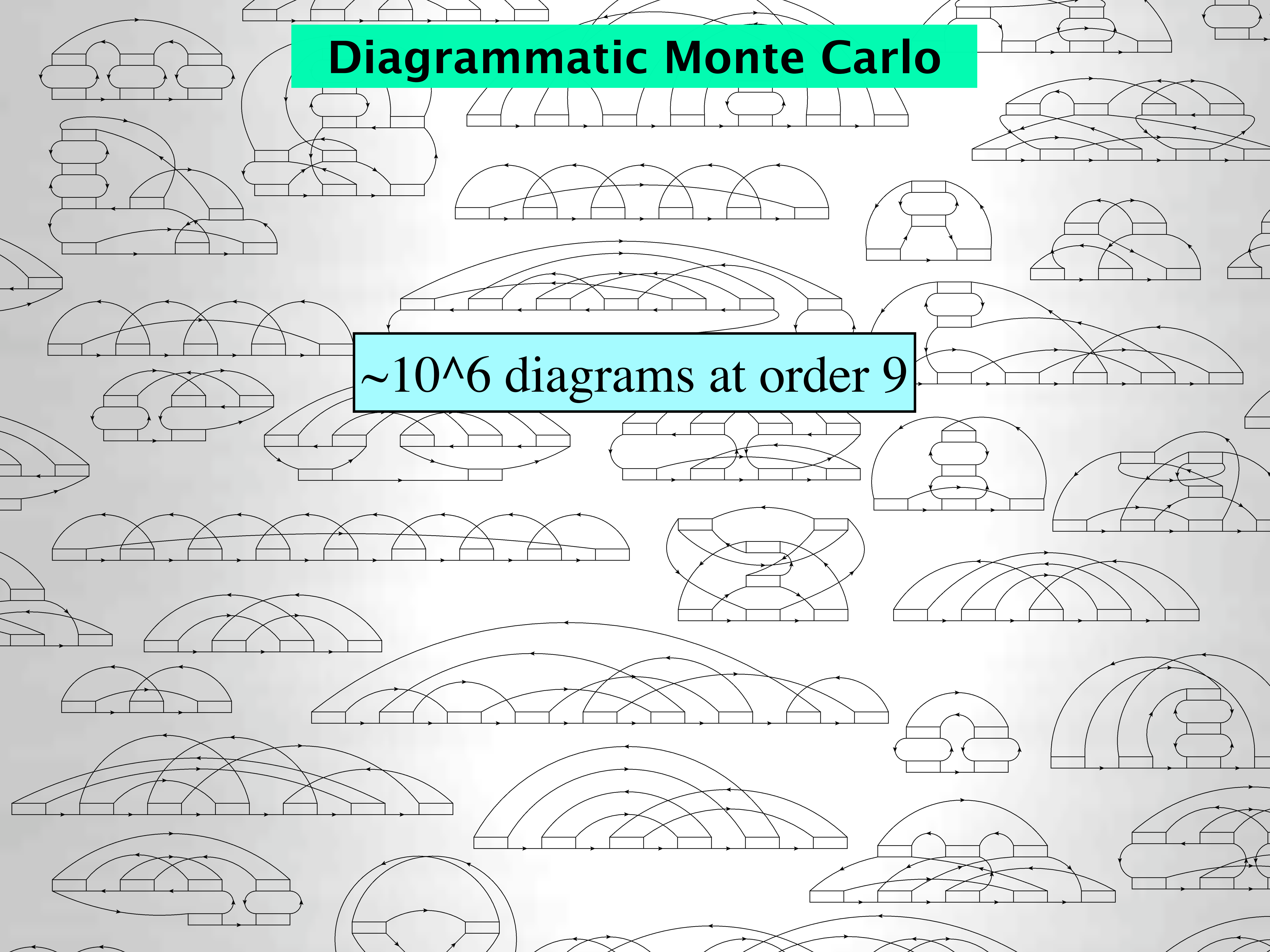
$$n\lambda^3 \underset{\beta\mu\to-\infty}{=} 2(e^{\beta\mu} + 2b_2 e^{2\beta\mu} + 3b_3 e^{3\beta\mu} + \ldots)$$

$$b_n \text{ comes from the } n - \text{body problem}$$

$$\boxed{b_3 = -0.29095295\ldots.}$$

Liu Hu Drummond 2009

Diagrammatic Monte Carlo

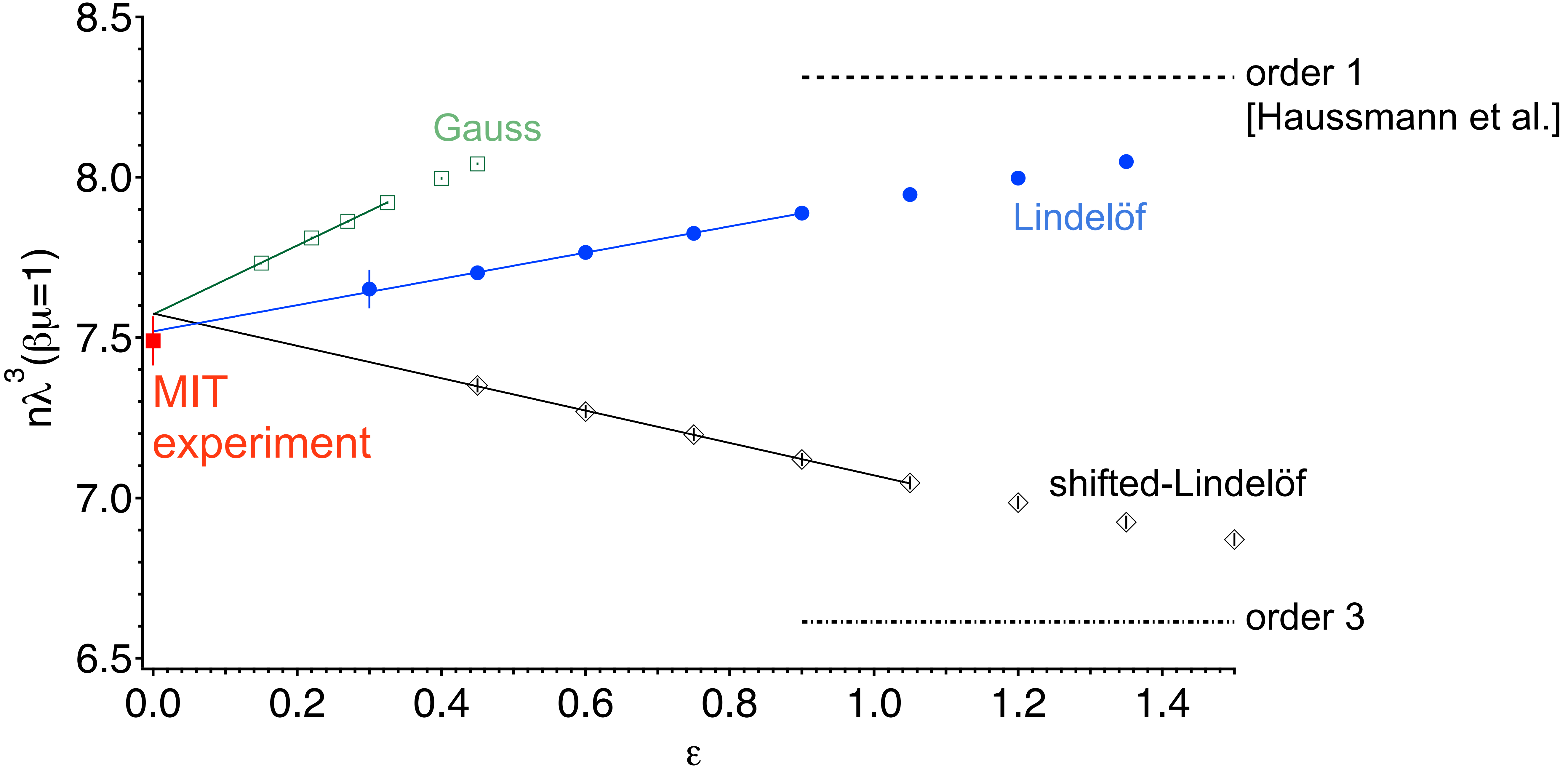


~10^6 diagrams at order 9

ABELIAN RESUMMATION

$$\Sigma = \lim_{\epsilon\to 0} \sum_{N\geq 1} \Sigma^{(N)} \cdot e^{-\epsilon\,\lambda_{N-1}}$$

$\lambda_n = n\ \log(n)$	(Lindelöf)
$\lambda_n = n^2$	(Gaussian)
$\lambda_n = (n-1)\log(n-1)$ with $\lambda_0 = \lambda_1 = 0$	(Shifted Lindelöf)

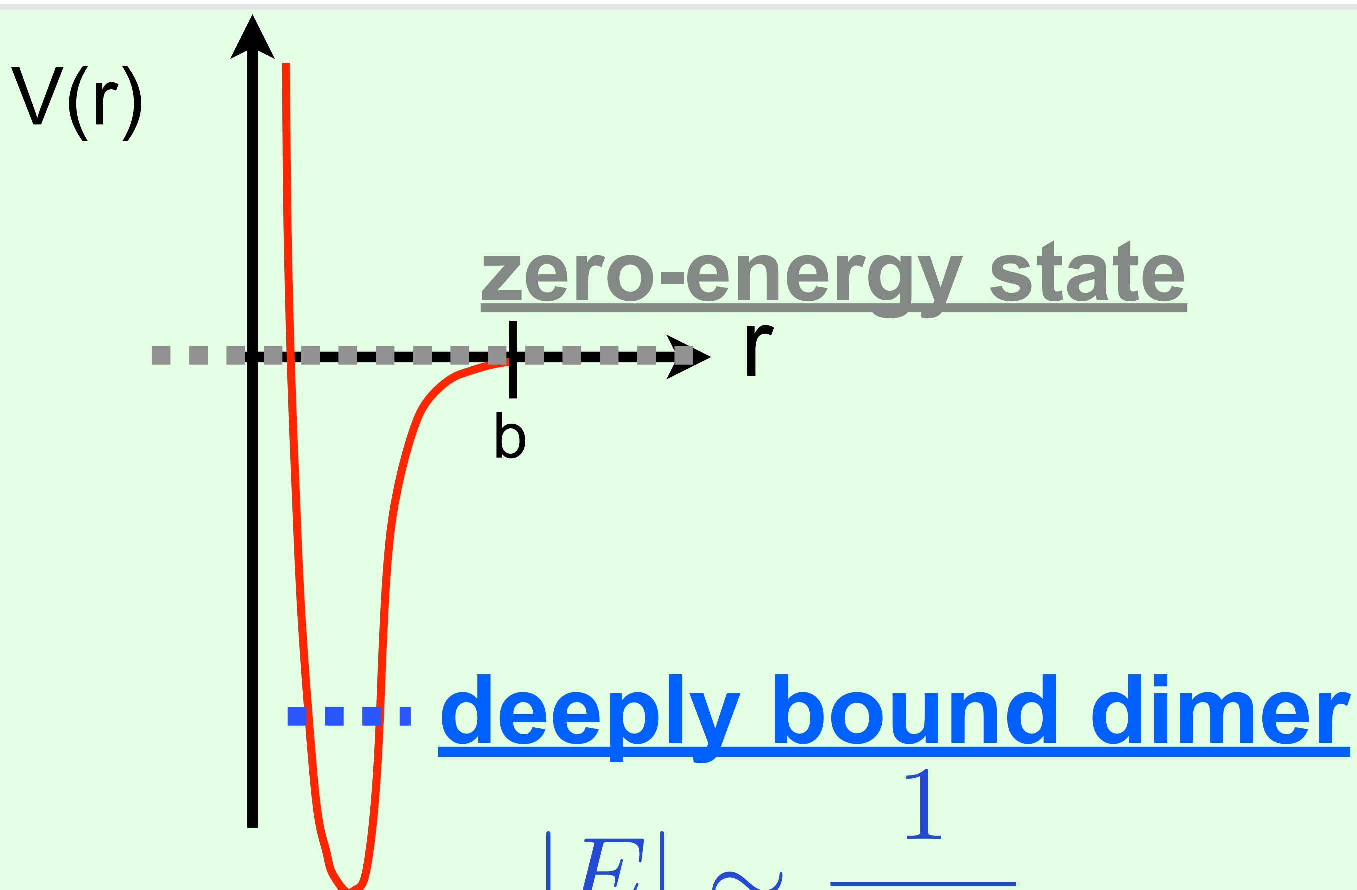


3-body losses

continuous-space finite-range model

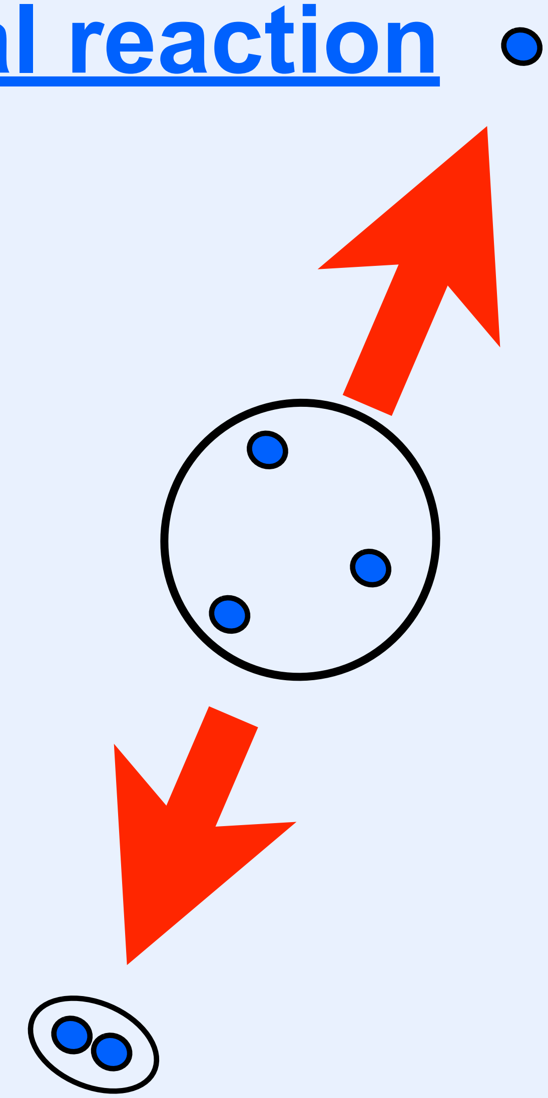
$$H = \sum_{i=1}^{N} \left[-\frac{1}{2m_i} \Delta_{\vec{r}_i} + U(\vec{r}_i) \right] + \sum_{1 \le i < j \le N} V(r_{ij})$$

For $a = \infty$: $\quad V(r) = \frac{1}{mb^2} \, f\left(\frac{r}{b}\right)$



$$|E| \sim \frac{1}{mb^2}$$

chemical reaction



Fermions

3-body loss rate vanishes in the zero-range limit

Example: Unitary Fermi gas $T = 0$

$$\Gamma \equiv -\frac{\dot{n}}{n}$$

$$\Gamma \underset{b \to 0}{\sim} K \; b^{2s} \; n^{(2s+2)/3}$$

K model-dependent constant

$s = 1.772724267\ldots$

[Petrov Salomon Shlyapnikov 2004]

Bosons

3-body loss rate does not vanish in the zero-range limit

***modified** 3-body contact condition* [Braaten Hammer 2003]

$$F(R) \underset{R\to 0}{\propto} \left(\frac{R}{R_t}\right)^{-s} - e^{-2\eta_*}\left(\frac{R}{R_t}\right)^{s}$$

η_* inelasticity parameter

Efimov trimers

decay rate

$$\Gamma \equiv -\frac{2}{\hbar}\,\mathrm{Im}\,E$$

$$\hbar\Gamma \simeq \frac{4\eta_*}{|s|}\,|E(\eta_*=0)| \qquad \eta_* \ll 1$$

Unitary Bose gas

$$\text{high}-\text{temperature regime} \quad T \gg \frac{n^{2/3}}{m}$$

$$\dot{n} = -n^3 \quad L_3(T)$$

$$L_3(T) = \frac{1}{m^3 T^2}\, f(R_t \sqrt{T}, \eta_*)$$

f dimensionless function

periodic function of $\ln(R_t\sqrt{T})$, period π/s_0

$$s_0 = 1.0062378\ldots$$

analytical expression for f [Petrov FW 2013]

$$f \approx 36\sqrt{3}\pi^2(1 - e^{-4\eta_*})$$

comparison with ENS experiment [Rem et al. 2013]

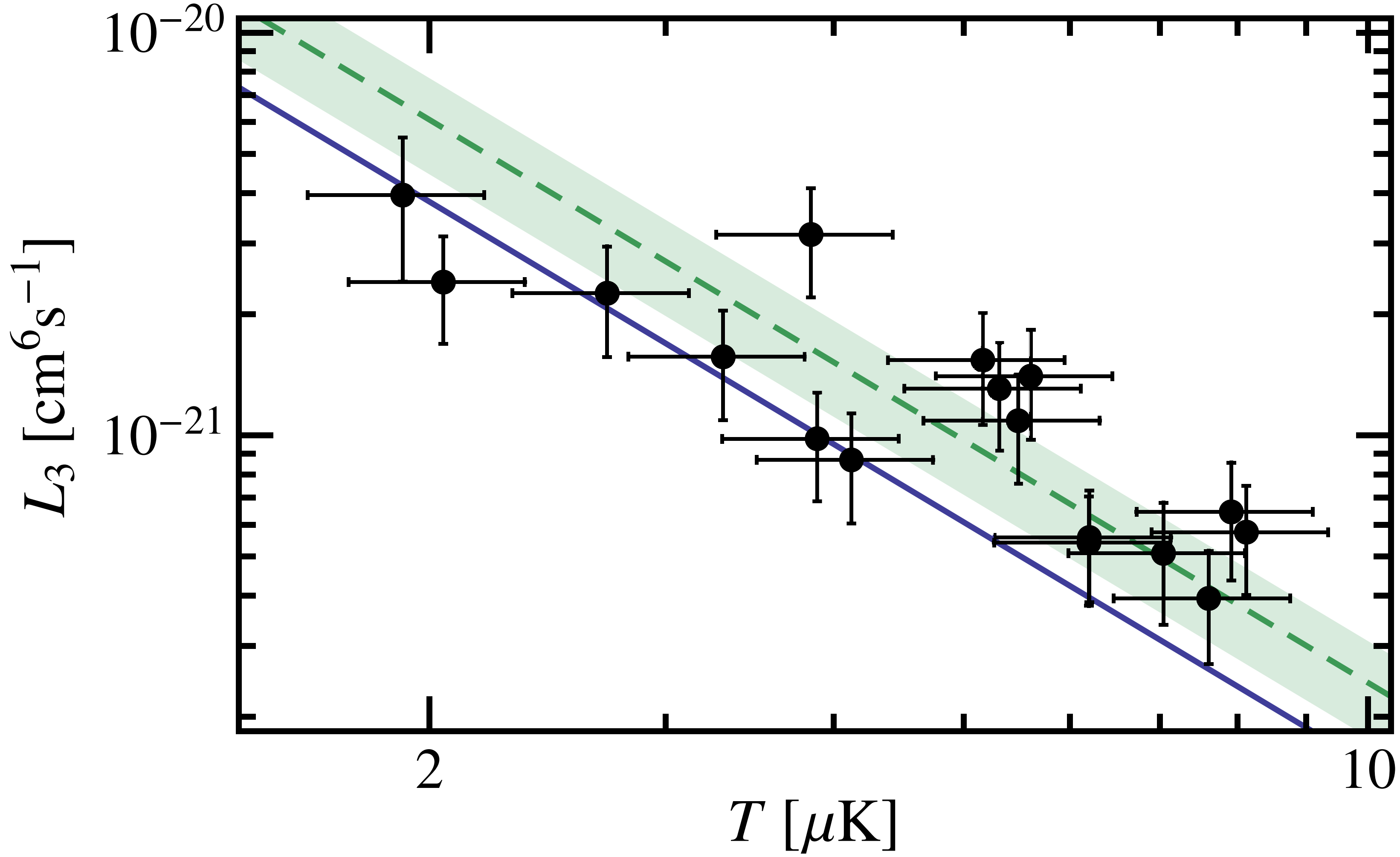


FIG. 2 (color online). Temperature dependence of the three-body loss rate L_3. Filled circles, experimental data; green dashed line, best fit to the data $L_3(T) = \lambda_3/T^2$ with $\lambda_3 = 2.5(3)_{\mathrm{stat}}(6)_{\mathrm{syst}} \times 10^{-20}\,(\mu\mathrm{K})^2\mathrm{cm}^6\mathrm{s}^{-1}$; the shaded green band shows the 1σ quadrature sum of uncertainties. Solid line, prediction from Eq. (5), $\lambda_3 = 1.52 \times 10^{-20}$ $(\mu\mathrm{K})^2\,\mathrm{cm}^6\,\mathrm{s}^{-1}$ with $\eta_* = 0.21$ from Refs. [30,39].

Appendix 1: Partial derivatives of the energy

$$\psi(\vec{r}_1, \ldots, \vec{r}_N) \underset{r_{ij}\to 0}{=} \left(\frac{1}{r_{ij}} - \frac{1}{a}\right) A_{ij}(\vec{R}_{ij}, (\vec{r}_k)_{k\neq i,j}) + O(r_{ij})$$

$$\left(\frac{\partial E}{\partial(-1/a)}\right)_{R_t} = \sum_{i<j} \int d\vec{R}_{ij} \Big(\prod_{k\neq i,j} d\vec{r}_k\Big) \left|A_{ij}(\mathbf{R}_{ij}, (\mathbf{r}_k)_{k\neq i,j})\right|^2$$

<u>For universal states:</u>

$$\left(\frac{\partial E}{\partial r_e}\right)_a = 2\pi \sum_{i<j} \int d\vec{R} \int \Big(\prod_{k\neq i,j} d\vec{r}_k\Big) A_{ij}(\vec{R}, (\vec{r}_k)_{k\neq i,j})$$

$$\cdot \left[E + \frac{\hbar^2}{4m}\Delta_{\vec{R}} + \frac{\hbar^2}{2m}\sum_{k\neq i,j}\Delta_{\vec{r}_k} - \sum_{l=1}^{N} U(\vec{r}_l)\right] A_{ij}(\vec{R}, (\vec{r}_k)_{k\neq i,j})$$

Applications: 3 particles, $a = \infty$

Efimov trimers:

$$\left(\frac{\partial E}{\partial(-1/a)}\right)_{R_t} = \sqrt{-E\frac{\hbar^2}{m}} \cdot \frac{\pi\tan(s\pi)\sin\left(s\frac{\pi}{2}\right)}{\cos\left(s\frac{\pi}{2}\right) - s\frac{\pi}{2}\sin\left(s\frac{\pi}{2}\right) - \frac{4\pi}{3\sqrt{3}}\cos\left(s\frac{\pi}{6}\right)}$$

Universal states in isotropic harmonic trap:

$$\frac{\partial E}{\partial(-1/a)} = \sqrt{\frac{2\hbar^3\omega}{m}} \cdot \frac{\Gamma(s+\frac{1}{2})\, s\sin\left(s\frac{\pi}{2}\right)}{\Gamma(s+1)\left[\cos\left(s\frac{\pi}{2}\right) - s\frac{\pi}{2}\sin\left(s\frac{\pi}{2}\right) - \eta\frac{2\pi}{3\sqrt{3}}\cos\left(s\frac{\pi}{6}\right)\right]}$$

where $\eta = 2$ for bosons, -1 for fermions

$$\left(\frac{\partial E}{\partial r_e}\right)_a = \sqrt{\frac{\hbar^3\omega}{8m}} \cdot \frac{\Gamma(s-\frac{1}{2})\, s(s^2-\frac{1}{2})\sin\left(s\frac{\pi}{2}\right)}{\Gamma(s+1)\left[-\cos\left(s\frac{\pi}{2}\right) + s\frac{\pi}{2}\sin\left(s\frac{\pi}{2}\right) + \eta\frac{2\pi}{3\sqrt{3}}\cos\left(s\frac{\pi}{6}\right)\right]}$$

Agreement with numerical results of: Braaten & Hammer; von Stecher, Greene & Blume; Werner & Castin

Appendix 2: More on Diagrammatic Monte Carlo

FEYNMAN DIAGRAMS for the unitary gas

Single-particle propagator: $G_\sigma(\vec{p},\tau) \equiv -\langle \mathrm{T}_\tau \; c_{\vec{p},\sigma}(\tau) c^\dagger_{\vec{p},\sigma}(0)\rangle$

$c_{\mathbf{p},\sigma}(\tau) \equiv e^{\tau(H-\mu N)} c_{\mathbf{p},\sigma}\, e^{-\tau(H-\mu N)}$

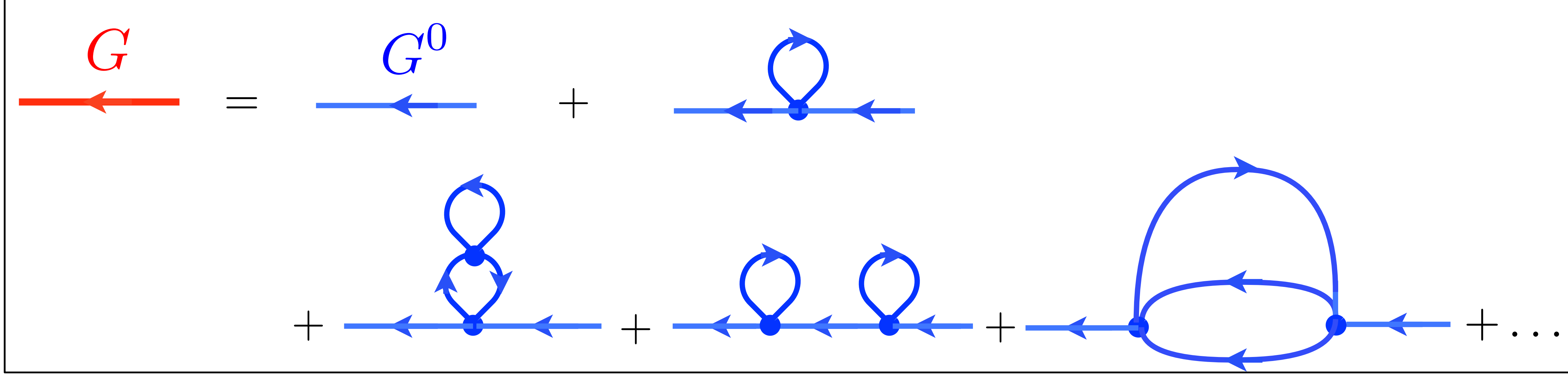
Momentum distribution:

$G_\sigma(\mathbf{p},\tau=0^-) = \langle c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma}\rangle = n_\sigma(\mathbf{p})$

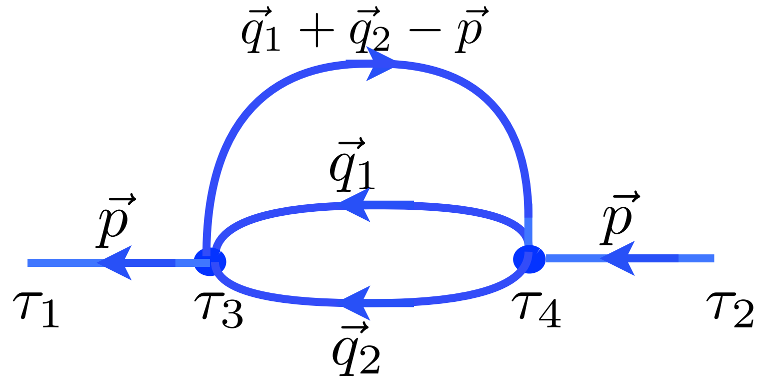
Density (equation of state): $\sum_{\sigma=\uparrow,\downarrow} \int \frac{d\mathbf{p}}{(2\pi)^3}\, n_\sigma(\mathbf{p}) = n$

Expansion of G in powers of g_0 :

G^0 = ideal gas propagator

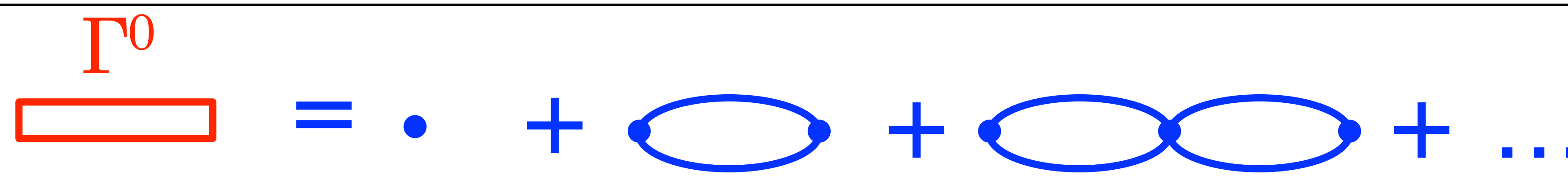


Feynman rules: example:



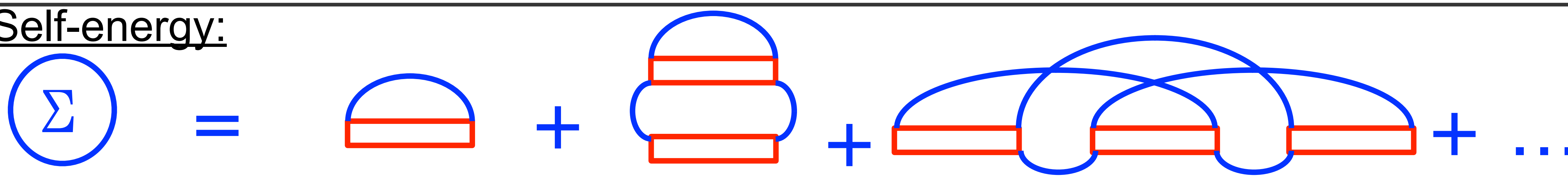
$$\{\text{contribution to } G(\vec{p}, \tau_1 - \tau_2)\,\} = (g_0)^2\,(-1)^{2+1} \int \frac{d\vec{q}_1}{(2\pi)^3}\frac{d\vec{q}_2}{(2\pi)^3} \int_0^\beta d\tau_3 d\tau_4 \;\; G^0(\vec{p}, \tau_1 - \tau_3)$$
$$\times\, G^0(\vec{q}_2, \tau_3 - \tau_4) G^0(\vec{q}_1, \tau_3 - \tau_4) G^0(\vec{q}_1 + \vec{q}_2 - \vec{p}, \tau_4 - \tau_3) G^0(\vec{p}, \tau_4 - \tau_2)$$

Ladder summation:

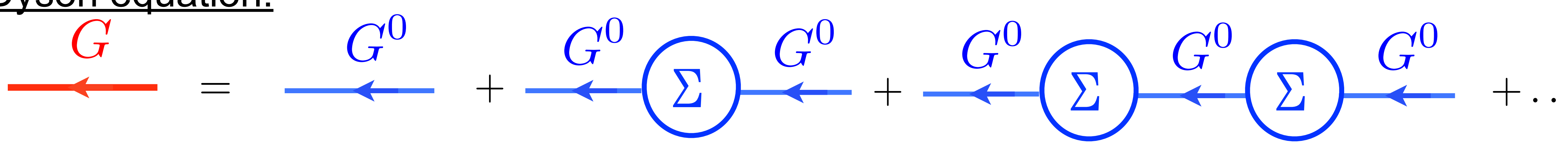


$\Rightarrow\ \Gamma^0$ is well-defined in the continuum limit, which can be taken analytically

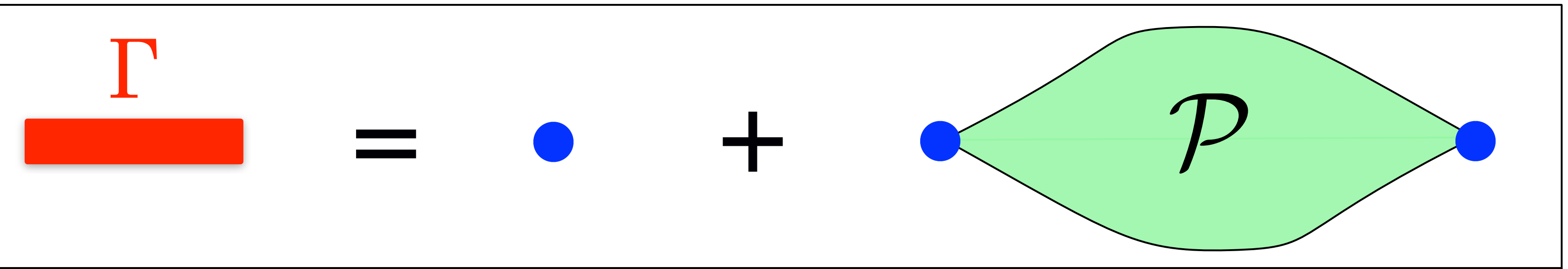
Self-energy:



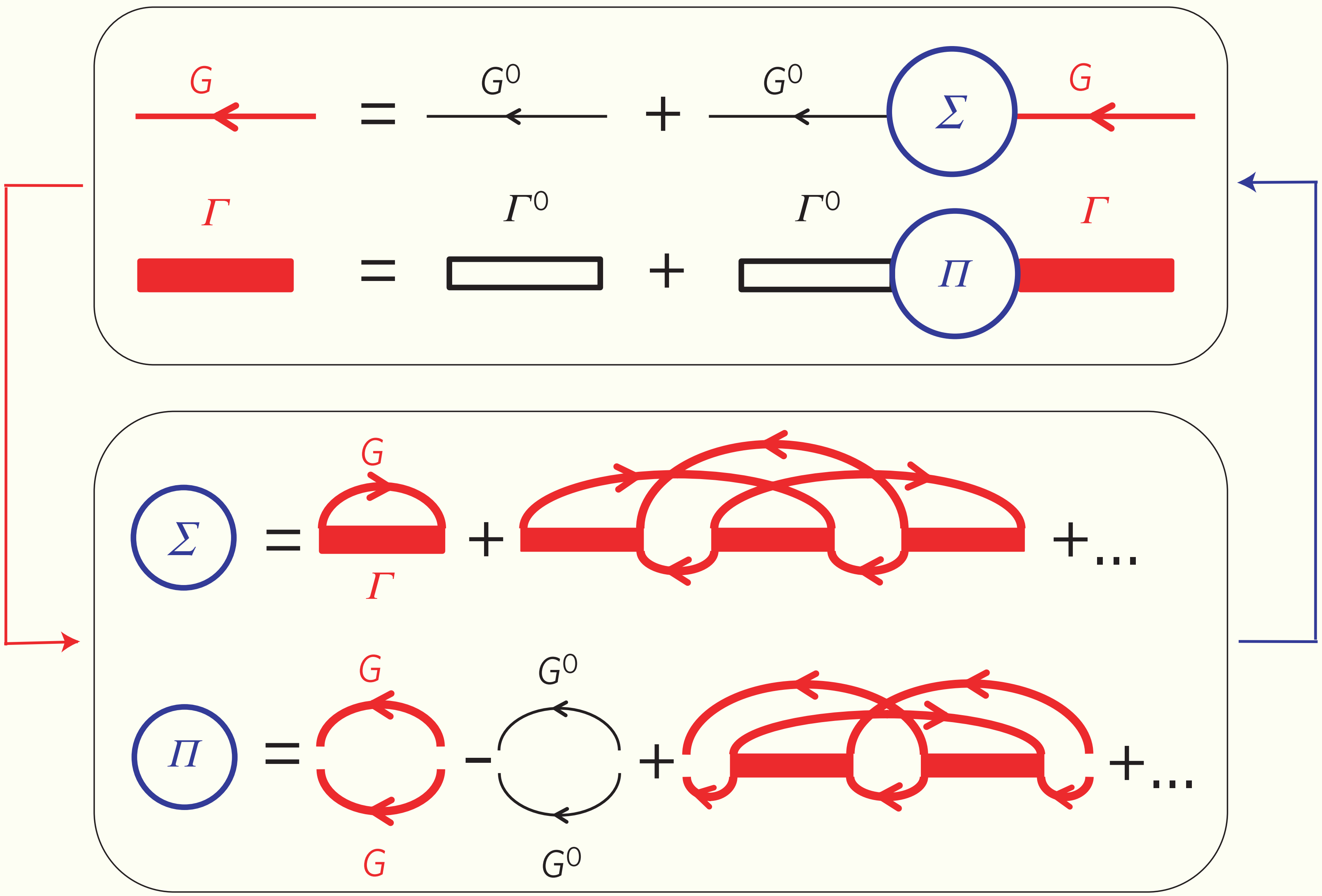
Dyson equation:



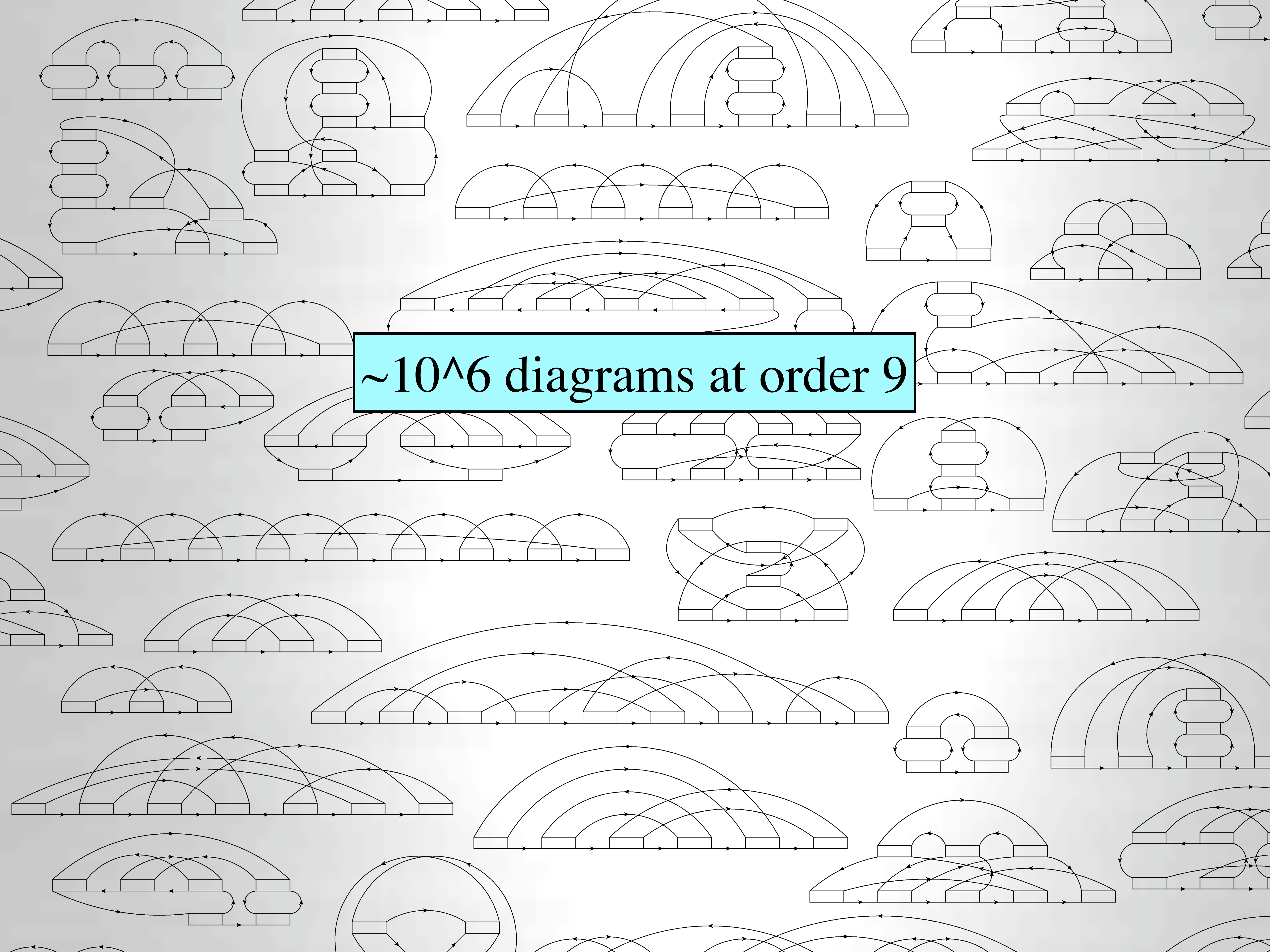
Pair propagator (fully dressed):



$$\mathcal{P}(\vec{r},\tau) \equiv -\left\langle \mathrm{T}\ (\Psi_\downarrow \Psi_\uparrow)(\vec{r},\tau)(\Psi^\dagger_\uparrow \Psi^\dagger_\downarrow)(\vec{0},0) \right\rangle$$



SOLVE THESE EQUATIONS INCLUDING THE "..."
UP TO ORDER 9 (~10^6 SKELETON DIAGRAMS)
USING DIAGRAMMATIC MONTE CARLO


~10^6 diagrams at order 9

ABELIAN RESUMMATION

$$\Sigma = \lim_{\epsilon\to 0} \sum_{N\geq 1} \Sigma^{(N)} \cdot e^{-\epsilon\,\lambda_{N-1}}$$

$\lambda_n = n \log(n)$	(Lindelöf)
$\lambda_n = n^2$	(Gaussian)
$\lambda_n = (n-1)\log(n-1)$ with $\lambda_0 = \lambda_1 = 0$	(Shifted Lindelöf)