Nonlinear Maximal Monotone extensions of symmetric operators

Andrea Posilicano
DiSAT, Università dell'Insubria, Como, Italy

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1. Introduction

Let $S : \mathscr{D}(S) \subseteq \mathscr{H} \to \mathscr{H}$, $S \geq 0$, be a linear symmetric positive operator on the Hilbert space $\mathscr{H}$.

By the famed Birman-Kreĭn-Vishik theory we know how to find its positive self-adjoint extensions.

Question: is there some nonlinear analogue of such a theory?

If $A \geq 0$ is a linear self-adjoint extension of S then e^{-tA}, $t \geq 0$, is a continuous semi-group of contractions in $\mathscr{H}$, i.e.

$$\|e^{-tA}u\| \leq \|u\| \quad (\text{equivalently } \|e^{-tA}u - e^{-tA}v\| \leq \|u - v\|\,)\,.$$

Thus in the nonlinear case we are lead to look for nonlinear extensions which are generators of continuous nonlinear semi-groups of contractions, i.e.

$$S_t,\ t \geq 0, \text{ such that } \|S_t(u) - S_t(v)\| \leq \|u - v\|.$$

By the theory of one-parameter continuous nonlinear semi-groups of contractions there follows that S_t has a nonlinear generator A given by a monotone operator which is the principal section $\mathcal{A}^0$ of a maximal monotone relation $\mathcal{A} \subset \mathscr{H} \times \mathscr{H}$.

Since maximal monotonicity can be characterized in terms of nonlinear resolvents and since, in the linear case, the theory of self-adjoint extensions can be formulated in terms of the famed Kreĭn's resolvent formula, one is led to look for a nonlinear version of such a formula.

2. Maximal monotone nonlinear operators (Brezis, Kato, Komura, Minty, Moreau, Pazy, Rockafellar,....)

A nonlinear operator $A : \mathscr{D}(A) \subseteq \mathscr{H} \to \mathscr{H}$ in the *real* Hilbert space $\mathscr{H}$ is said to be *monotone of type* ω (*monotone* if $\omega = 0$) whenever

$$\forall\, u, v \in \mathscr{D}(A)\,, \quad \langle A(u) - A(v), u - v\rangle \geq -\omega \|u - v\|^2\,,$$

and *maximal monotone of type* ω if for some $\lambda > \omega$ (equivalently for any $\lambda > \omega$) one has

$$\mathrm{Range}\,(A + \lambda) = \mathscr{H}\,.$$

A nonlinear operator $\tilde{A} : \mathscr{D}(\tilde{A}) \subseteq \tilde{\mathscr{H}} \to \tilde{\mathscr{H}}$, $\tilde{\mathscr{H}}$ the *complex* Hilbert space $\tilde{\mathscr{H}} = \mathscr{H} + i\,\mathscr{H}$, is said to be monotone of type ω whenever

$$\forall\, u, v \in \mathscr{D}(\tilde{A})\,, \quad \mathrm{Re}\langle \tilde{A}(u) - \tilde{A}(v), u - v\rangle \geq -\omega \|u - v\|^2\,.$$

Defining A_1, A_2 by

$$\tilde{A}(u_1 + iu_2) = A_1(u_1, u_2) + iA_2(u_1, u_2)\,,$$

one has that $\tilde{A}$ is monotone in $\mathscr{H}$ if and only if A defined by

$$A(u_1 \oplus u_2) := A_1(u_1, u_2) \oplus A_2(u_1, u_2)$$

is monotone in the real Hilbert space $\mathscr{H} \oplus \mathscr{H}$. Similarly $\tilde{A}$ is maximal monotone if and only if A is maximal monotone. Thus the whole theory of maximal monotone operators in real Hilbert spaces extends, with the obvious modifications, to complex Hilbert spaces.

If A is monotone of type ω then

$$\langle (A+\lambda)(u) - (A+\lambda)(v), u - v\rangle \geq (\lambda - \omega)\|u - v\|^2 \,.$$

and so if A is maximal then

$$(A+\lambda) : \mathscr{D}(A) \to \mathscr{H}$$

is bijective for any $\lambda > \omega$ and the nonlinear resolvent

$$(A+\lambda)^{-1} : \mathscr{H} \to \mathscr{H} \,, \quad \lambda > \omega \,,$$

is monotone and Lipschitz with Lipschitz constant $(\lambda - \omega)^{-1}$.

The notion of maximal monotone operator can be generalized to multi-valued maps:

$\mathcal{A} \subset \mathscr{H} \times \mathscr{H}$ is said to be a *monotone relation of type* ω (*monotone relation* in case $\omega = 0$) if

$$\forall\,(u,\tilde{u}),(v,\tilde{v}) \in \mathcal{A}\,, \quad \langle \tilde{u}-\tilde{v}, u-v\rangle \geq -\omega\|u-v\|^2$$

and is said to be a *maximal monotone relation of type* ω if it is not properly contained in any other monotone relation of type ω.

The graph

$$\mathrm{Graph}(A) := \{(u,\tilde{u}) \in \mathscr{H} \times \mathscr{H} : u \in \mathscr{D}(A),\ \tilde{u} = A(u)\}$$

of a maximal monotone operator of type ω is a maximal monotone relation of type ω.

Any $\mathcal{A} \subset \mathscr{H} \times \mathscr{H}$ defines a set-valued operator by

$$u \mapsto \mathcal{A}(u) := \{\tilde{u} \in \mathscr{H} : (u, \tilde{u}) \in \mathcal{A}\}$$

with domain

$$\mathscr{D}(\mathcal{A}) := \{u \in \mathscr{H} : \mathcal{A}(u) \neq \emptyset\}$$

If $\mathcal{A}$ is maximal monotone then $\mathcal{A}(u)$ is closed and convex and so

$$\exists! u_{min} \in \mathscr{H} \text{ such that } \|u_{min}\| = \inf\{\, \|v\| : v \in \mathcal{A}(u)\,\}.$$

Therefore the single-valued nonlinear operator

$$\mathcal{A}^0 : \mathscr{D}(\mathcal{A}) \subseteq \mathscr{H} \to \mathscr{H}\,, \quad \mathcal{A}^0(u) := u_{min}$$

is well defined; it is called the *principal section* of $\mathcal{A}$.

The principal section is unique: $\mathcal{A}_1^0 = \mathcal{A}_2^0 \Longrightarrow \mathcal{A}_1 = \mathcal{A}_2$.

While the domain of a linear maximal monotone relation is necessarily dense, in the nonlinear case this can be false.

$$\mathcal{A} \text{ is maximal monotone} \Longrightarrow \overline{\mathscr{D}(\mathcal{A})} \text{ is a convex set.}$$

Let $\mathscr{C}$ be a closed convex nonempty subset of $\mathscr{H}$. The family of nonlinear operators $S_t : \mathscr{C} \to \mathscr{C}$, $t \geq 0$, is said to be a one-parameter nonlinear continuous semi-group of type ω (of contractions in case $\omega = 0$) on $\mathscr{C}$ if

$$S_0 = \mathrm{Id}\,, \quad S_{t_1} \circ S_{t_2} = S_{t_1+t_2}\,,$$

$$\forall u \in \mathscr{C}\,, \quad \lim_{t \downarrow 0} \|S_t(u) - u\| = 0\,,$$

$$\forall\, u, v \in \mathscr{C}\,, \qquad \|S_t(u) - S_t(v)\| \leq e^{\omega t}\|u - v\|\,.$$

The generator of the semigroup S_t is defined by

$$A : \mathscr{D}(A) \subseteq \mathscr{H} \to \mathscr{H}\,, \qquad -A(u) := \lim_{t \downarrow 0} \frac{1}{t}\left(S_t(u) - u\right),$$

where $\mathscr{D}(A) \subseteq \mathscr{C}$ is the set of u such that the above limit exists.

$\mathscr{D}(A)$ is dense in $\mathscr{C}$ and invariant.

For all $u \in \mathscr{D}(A)$, $u(t) := S_t(u)$ is the unique solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt}\, u(t) = -A(u(t))\,, & \text{a.e. } t > 0 \\ u(0) = u\,. \end{cases}$$

Theorem. (Komura-Kato)

A maximal monotone of type ω

$$\Downarrow$$

A generates a strongly continuous semigroup of type ω on $\overline{\mathscr{D}(A)}$

$$\Downarrow$$

A is the principal section of a maximal monotone relation $\mathcal{A}$ of type ω.

Given A maximal monotone the corresponding one-parameter nonlinear continuous semi-group S_t is constructed in the following way: defining the nonlinear Yosida approximation

$$A_\lambda := \frac{1}{\lambda}\left(1 - (1+\lambda A)^{-1}\right),$$

maximal monotonicity implies that A_λ is a Lipschitz map and that

$$\forall u \in \mathscr{D}(A)\,, \quad \lim_{\lambda\to 0} A_\lambda(u) = A(u)\,.$$

By the Lipschitz property the Cauchy problem

$$\begin{cases} \frac{d}{dt} u_\lambda(t) = A_\lambda(u_\lambda(t)) \\ u_\lambda(0) = u \in \mathscr{H} \end{cases}$$

has a unique solution $t \mapsto u_\lambda(t)$ which defines the semi-group $S_t^\lambda(u) := u_\lambda(t)$. Finally

$$\forall T \geq 0\,, \quad \forall u \in \overline{\mathscr{D}(A)}\,, \quad \lim_{\lambda\to 0} \sup_{0\leq t\leq T} \|S_t^\lambda(u) - S_t(u)\| = 0\,.$$

Let $\varphi : \mathscr{H} \to (-\infty, +\infty]$ be a proper (i.e. not identically $+\infty$) convex function. Its *sub-differential* $\partial\varphi \subset \mathscr{H} \times \mathscr{H}$ is defined by

$$\partial\varphi := \{(u, \tilde{u}) \in \mathscr{H} \times \mathscr{H} : \forall v \in \mathscr{H}\,,\ \varphi(u) \le \varphi(v) + \langle \tilde{u}, u - v \rangle\}$$

Notice that $(u, 0) \in \partial\varphi$ if and only if u is a minimum point of φ.

If φ is Gâteaux-differentiable at u then $\partial\varphi(u) = \nabla\varphi(u)$.

Sub-differentials are the main source of maximal monotone operators:

$$\varphi \text{ convex, lower semi-continuous} \implies \partial\varphi \text{ is maximal monotone.}$$

Let S_t^φ be the nonlinear semigroup generated by $A = \partial\varphi$. Then one has the following regularity results:

$$\forall u \in \overline{\mathscr{D}(A)}\,,\ \forall t > 0\,, \quad S_t^\varphi(u) \in \mathscr{D}(A)\,,$$

$$\forall u \in \overline{\mathscr{D}(A)}\,, \forall v \in \mathscr{D}(A)\,, \forall t > 0\,, \quad \left\| \frac{d}{dt} S_t^\varphi(u) \right\| \leq \|Av\| + \frac{1}{t}\,\|u - v\|\,,$$

$$\forall u \in \overline{\mathscr{D}(A)}\,,\ \forall T > 0\,, \quad \int_0^T t \left\| \frac{d}{dt} S_t^\varphi(u) \right\|^2 dt < +\infty\,,$$

$$\forall u : \varphi(u) < +\infty\,,\ \forall T > 0\,, \quad \int_0^T \left\| \frac{d}{dt} S_t^\varphi(u) \right\|^2 dt < +\infty\,,$$

$$\forall u \in \overline{\mathscr{D}(A)}\,,\ \forall T > 0\,, \quad \int_0^T |\varphi(S_t^\varphi(u))|\, dt < +\infty\,,$$

$$\forall u : \varphi(u) < +\infty\,,\ \forall T > 0\,, \quad \int_0^T \left| \frac{d}{dt}\, \varphi(S_t^\varphi(u)) \right| dt < +\infty\,.$$

3. Nonlinear maximal monotone extensions

Let $S \geq -\omega$ be a densely defined, symmetric lower bounded operator. It is linear monotone of type ω but is not maximal monotone since its Friedrich's extensions $A_0 \geq -\omega$ is a proper monotone extension. We want to define *nonlinear* maximal monotone operators A such that

$$S \subset A \subset S^* .$$

Without loss of generality we can suppose that $S = A|\mathscr{N}$, where $\mathscr{N}$ is the (dense in $\mathscr{H}$) kernel of a continuos (w.r.t. the graph norm of A_0) surjective linear map

$$\tau : \mathscr{D}(A_0) \to \mathfrak{h} ,$$

$\mathfrak{h}$ being an auxiliary Hilbert space.

For any $\lambda > \omega$ we pose $R^0_\lambda := (A_0 + \lambda)^{-1}$ and define the bounded linear operator

$$G_\lambda : \mathfrak{h} \to \in \mathscr{H} \,, \qquad G_\lambda := (\tau R^0_\lambda)^* \,.$$

By the denseness hypothesis on $\mathscr{N}$ one has

$$\mathrm{Range}(G_\lambda) \cap \mathscr{D}(A_0) = \{0\}$$

and, by first resolvent identity,

$$(\lambda - \mu)\, R^0_\mu G_\lambda = G_\mu - G_\lambda \,.$$

We try to define a nonlinear extension A by producing its nonlinear resolvent. Given a nonlinear resolvent $R_\lambda = (A+\lambda)^{-1}$, one has

$$R_\lambda^{-1} - \lambda \; = \; A \; = \; R_\mu^{-1} - \mu \, ,$$

which is equivalent to the nonlinear resolvent identity

$$R_\lambda = R_\mu \circ (1 - (\lambda - \mu) R_\lambda) \, .$$

Thus if $R_\lambda : \mathscr{H} \to \mathscr{H}$, $\lambda > \omega$, is a family of monotone and injective nonlinear maps which satisfies the nonlinear resolvent identity, then

$$A := (R_\lambda^{-1} - \lambda) : \mathscr{D}(A) \subseteq \mathscr{H} \to \mathscr{H} \, , \quad \mathscr{D}(A) := \mathrm{Range}(R_\lambda) \, ,$$

is a λ-independent, maximal monotone nonlinear operator of type ω.

Therefore we need to produce a family R_λ, $\lambda > \omega$, of monotone and injective nonlinear maps which satisfies the nonlinear resolvent identity.

Kreĭn's linear resolvent formula suggests us to write the presumed resolvent as

$$R_\lambda = R^0_\lambda + G_\lambda V_\lambda \circ G^*_\lambda\,,$$

where the nonlinear map $V_\lambda : \mathfrak{h} \to \mathfrak{h}$ has to be determined. Since R^0_λ is monotone,

$$\langle R_\lambda(u) - R_\lambda(v), u - v\rangle \geq \langle V_\lambda(G^*_\lambda u) - V_\lambda(G^*_\lambda v), G^*_\lambda u - G^*_\lambda v\rangle\,,$$

so that R_λ is monotone whenever

$$\forall\, \xi, \zeta \in \mathfrak{h}\,, \quad \langle V_\lambda(\xi) - V_\lambda(\zeta), \xi - \zeta\rangle \geq 0\,,$$

namely whenever V_λ is monotone.

Lemma. Let $V_\lambda : \mathfrak{h} \to \mathfrak{h}$ be monotone. Then

$$R_\lambda = R^0_\lambda + G_\lambda V_\lambda \circ G^*_\lambda$$

satisfies the nonlinear resolvent identity if and only if there exists a family of maximal monotone relations $\Gamma_\lambda \subset \mathfrak{h} \times \mathfrak{h}$ such that $\Gamma_\lambda^{-1} = V_\lambda$ and

$$\Gamma_\lambda - \Gamma_\mu = (\lambda - \mu)\, G^*_\mu G_\lambda \,. \tag{1}$$

Lemma. Let $\Theta \subset \mathfrak{h} \times \mathfrak{h}$ be a maximal monotone relation and let $\lambda_0 > \omega$. Then

$$\Gamma^\Theta_\lambda := \Theta + (\lambda - \lambda_0) G^* G_\lambda \,, \quad \lambda > \omega \,, \quad G := G_{\lambda_0} \,,$$

is a maximal monotone relation for any $\lambda \geq \lambda_0$. It fulfills (1) and it has a single-valued monotone inverse for any $\lambda > \lambda_0$.

By collecting the above results one gets the following nonlinear version of Kreĭn's resolvent formula:

Theorem.
Let $\lambda_0 > \omega$ and let $\Theta \subset \mathfrak{h} \times \mathfrak{h}$ be a maximal monotone relation. Then

$$R_\lambda^\Theta := R_\lambda^0 + G_\lambda(\Theta + (\lambda - \lambda_0)G^* G_\lambda)^{-1} \circ G_\lambda^* \,, \qquad \lambda > \lambda_0$$

is the resolvent of a nonlinear maximal monotone operator A_Θ of type λ_0; A_Θ is monotone of type ω whenever Θ^{-1} is single-valued. Such an operator is defined by

$$\mathscr{D}(A_\Theta) := \{u \in \mathscr{H} : u = u_0 + G\xi_u \,,\ u_0 \in \mathscr{D}(A_0) \,,\ (\xi_u, \tau u_0) \in \Theta\} \,,$$

$$A_\Theta(u) := A_0 u_0 - \lambda_0 G \xi_u \,.$$

Remarks.

$$A_\Theta \subset S^* ,$$

$$\mathscr{D}(A_0) \cap \mathscr{D}(A_\Theta) \neq \emptyset \iff 0 \in \mathscr{D}(\Theta) ,$$

$$\mathscr{D}(A_0) \cap \mathscr{D}(A_\Theta) \text{ is convex and closed in } \mathscr{D}(A_0) ,$$

$$\forall u \in \mathscr{D}(A_0) \cap \mathscr{D}(A_\Theta) , \quad A_\Theta(u) = A_0 u ,$$

$$S \subset A_\Theta \iff (0,0) \in \Theta \implies \overline{\mathscr{D}(A_\Theta)} = \mathscr{H} .$$

Theorem.
Suppose $\Theta = \partial\varphi$ and $\mathrm{Range}(G) \cap \mathscr{D}((A_0 + \lambda_0)^{\frac{1}{2}}) = \{0\}$. Define the proper convex function $\Phi : \mathscr{H} \to (-\infty, +\infty]$ by

$$\Phi(u) := \begin{cases} \frac{1}{2}\,\|(A_0 + \lambda_0)^{\frac{1}{2}} u_0\|^2 + \varphi(\xi) & u \in \mathscr{D}(\Phi) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathscr{D}(\Phi) := \{u \in \mathscr{H} : u = u_0 + G\xi\,,\ u_0 \in \mathscr{D}((A_0 + \lambda_0)^{\frac{1}{2}})\,,\ \varphi(\xi) < +\infty\}\,.$$

Then

$$A_\Theta + \lambda_0 = \partial\Phi = \partial\bar{\Phi}\,,$$

where $\bar{\Phi}$ denotes the lower semi-continuous regularization of Φ i.e. Φ is the largest lower semi-continuous minorant of Φ:

$$\bar{\Phi}(v) := \liminf_{u \to v} \Phi(u)\,.$$

Corollary.
Let $A_0 > 0$ and take $\lambda_0 = 0$. Suppose $\Theta = \partial\varphi$ and that ξ_0 is the unique minimum point of φ. Then $A_\Theta G\xi_0 = 0$ and

$$\forall u \in \overline{\mathscr{D}(A_\Theta)}\,, \quad \text{w-}\lim_{t\to+\infty} S_t(u) = G\xi_0\,.$$

If φ is an even function then the above weak limit becomes a strong one.

4. Examples.

Laplacians with nonlinear singular perturbations supported on null sets.

Let $A_0 = -\Delta : H^2(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and let $N \subset \mathbb{R}^n$ be a d-set with $2 < n - d < 4$. A Borel set $N \subset \mathbb{R}^n$ is called a d-set, if

$$\exists\, c_1,\, c_2 > 0 \,:\, \forall\, x \in N,\, \forall\, r \in (0,1), \quad c_1 r^d \le \mu_d(B_r(x) \cap M) \le c_2 r^d\,,$$

where μ_d is the d-dimensional Hausdorff measure and $B_r(x)$ is the closed n-dimensional ball of radius r centered at the point x. Examples of d-sets for d integer are finite unions of d-dimensional Lipschitz submanifolds and, in the not integer case, self-similar fractals of Hausdorff dimension d. Then we take $\tau = \gamma_N$, where

$$\gamma_N : H^2(\mathbb{R}^n) \to H^s(N)\,, \qquad s = 2 - \frac{n-d}{2}$$

is the unique linear continuous and surjective map with coincide on smooth functions with the evaluation at points in N.

Here $H^s(N)$, $0 < s < 1$, is defined as the Hilbert space of functions $f \in L^2(N; \mu_N)$ having finite norm

$$\|f\|^2_{H^2(N)} := \|f\|^2_{L^2(N;\mu_N)} + \int_{|x-y|<1} \frac{|f(x) - f(y)|^2}{|x-y|^{d+2s}} \, d\mu_N(x) \, d\mu_N(y) \,,$$

where μ_N denotes the restriction of the d-dimensional Hausdorff measure μ_d to the set N.
Given $f \in H^s(N)$, let $\nu_N(f) \in H^{-2}(\mathbb{R}^n)$ be the signed measure with $\mathrm{supp}(\nu_N(f)) = N$ defined by

$$(\nu_N(f), u)_{-2,2} = \langle f, \gamma_N u \rangle_{H^s(M)} \,,$$

where $(\cdot, \cdot)_{-2,2}$ denotes the H^{-2}-H^2 duality.
Given $\lambda > 0$, let g_λ be the kernel of $(-\Delta + \lambda)^{-1}$. Then

$$G_\lambda : H^s(N) \to L^2(\mathbb{R}^n) \,, \qquad G_\lambda f := g_\lambda * \nu_N(f) \,.$$

Therefore, given $\lambda_0 > 0$ and posing $g := g_{\lambda_0}$, for any nonlinear maximal monotone relation $\Theta \subset H^s(N) \times H^s(N)$, one gets a nonlinear maximal monotone operator $-\Delta_\Theta$ of type λ_0 defined by

$$-\Delta_\Theta u = -\Delta u_0 - \lambda_0\, g * \nu_N(f_u)\,,$$

$$\begin{aligned}&\mathscr{D}(-\Delta_\Theta)\\ :=&\left\{u \in L^2(\mathbb{R}^n) : u = u_0 + g * \nu_N(f_u),\ u_0 \in H^2(\mathbb{R}^n),\ (f_u, \gamma_N u_0) \in \Theta\right\}\end{aligned}$$

and with nonlinear resolvent

$$(-\Delta_\Theta+\lambda_0)^{-1} = (-\Delta+\lambda_0)^{-1}+g_\lambda*\nu_N((\Theta+\Gamma_\lambda)^{-1}\circ(\gamma_N(-\Delta+\lambda)^{-1}))\,,$$

where

$$\Gamma_\lambda f = (\lambda - \lambda_0)\,\gamma_N(g * g_\lambda * \nu_N(f))\,.$$

Notice that, since $(-\Delta+\lambda_0)g=\delta_0$, $-\Delta_\Theta$ can be alternatively defined by

$$(-\Delta_\Theta+\lambda_0)u:=(-\Delta+\lambda_0)u-\nu_N(f_u)\,.$$

When N is a Riemannian manifold with volume form dv, since

$$\nu_N(f)=((-\Delta_{LB}+\lambda_0)^s f)\delta_N\,,$$

where, for any $f\in H^{-s}(N)$,

$$(f\delta_N,u)_{-2,2}=\int_N(-\Delta_{LB}+\lambda_0)^{-s/2}f(x)\,((-\Delta_{LB}+\lambda_0)^{s/2}\gamma_N u)(x)\,dv(x)\,,$$

one has

$$(-\Delta_\Theta+\lambda_0)u=(-\Delta+\lambda_0)u-((-\Delta_{LB}+\lambda_0)^s f_u)\,\delta_N\,.$$

In the case Θ^{-1} is single-valued one can also write

$$(-\Delta_\Theta + \lambda_0)u = (-\Delta + \lambda_0)u - ((-\Delta_{LB} + \lambda_0)^s \Theta^{-1}(\gamma_N u_0))\, \delta_N \,.$$

If $\Theta = \partial\varphi$, where $\varphi : H^s(N) \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $-\Delta_\Theta + \lambda_0 = \partial\Phi$, where

$$\Phi(u) := \begin{cases} \frac{1}{2}\, \|(-\Delta + \lambda_0)^{\frac{1}{2}} u_0\|^2 + \varphi(f) & u \in \mathscr{D}(\Phi) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathscr{D}(\Phi) := \{u \in L^2(\mathbb{R}^n) : u = u_0 + g * \nu_N(f)\,,\ u_0 \in H^1(\mathbb{R}^n)\,,\ \varphi(f) < +\infty\}\,.$$

The Laplacian with nonlinear boundary conditions on a bounded domain.
Let $\Omega \subset \mathbb{R}^n$, $n > 1$, be a bounded open set with a regular boundary $\partial\Omega$. The continuous and surjective linear operator

$$\gamma : H^2(\Omega) \to H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)\,, \quad \gamma u := (\gamma_0 u, \gamma_1 u)\,,\,,$$

is defined as the unique bounded linear operator such that, in the case $u \in C^\infty(\bar{\Omega})$,

$$\gamma_0 u\,(x) = u\,(x)\,, \quad \gamma_1 u\,(x) = \frac{\partial u}{\partial n}\,(x)\,, \quad x \in \partial\Omega\,,$$

where n is the inner normal vector on $\partial\Omega$. The map γ can be extended to a bounded linear operator

$$\hat{\gamma} : \mathscr{D}(\Delta_{max}) \to H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega)\,, \quad \hat{\gamma}\phi = (\hat{\gamma}_0 u\phi, \hat{\gamma}_1 u)\,,$$

where

$$\mathscr{D}(\Delta_{max}) := \left\{ u \in L^2(\Omega) \,:\, \Delta u \in L^2(\Omega) \right\}\,.$$

Let $A_0 = -\Delta_D$ be the self-adjoint operator in $L^2(\Omega)$ given by the Dirichlet Laplacian, i.e.

$$\mathscr{D}(\Delta_D) := H^2(\Omega) \cap H_0^1(\Omega)\,, \quad H_0^1(\Omega) := \left\{ u \in H^1(\Omega) \,:\, \gamma_0 u = 0 \right\}\,.$$

We take

$$\mathfrak{h} = H^{1/2}(\partial\Omega) \quad \text{and} \quad \tau = \gamma_1 | \mathscr{D}(\Delta^D).$$

Thus we are looking for nonlinear maximal monotone extensions of the strictly positive symmetric operator $S = -\Delta_{min}$ given by the minimal Laplacian with domain

$$\mathscr{D}(\Delta_{min}) := \left\{ u \in H^2(\Omega) \,:\, \gamma_0 u = \gamma_1 u = 0 \right\}\,.$$

Let $\varphi : L^2(\Omega) \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function such that

$$\operatorname{int}\big(\{f \in L^2(\partial\Omega) : \varphi(f) < +\infty\}\big) \cap H^{1/2}(\partial\Omega) \neq \emptyset.$$

Defining the maximal monotone relation

$$\Theta_\varphi := (\partial\varphi - P) \circ (-\Delta_{LB} + 1)^{1/2} \subset H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

where P is the Dirichlet-to-Neumann operator, one obtains the nonlinear maximal monotone operator $-\Delta_\varphi := -\Delta_{\Theta_\varphi}$ defined by

$$-\Delta_\varphi : \mathscr{D}(-\Delta_\varphi) \subseteq L^2(\Omega) \to L^2(\Omega)\,, \quad -\Delta_\varphi u = -\Delta u\,,$$

$$\mathscr{D}(-\Delta_\varphi) = \{u \in \mathscr{D}(\Delta_{max}) : (\hat{\gamma}_0 u, \hat{\gamma}_1 u) \in \partial\varphi\}\,.$$

Moreover $-\Delta_\varphi = \partial\Phi$, where

$$\Phi(u) = \begin{cases} \frac{1}{2}\,\|\nabla u\|^2 + \varphi(\gamma_0 u)\,, & u \in \mathscr{D}(\Phi) \\ +\infty\,, & \text{otherwise}\,, \end{cases}$$

$$\mathscr{D}(\Phi) = \{u \in H^1(\Omega) : \varphi(\gamma_0 u) < +\infty\}\,.$$

If φ has an unique minimum point $f_0 \in L^2(\partial\Omega)$ then, denoting by S_t^φ the nonlinear semigroup of contractions generated by $-\Delta_\varphi$, one has

$$\forall u \in \overline{\mathscr{D}(-\Delta_\varphi)}\,, \quad \text{w-}\lim_{t\to+\infty} S_t^\varphi(u) = u_0\,,$$

where u_0 is the unique harmonic function in Ω such that $\gamma_0 u_0 = f_0$. If φ is an even function then the above limit holds in strong sense.