Non-linear Schrödinger equation with point interactions: localised defects and concentrated nonlinearities

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Mathematical challenges of zero-range Physics: rigorous results and open problems

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1. NLS: the equation

$$i\partial_t u(x,t) = -\Delta u(x,t) + V(x)u(x,t) \pm Q(x)|u(x,t)|^{2\sigma} u(x,t)$$

   where $x \in \mathbb{R}^n$, $t > 0$, $u : \mathbb{R}^n \to \mathbb{C}$, $\sigma > 0$ and $V, Q : \mathbb{R}^n \to \mathbb{R}$; $Q > 0$ and $-/+$ focusing/defocusing;

2. NLS: paradigm of nonlinear wave propagation: dispersion, scattering, bound states, breathers, solitons, stability...;

3. NLS: many physical systems described by NLS: e.m. pulse propagation in Kerr media, dynamics of BEC (Gross-Pitaevskii equation)...;

4. NLS: V and Q model inhomogeneities of the medium (impurities, external fields...) and give rise often to stable localized structures in the form of *standing waves*.

5. NLS: point defects ($-\Delta + V \to -\Delta + \alpha\delta$, $-\Delta + \beta\delta' \cdots$) and concentrated nonlinearities ($Q \to \delta$)

Here we are interested in *spatially modulated* nonlinearity

$$i\partial_t u(x,t) = -\Delta u(x,t) - Q(x)|u(x,t)|^{2\sigma} u(x,t)$$

where again $x \in \mathbb{R}^n$, $t > 0$, $u : \mathbb{R}^n \to \mathbb{C}$ and $Q : \mathbb{R}^n \to \mathbb{R}$;

if Q has very small support, ideally pointlike, we have *concentrated* nonlinearities

In the same order of ideas: mean field nonlinear interaction

$$i\partial_t u(x,t) = -\Delta u(x,t) - |\langle \rho, u(\cdot,t)\rangle|^{2\sigma} \langle \rho, u(\cdot,t)\rangle \rho(x)$$

or more generally

$$i\partial_t u(x,t) = -\Delta u(x,t) - F(\langle \rho, u(\cdot,t)\rangle)\rho(x)$$

Do make sense nonlinearities concentrated at a point?

The linear case $\sigma = 0$; dimension 1

consider $V \in L^1(\mathbb{R}^n)$ the Schrödinger operator

$$-\Delta u(x,t) + \frac{1}{\epsilon} V(\frac{x}{\epsilon}) u(x,t) := -\Delta + V_\epsilon$$

for $n = 1$, $\quad -\frac{d^2}{dx^2} + \frac{1}{\epsilon} V(\frac{x}{\epsilon}) \xrightarrow{\mathcal{R}} -\frac{d^2}{dx^2} + \alpha\delta_0 \qquad (\alpha = \int_{\mathbb{R}} V(x)\, dx)$;

the well known Schrödinger operator with "δ potential" ;

domain: $\quad u \in H^2(\mathbb{R} \setminus \{0\}) \cap H^1(\mathbb{R})$ s.t. $u'(0^+) - u'(0^-) = \alpha u(0)$

action: $\quad (-\frac{d^2}{dx^2} + \alpha\delta_0) u = -\frac{d^2}{dx^2} u, \quad x \neq 0$ and interaction shifted to the boundary condition

The linear case $\sigma = 0$; dimension 3

$$-\Delta u(x,t) + \frac{1}{\epsilon^3} V(\frac{x}{\epsilon}) u(x,t) := -\Delta + V_\epsilon$$

for $n = 3$ and every reasonable V the previous operator is trivial ($\equiv -\Delta$) in the limit $\epsilon \to 0$;

instead, if the potential V

- - has a zero energy resonance;
- - undergoes the "anomalous" scaling $\frac{1+\epsilon\mu}{\epsilon^2} V(\frac{x}{\epsilon}), \quad \mu \in \mathbb{R}$;

then

$$-\Delta + V_\epsilon \xrightarrow{\mathcal{R}} H_\alpha \ ;$$

the s.a. H_α is called "point" or "delta" interaction with strength α (depending on μ, V and its resonance function ξ: $\alpha = -\mu\langle |V|^{\frac{1}{2}}, \xi\rangle$).

In any case:

$$D(H_\alpha) = \left\{ u \in L^2(\mathbb{R}^3) : \; u = \phi + qG_0, \; \begin{array}{l} \phi \in L^2_{loc}(\mathbb{R}^3), \nabla\phi \in H^1(\mathbb{R}^3), \; q \in \mathbb{C} \\ \phi(0) = \alpha q \end{array} \right\}$$

where $G_0(x) = \frac{1}{4\pi|x|}$; q is called charge of the element domain u and ϕ regular part; the action of the operator is

$$H_\alpha u = -\Delta\phi \ . \tag{1}$$

The linear case $\sigma = 0$; approximation via finite rank perturbation

in $d = 3$ consider the mean field operator

$$-\Delta u(x,t) - \alpha_\epsilon \langle \rho_\epsilon, u(\cdot,t) \rangle \rho_\epsilon(x) \quad \text{with} \quad \rho_\epsilon(\cdot) = \frac{1}{\epsilon^3}\rho(\frac{\cdot}{\epsilon}) \xrightarrow{w} \delta_0 \ ; \tag{2}$$

the limit exists and coincides with H_α after a suitable "renormalization":

$$\frac{1}{\alpha} = \frac{1}{\alpha_\epsilon} - \frac{1}{\epsilon}\langle -\Delta^{-1}\rho, \rho \rangle \ .$$

Now a largely open problem comes:

What about the limit if existing, of the family of *nonlinear* operators

$$-\Delta u(x,t) - V_\epsilon(x) F(u(x,t)) \qquad \text{as} \quad \epsilon \longrightarrow 0 \ ?$$

and analogously which is the limit, if existing, of the family

$$-\Delta u(x,t) - F(\langle \rho_\epsilon, u(\cdot,t)\rangle)\rho_\epsilon(x) \quad \text{as} \quad \rho_\epsilon \xrightarrow{w} \delta_0 \ ?$$

Corresponding problems for evolution equations (e.g. Schrödinger, Heat, Wave) in the limit.

So, how to proceed? Euristics suggests that the nonlinearity effects concentrate at the singularity; it is not unreasonable to embody them in the boundary condition.

In $d = 1$ case $q = u(0)$, and one sets

$$u'(0^+) - u'(0^-) = \alpha(q)q \ = \alpha|q|^\sigma, \qquad q = u(0)$$

Large activity on the one dimensional concentrated nonlinearities, especially in physical applications in the effort to model localized inhomogeneities with a nonlinear response to propagation of light pulses or matter waves.

Rigorous results in Adami-Teta (2001): well posedness for $\alpha > 0, \sigma > 0$ and $\alpha < 0, \sigma \in (0,1)$; blow-up for $\alpha < 0, \sigma \geq 1$.
Weaker results in Komech-Komech (2007), but consideration of other nonlinearities.

Let us consider the approximation issue (Cacciapuoti,Finco,N.,Teta (2014))
Regularized problem in weak form:

$$\psi^{\varepsilon}(t,x) = U(t)\psi^{\varepsilon}(0,x) + \\ - i\int_0^t ds \int dy\; U(t-s,x-y)\,\frac{1}{\epsilon}V(\frac{y}{\epsilon})\,|\psi^{\varepsilon}(s,y)|^{2\sigma}\psi^{\varepsilon}(s,y)$$

Limit problem in weak form:

$$\psi(t,x) = U(t)\psi(0,x) - i\alpha\int_0^t ds\; U(t-s,x)\,|\psi(s,0)|^{2\sigma}\psi(s,0)$$

and pick initial data $\psi^{\varepsilon}(0,x) = \psi(0,x) = \psi_0 \in H^1(\mathbb{R})$

If $V \in L^1(\mathbb{R},(1+|x|)dx) \cap L^{\infty}(\mathbb{R})$ and $V \geq 0$ or $\sigma \in (0,1)$, then

$$\sup_{t\in[0,T]} \|\psi^{\varepsilon}(t) - \psi(t)\|_{H^1} \to 0 \quad \text{as } \varepsilon \to 0 \quad \forall\; T \in \mathbb{R}^+ \,. \tag{3}$$

Some comments:

1. Well posedness in the regularized problem holds for $\alpha > 0, \sigma > 0$ and for $\alpha < 0, \sigma \in (0,2)$: in the limit there is a (consistent) loss in well posedness; what is the fate of regularized solutions in $\alpha < 0, \sigma \in [1,2)$?
2. If $\int V > 0$ but V is not everywhere positive convergence is for $\sigma \in (0,1)$ and not for $\sigma > 0$, at variance with the repulsive limit problem with $\alpha > 0$; is this just a technical issue?
3. Present proof: convergence is in $H^1([0, T])\ \forall T$ but not in $H^1(\mathbb{R})$
4. A completely analogous result holds true for NLS with concentrated nonlinearities at N points $y_1\ , \cdots\ , y_N$.
5. Work in progress on the $3d$ case.

In $d = 3$ the definition is slightly less transparent; recall that

$$u = \phi + qG, \quad \phi(0) = \alpha q \ \text{ and } \ H_\alpha u = -\Delta\phi \ ;$$

a power nonlinearity is then defined as

$$\alpha = \alpha(q) = \gamma|q|^{2\sigma} \quad \sigma \geq 0, \gamma \in \mathbb{R}.$$

From the boundary condition one sees that the nonlinearity is concentrated at the origin.

There is few physical work; on the mathematical side,

- Adami, Dell'Antonio, Figari, and Teta '03, '04; well posedness and blow-up of the NLS model
- N-Posilicano '05; analogous propertes for NL wave model
- Adami-N-Ortoleva '13; Adami-N-Ortoleva '14; stability and asymptotic stability of standing waves

NLS with concentrated nonlinearity

$$i\frac{d}{dt}u = H_{\alpha(q)}u. \qquad \text{(C-NLS)}$$

Local existence, uniqueness and continuity with respect to data hold true for $\sigma > 0$; for $\gamma > 0$ or $0 < \sigma < 1$ there is global existence for all data, and mass M and energy E conservation:

$$M(u(t)) = \frac{1}{2}\|u(t)\|_{L^2}^2$$

$$E(u(t)) = \frac{1}{2}\|\nabla\phi(t)\|_{L^2}^2 + \frac{\gamma}{2\sigma+2}|q(t)|^{2\sigma+2}.$$

Analogous properties hold for finite energy solutions, in the energy domain

$$V = \left\{u \in L^2(\mathbb{R}^3) : \ u = \phi + qG_0, \phi \in H^1_{loc}(\mathbb{R}^3), \nabla\phi \in L^2(\mathbb{R}^3)\right\}.$$

Stationary states

NLS with concentrated nonlinearity

$$i\frac{d}{dt}u = H_{\alpha(q)}u. \qquad \text{(C-NLS)}$$

Standing waves are solutions of the form

$$u(t,x) = e^{i\omega t}\Phi_\omega(x) \ ;$$

these solve the *stationary equation*

$$H_{\alpha(q)}\Phi_\omega(x) + \omega\Phi_\omega(x) = 0 \quad x \in \mathbb{R}^3.$$

There are standing solutions if and only if $\omega > 0$ and $\gamma < 0$; moreover

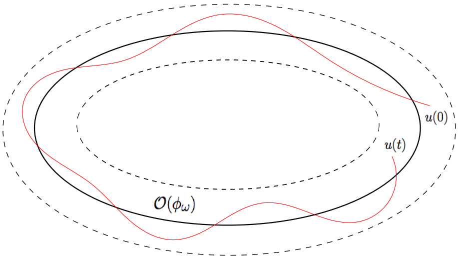
$$\Phi_\omega(x) = q_\omega \frac{e^{-\sqrt{\omega}|x|}}{4\pi|x|} \qquad \text{and} \qquad q_\omega = \left(\frac{\sqrt{\omega}}{4\pi|\gamma|}\right)^{1/2\sigma}$$

Orbital stability

Recall that NLS has $U(1)$, or phase, symmetry.
There cannot be stability of equilibrium *points*, but (maybe) of equilibrium *orbits*:

$e^{i\omega t}\Phi_\omega(x)$ is *orbitally stable* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall u_0 \in D(H_{\alpha(q)})$ with $\|u_0 - e^{i\theta}\Phi_\omega\|_V < \delta$, (C-NLS) has a global solution $u(t)$ with initial datum u_0 and

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\Phi_\omega(x)\|_V < \epsilon.$$



Lyapunov stability up to symmetries (only $U(1)$ symmetry here).

Φ_ω is unstable if it is not stable.

Main strategy to prove orbital stability: constrained linearization with control of nonlinear remainders.

Main references: Weinstein (NLS, '83), Grillakis-Shatah-Strauss (general theory of stability and instability and application to various dispersive hamiltonian equations, 87-90);

Previous more heuristic but seminal analysis: Vakhitov-Kolokolov '70, Zakharov '70, Benjamin (KdV, early '70)

Sometimes a standing wave Φ_ω is a *ground state*: absolute minimum of energy at constant mass (Cazenave-Lions '82 via concentration compactness). For orbital stability it is sufficient (and necessary) a *local* constrained minimum of energy at constant mass.

One would like to construct a Lyapunov function for the Hamiltonian system with symmetry using conserved quantities
In the present case a Lyapunov function is the so called Action

$$S_\omega(\Psi) = E(\Psi) + \omega||\Psi||_2^2$$

Notice that $S'_\omega \Psi = 0$ is just the stationary equation, solved by Φ_ω.

For every ϵ if δ is small enough one has

$$\begin{aligned}\epsilon^2 &> S_\omega(u_0) - S_\omega(\Phi_\omega) = S_\omega(u(t)) - S_\omega(\Phi_\omega) \\ &= S_\omega(e^{i\theta}u(t)) - S_\omega(\Phi_\omega) = S_\omega(\Phi_\omega + v(t) + iw(t))) - S_\omega(\Phi_\omega) \\ &= \langle L_+ v(t), v(t)\rangle + \langle L_- w(t), w(t)\rangle + R\end{aligned}$$

If there exist positive C_- , C_+ such that

$$(L_+ v(t), v(t)) \geq C_+||v(t)||_V^2, \qquad (L_- w(t), w(t)) \geq C_-||w(t)||_V^2$$

and nonlinear remainder R is controlled, the result follows.
But much more complicated state of affairs...

Linearization around Φ_ω: $u = e^{i\omega t}(\Phi_\omega + R)$, and $\frac{d}{dt}R = LR$

$$L = JD = J\begin{bmatrix} H_{\alpha_1} + \omega & 0 \\ 0 & H_{\alpha_2} + \omega \end{bmatrix} = \begin{bmatrix} 0 & H_{\alpha_2} + \omega \\ -H_{\alpha_1} + \omega & 0 \end{bmatrix}$$

$\alpha_1(q) = \gamma(2\sigma + 1)|q_\omega|^{2\sigma}$ and $\alpha_2(q) = \gamma|q_\omega|^{2\sigma}$ are now fixed;
J is the standard symplectic matrix, $D = (L_+ \ , L_-)$.
Notice that D is selfadjoint but L is not selfadjoint nor skewadjoint.

Spectral information:

- $\sigma(H_{\alpha_1} + \omega) = \{-4\sigma(\sigma + 1)\} \cup [\omega, \infty)$,
- $\sigma(H_{\alpha_2} + \omega) = \{0\} \cup [\omega, \infty)$,
- hence $\sigma(D) = \{-4\sigma(\sigma + 1),\ 0\} \cup [\omega, \infty)$;

So, L_- has a nontrivial ($1d$) kernel and L_+ has a negative eigenvalue: dangerous directions.
Nothwistanding, Weinstein and GSS with these spectral conditions give stability/instability if *Vakhitov-Kolokolov* condition holds

$$\frac{d}{d\omega}||\Phi_\omega||^2 > 0 \ / \ \frac{d}{d\omega}||\Phi_\omega||^2 < 0$$

.

Orbital stability result

More precisely in this case (Adami, N., Ortoleva '13):
$e^{i\omega t}\Phi_\omega(x)$ is:

- orbitally stable if $0 < \sigma < 1$;
- orbitally unstable if $\sigma > 1$;
- unstable by blow up if $\sigma = 1$

Completely analogous results in $1d$.
Orbital stability holds true in the same range of global existence.

Asymptotic stability

The solution near an orbitally stable standing wave in principle could wander near the orbit without relaxing toward anything.
Let us define the solitary (or soliton) manifold:

$$\mathcal{M} = \{u \in D(H_{\alpha(q)}) : \ u(x) = e^{i\theta}\Phi_\omega(x) \ \ with \ \ \theta \in \mathbb{R}, \ \omega > 0\}.$$

Question:

$$\exists \epsilon > 0 \ : dist_V(u_0, \mathcal{M}) < \epsilon \Rightarrow \lim_{t\to+\infty} dist_V(u(t), \mathcal{M}) = 0 \ ?$$

More precisely:

$\mathcal{M}$ is *asymptotically stable* if $\forall\omega > 0 \ \exists\epsilon > 0 \ \exists\omega_+ > 0$ (depending on ω) such that given $u_0 = u(0) \in D(H_{\alpha(q)})$ with $\|u_0 - \Phi_\omega\|_V < \epsilon$, the evolution $u(t)$ satisfies

$$\lim_{t\to+\infty} \|u(t) - e^{i\omega_+ t}\Phi_{\omega_+}\|_V = 0.$$

General theory: Soffer and Weinstein '90-'92 and Buslaev and Perelman '92-'95 and many more recent contributions.
1d NLS with concentrated nonlinearity: Buslaev, Komech, Kopylova, and Stuart '08-'12;
3d NLS with concentrated nonlinearity here discussed: Adami, N., Ortoleva '13-'14.

The following Ansatz is made:

$$u(t,x) = e^{i\Theta(t)} \left(\Phi_{\omega(t)}(x) + \chi(t,x)\right),$$

where

$$\Theta(t) = \int_0^t \omega(s)ds + \gamma(t).$$

The solution is decomposed in a finite dimensional component (ω, γ) along the solitary manifold and an infinite dimensional "fluctuation" χ.
There are three unknown quantities and a single original equation: one has to put some restriction to make determined the above representation.

Modulation equations

Remark

$L^2(\mathbb{R}^3, \mathbb{C})$ is a symplectic manifold when considered as a real Hilbert space of couples $u \equiv (\mathrm{Re}\ u, \mathrm{Im}\ u)$ with

$$\Omega(u, v) = \int_{\mathbb{R}^3} (\mathrm{Re}\,(u)\,\mathrm{Im}\,(v) - \mathrm{Im}\,(u)\,\mathrm{Re}\,(v))dx.$$

$$\Downarrow$$

$\mathcal{M}$ is a symplectic submanifold of $L^2(\mathbb{R}^3, \mathbb{C})$ invariant for the flow, with tangent space (if $\sigma \neq 1$)

$$T_{\Phi_\omega} = N_g(L) = \left\{ \left(J\Phi_\omega, \frac{d\Phi_\omega}{d\omega} \right) \right\}.$$

$$\Downarrow$$

project the flow onto $\mathcal{M}$ and its symplectic complement ($\equiv P^c V$).

In the above $N_g(L)$ is the generalized kernel of L .

Projecting on $N_g(L)^{\perp} = P_c V$ one obtains *modulation equations*:

If $\chi(x,t)$ is such that $\chi \in P^c V$ for all $t \geq 0$ and $\omega(t)\ ,\gamma(t) \in C^1$, then ($\varphi_\omega = \frac{d\Phi_\omega}{d\omega}$, $Q_{\alpha,Lin} =$ quadratic form of L, $N =$ nonlinear remainder)

$$\left(i\frac{d\chi}{dt}(t), v\right)_{L^2} = Q_{\alpha,Lin}(\chi(t), v) + \dot{\gamma}(t)(\Phi_{\omega(t)} + \chi(t), v)_{L^2} + \\ + \dot{\omega}(t)\left(-i\frac{d\Phi_{\omega(t)}}{d\omega}, v\right)_{L^2} + N(q_\chi, q_v) \quad \forall v \in V \ .$$

$$\dot{\omega} = \frac{((\chi,\varphi_\omega)_{L^2} + (\varphi_\omega,\Phi_\omega)_{L^2})N(q_\chi, q_{i\Phi_\omega}) - (\chi, i\Phi_\omega)_{L^2}N(q_\chi, q_{\varphi_\omega})}{(\varphi_\omega,\Phi_\omega)^2_{L^2} - (\chi,\varphi_\omega)^2_{L^2}}$$

$$\dot{\gamma} = \frac{((\chi,\varphi_\omega)_{L^2} - (\varphi_\omega,\Phi_\omega)_{L^2})N(q_\chi, q_{\varphi_\omega}) + (\chi, i\frac{d}{d\omega}\varphi_\omega)_{L^2}N(q_\chi, q_{i\Phi_\omega})}{(\varphi_\omega,\Phi_\omega)^2_{L^2} - (\chi,\varphi_\omega)^2_{L^2}}$$

The nonlinear remainder N is complicated but it depends only on charges.

The general scheme

$A(t) = \{\omega(t), \gamma(t)\}$ finite dimensional component, χ fluctuating part
Finite dimensional part ($\| \cdot \|_{w^{-1}}$ a weighted space):

$$\dot{A}(t) = R_1(A(t), \chi(t)), \qquad |R_1(A(t), \chi(t))| \leqslant \|\chi(t)\|^2_{w^{-1}}$$

Finite dimensional parameters along the solitary manifold change slowly if χ is small and dispersive: they are adiabatic invariants of the dynamics.
Infinite dimensional part:

$$\dot{\chi}(t) = L(t)\chi(t) + R_2(A(t), \chi(t)), \quad \|R_2(A(t), \chi(t))\|_w \leqslant \|\chi(t)\|^2_{w^{-1}}$$

1) The goal is to show that the solution of the χ equation disperses as it needs for $t \to \infty$
2) Linearized operator is non autonomous: choose t_1 and freeze the dynamics at t_1 posing $L(t) = L(t_1) + (L(t) - L(t_1))$, use the propagator of the freezed dynamics and then show uniformity in time.

Spectral analysis of L

Proposition

(a) $\sigma_{ess}(L) = \{\lambda \in \mathbb{C} : \ \mathrm{Re}\,(\lambda) = 0 \, and \, |\,\mathrm{Im}\,(\lambda)| \geq \omega\}$

(b) *If* $\sigma \in (0, 1/\sqrt{2})$, *the only eigenvalue of L is* 0 *with algebraic multiplicity* 2.

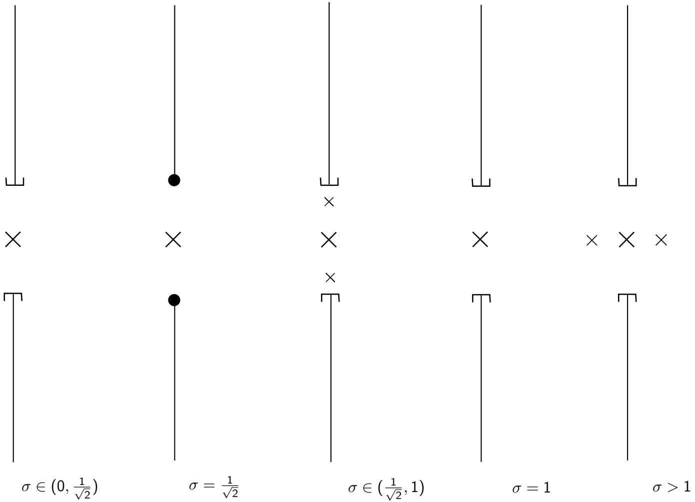
(c) *If* $\sigma = 1/\sqrt{2}$, *L has resonances* $\pm i\omega$ *at the border of the essential spectrum and the eigenvalue* 0 *with algebraic multiplicity* 2.

(d) *If* $\sigma \in (1/\sqrt{2}, 1)$, *L has two simple eigenvalues* $\pm i\xi == \pm i2\sigma\sqrt{1-\sigma^2}\omega$ *and the eigenvalue* 0 *with algebraic multiplicity* 2.

(e) *If* $\sigma = 1$, *the only eigenvalue of L is* 0 *with algebraic multiplicity* 4.

(f) *If* $\sigma \in (1, +\infty)$, *L has two simple eigenvalues* $\pm 2\sigma\sqrt{\sigma^2 - 1}\omega$ *and the eigenvalue* 0 *with algebraic multiplicity* 2.

σ flow of eigenvalues



Dispersive estimates on the propagator of L

The following weighted L^p spaces are needed

- $L^1_w(\mathbb{R}^3) = \left\{ f : \mathbb{R}^3 \to \mathbb{C} : \int_{\mathbb{R}^3} w(x)|f(x)|dx < +\infty \right\}$,
- $L^\infty_{w^{-1}}(\mathbb{R}^3) = \left\{ f : \mathbb{R}^3 \to \mathbb{C} : \operatorname{esssup}_{x\in\mathbb{R}^3}(w(x))^{-1}|f(x)| < +\infty \right\}$,

where $w(x) = 1 + \frac{1}{|x|}$.

Theorem

Assume $\sigma \neq 1/\sqrt{2}$, then there exist a constant $C > 0$ such that

$$\|P^c f\|_{L^\infty_{w^{-1}}} \leqslant C t^{-\frac{3}{2}} \|f\|_{L^1_w},$$

for any $f \in L^1_w(\mathbb{R}^3)$, where P^c is the projection onto the essential spectrum of L.

Time decay of solutions

One want to prove time decay transversal component $\chi(t)$.
Suppose first that $\sigma \in (0, \frac{1}{\sqrt{2}})$ i.e. no nonvanishing eigenvalues

Majorant method (Buslaev and Perelman '93, '95; Buslaev-Sulem '02; other subsequent refinements).

For any $T > 0$ define the majorant

$$M(T) = \sup_{0 \leqslant t \leqslant T} \left[(1+t)^{3/2} \|\chi(t)\|_{L^\infty_{w^{-1}}} + (1+t)^3 (|\dot{\gamma}(t)| + |\dot{\omega}(t)|) \right].$$

It is shown that the majorant is in fact uniformly bounded in T.

Concluding the proof of asymptotic stability

- Majorants $\Rightarrow \omega(t) \to \omega_\infty$, and $\Theta(t) - \omega_\infty t \to 0$, as $t \to +\infty$.
- Ansatz: $z(t,x) = u(t,x) - e^{i\Theta(t)}\Phi_{\omega(t)}(x)$.

$$\Downarrow$$

$$z(t,x) = \mathcal{U}_t * z_0(x) + i\int_0^t \mathcal{U}_{t-\tau}(x) q_z(\tau) d\tau - i\int_0^t \mathcal{U}_{t-\tau} * f(s(\tau)) d\tau,$$

where $f(s) = \dot{\gamma}s - i\dot{\omega}\frac{ds}{d\omega}$.

- The thesis follows estimating the L^2 norm of
  - $\phi_\infty = z_0 + \int_0^\infty U_{-\tau}(x) q_z(\tau) d\tau + \int_0^\infty U_{-\tau} * f(s(\tau)) d\tau$,
  - $r_\infty = \int_t^\infty U_{t-\tau}(x) q_z(\tau) d\tau + -\int_t^\infty U_{t-\tau} * f(s(\tau)) d\tau$.

Asymptotic stability result in the case $\sigma \in (0, 1/\sqrt{2})$

Theorem

Assume $\sigma \in (0, 1/\sqrt{2})$. *Let* $u(t) \in C(\mathbb{R}^+, V)$ *be a solution of equation* (C-NLS) *with* $u(0) = u_0 \in D(H_\alpha) \cap L^1_w$ *and denote*

$$d = \|u_0 - e^{i\theta_0}\Phi_{\omega_0}\|_{V \cap L^1_w},$$

for some $\omega_0 > 0$ *and* $\theta_0 \in \mathbb{R}$. *Then if* d *is sufficiently small, the solution* $u(t)$ *can be asymptotically decomposed as follows*

$$u(t) = e^{i\omega_\infty t}\Phi_{\omega_\infty} + U_t * \phi_\infty + r_\infty, \quad as \;\; t \to +\infty,$$

where $\omega_\infty > 0$ *and* ϕ_∞, $r_\infty \in L^2(\mathbb{R}^3)$ *with*

$$\|r_\infty\|_{L^2} = O(t^{-5/4}) \quad as \;\; t \to +\infty.$$

The case $\sigma \in (1/\sqrt{2}, 1)$

Now the ansatz is

$$u(t,x) = e^{i\Theta(t)} \left(\Phi_{\omega(t)}(x) + \chi(t,x)\right), \tag{4}$$

with

$$\Theta(t) = \int_0^t \omega(s)ds + \gamma(t), \tag{5}$$

$$\chi(t,x) = z(t)\Psi_1(t,x) + \overline{z}(t)\Psi_2(t,x) + f(t,x) \equiv \psi(t,x) + f(t,x), \tag{6}$$

with Ψ_1 and Ψ_2 eigenvectors associated to $\pm i\xi$ and $\omega(t)$, $\gamma(t)$, $z(t)$ and $f(t,x)$ satisfying their modulation equations.

Comments:

- Extra modulation equations for terms associated to the purely imaginary eigenvalues.
- Modulation equations are better handled when rewritten in Poincar (or Birkhoff) normal form; the oscillator $z(t)$ corresponding to $\pm i\xi$ has to be shown to be "dissipative" in the future.
  A nonlinear Fermi Golden Rule (introduced by Sigal '93) enters the game: suppose $2i\xi > \omega$ and $\langle N, u(2i\xi)\rangle > 0$ where N is the nonlinear remainder and $u(2\xi)$ is the improper eigenfunction of linearization L at $2i\xi$. This means that a higher (second in this case) harmonic of $i\xi$ falls in the continuous spectrum, and it has a nonvanishing interaction with the dispersive part releasing (> 0 in FGR) energy to it; dissipation is provided by dispersion.

Main references:
NLS general theory, Buslaev-Sulem '03; NLS with regular potentials, Gang-Sigal 05-07; recent work by Cuccagna-Mizumachi (2008) and Bambusi (2013) with strong improvements; concentrated NLS 1-d model, Komech-Kopylova-Stuart 08'-'12 .

Asymptotic stability in the case $\sigma \in (1, \frac{1+\sqrt{3}}{2\sqrt{2}}]$

Notice that $2\xi > \omega$ if $\sigma \in (1, \frac{\sqrt{3}+1}{2\sqrt{2}}]$. In this case one has the following

Theorem

Let $u(t) \in C(\mathbb{R}^+, V)$ be a solution of equation (C-NLS) with

$$u(0) = e^{i\omega_0 t+\gamma_0}\Phi_{\omega_0} + \underbrace{e^{i\omega_0 t+\gamma_0}[(z_0 + \overline{z_0})\Psi_1 + i(z_0 - \overline{z_0})\Psi_2] + f_0}_{\substack{\Cap \\ V \cap L^1_w(\mathbb{R}^3),}}$$

for some $\omega_0 > 0$, γ_0, $z_0 \in \mathbb{R}$ and $f_0 \in L^2(\mathbb{R}^3) \cap L^1_w(\mathbb{R}^3)$. Furthermore, assume that the initial datum u_0 is $\epsilon-$close in V to a solitary wave. Then the solution $u(t)$ can be asymptotically decomposed as follows

$$u(t) = e^{i(\omega_\infty t + b_1 \log(1+\epsilon k_\infty t))}\Phi_{\omega_\infty} + U_t * \phi_\infty + r_\infty, \quad as \;\; t \to +\infty,$$

where ω_∞, $\epsilon k_\infty > 0$, $b_1 \in \mathbb{R}$ and ϕ_∞, $r_\infty \in L^2(\mathbb{R}^3)$ with

$$\|r_\infty\|_{L^2} = O(t^{-1/4}) \quad as \;\; t \to +\infty \;\; in \;\; L^2(\mathbb{R}^3)$$

Some final comments

- Decay in the presence of non vanishing eigenvalues is slower
- The case $\sigma = \frac{1}{\sqrt{2}}$ where a threshold resonance appears in the linearization L is open. Treatment of threshold resonances is difficult and out of reach with the present technology independently of the model considered. This because of the poor dispersion, which does not allow decay of continuous part and convergence of solitary wave parameters.
- Asymptotic stability is proved in the interval $(1, \frac{1+\sqrt{3}}{2\sqrt{2}}]$ where $\frac{\sqrt{3}+1}{2\sqrt{2}} \cong 0.96$: it remains the small gap $\sigma \in (\frac{\sqrt{3}+1}{2\sqrt{2}}, 1)$. Requiring that $3i\xi \in [i\omega, i\infty)$ one can push the normal forms of modulation equations to a next order and cover in principle also this last gap.
- Finally notice that the asymptotic stability results obtained are for subcritical nonlinearities of concentrated type. For usual power nonlinearities asymptotic stability holds true if nonlinearity is flat at the origin and at least supercritical.