POINT INTERACTIONS IN THE PROBLEM OF THREE PARTICLES WITH INTERNAL STRUCTURE*

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Mathematical Challenges of Zero-Range Physics:
Rigorous Results and Open Problems
CAS^{LMU}, Munich, February 26, 2014

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*Based on joint work with K. A. Makarov and V. V. Melezhik

[Makarov, Melezhik, M.: Theor. Math. Phys. **102** (1995), 188–207]

doi: 10.1007/BF01040400

1 Introduction

[Bethe-Peirls 1931]: due to the small radius of (nuclear) forces many low-energy properties of a two-body system (deuteron) practically do not depend on the interaction details. Only one parameter is sufficient, the scattering length a. Assuming $\hbar = 1$ and $\mu = \frac{1}{2}$, the potential may be replaced by the boundary condition

$$\frac{d}{dr} \ln [r\psi(\mathbf{r})] \Big|_{r=0} = -\frac{1}{\mathrm{a}}, \qquad (1.1)$$

where $\mathbf{r}$ is the relative position vector of the particles.

[Berezin-Faddeev 1961]: one-parametric extensions of $-\Delta$ restricted to $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$.

Till now a source of explicitly solvable problems for various areas of physics (see, e.g., the fundamental book [Albeverio, Gesztesy, Høegh-Krohn and Holden 1988/2005]).

Zero-range interactions in a three-body problem produce mathematical difficulties [Minlos-Faddeev 1961] that are not present in the case of "regular" interactions. This comes from the fact that the supports of point interactions in two-body subsystems $\alpha = 1, 2, 3$, are 3-dim hyperplanes $\mathscr{M}_\alpha$. Codimension of $\mathscr{M}_\alpha$ w.r.t. the configuration space $\mathbb{R}^6$ is too high. The triple collision point $X = 0$, the only intersection point of $\mathscr{M}_\alpha$'s plays a crucial role. A natural switching on zero-range interactions produces a symmetric Hamiltonian [which is behind Skornyakov–Ter-Martirosyan equations (1956)] with non-zero deficiency indices. An extension is needed. Danilov conditions (1961) lead to a Hamiltonian that is not semibounded from below (Thomas effect 1935). Regularizing $\to$ three-body forces.

It is a priori clear that any generalization of the zero-range potential (that still remains non-trivial only at $r=0$) should produce the scattering wave functions $\psi(\mathbf{r},\mathbf{k})$ satisfying

$$\frac{d}{dr}\ln\left[r\psi(\mathbf{r},\mathbf{k})\right]\Big|_{r=0} = k\cot\delta(k),$$

where k is the modulus of the relative momentum and $\delta(k)$ the scattering phase shift. The low-energy expansion

$$k\cot\delta(k) \underset{E\downarrow 0}{=} -\frac{1}{\mathrm{a}} + \frac{1}{2}r_0E + Ar_0^2E^2 + \ldots \qquad (1.2)$$

where $E=k^2>0$ is the energy, and r_0 the effective radius (of the interaction).

[Shondin 1982], [LE Thomas 1984]: first example of a semibounded three-body Hamiltonian with δ-like interaction, efficiently with extra degrees of freedom: $L_2(\mathbb{R}^3)$ was extended to $L_2(\mathbb{R}^3)\oplus\mathbb{C}$; $r_0\neq 0$.

Another approach [Pavlov 1984], [Pavlov-Shushkov 1988]: a joint extension of

$$\Delta|_{C_0^\infty(\mathbb{R}^3\setminus\{0\})} \oplus A|_{D_A}, \quad D_A \subset \mathfrak{H}^{in}$$

where A is a (self-adjoint) operator on an auxiliary, rather arbitrary Hilbert space $\mathfrak{H}^{in}$ (describing "internal degrees of freedom"). Pavlov's "restriction-extension" model involves the deficiency elements of restricted channel operators. An equivalent direct description in [Makarov 1992] (boundary conditions) and [M. 1993] (singular potentials and singular coupling operators).

[M. 1993]: a two-channel operator matrix

$$\widehat{\mathbf{h}} = \begin{pmatrix} -\widehat{\Delta} + \widehat{V}_h & B \\ B^+ & A \end{pmatrix}, \tag{1.3}$$

where $\widehat{\Delta}$ is the Laplacian understood in the distributional sense; the operator A describes the internal degrees of freedom; $\widehat{V}_h$ is a generalized singular potential corresponding to the standard zero-range interaction; B and B^+ are (singular) coupling operators.

The spectral problem for $\widehat{\mathbf{h}}$ reduces to the "external" channel equation

$$\left(-\widehat{\Delta}+\widehat{w}(z)-z\right)\Psi=0$$

with the energy dependent interaction

$$\widehat{w}(z)=\widehat{V}_h-B(A-zI)^{-1}B^{+}. \tag{1.4}$$

If $\mathfrak{H}^{in}$ is a finite-dimensional (and, thus, A finite rank), the corresponding function $(-k\operatorname{ctg}\delta)$ is a rational Herglotz function of the energy z of the form

$$-k\operatorname{ctg}\delta(k)=\frac{P_N(z)}{Q_N(z)}, \qquad z=k^2, \tag{1.5}$$

where P_N and Q_N are polynomials of the power $N\leq\dim(\mathfrak{H}^{in})$ (notice that necessarily $r_0\leq 0$).

The question was how to include the point interaction with internal degrees of freedom into the three-body Hamiltonian. We followed an idea first developed in the case of a singular interaction with a surface support [Kuperin-Makarov-Merkuriev-M.-Pavlov, 1986].

Then — Faddeev equations. Two cases, depending on the asymptotic behavior of the two-body scattering matrices:

If $s_\alpha(E) \to -1$ as $E \to +\infty$, $\alpha = 1, 2, 3$ (or at least two of them) then the three-body Hamiltonian is not semibounded from below [Makarov 1992] and Faddeev equations are not Fredholm [Makarov-Melezhik-M., 1995].

If $s_\alpha(E) \to +1$ as $E \to +\infty$, $\alpha = 1, 2, 3$, we have both the opposite statements, in particular, the semiboundedness (cf. [Pavlov 1988]).

2 Two-body problem, some details

2.1 "Structureless" point interaction

First, recall the definition of the standard zero-range potential.

Let x, $x \in \mathbb{R}^3$ be the relative variable (Jacobi coordinate) for the system of two particles. Introduce a function class

$$\widehat{D} = \{\psi \in \widetilde{W}_2^2(\mathbb{R}^3 \setminus \{0\}), \quad \psi(x) \underset{x\to 0}{=} \frac{a}{4\pi|x|} + b + o(1)\}, \qquad \text{for some } a,b \in \mathbb{C}. \tag{2.1}$$

($\widehat{D}$ is simply the domain of the adjoint of $\Delta_0 := \Delta|_{C_0^\infty(\mathbb{R}^3\setminus\{0\})}$.)

The Hamiltonian h acts as the Laplacian $-\Delta$ on $\mathscr{D}(h) \subset \widehat{D}$ fixed by the condition

$$a = \gamma b \quad \text{for some } \gamma \in \mathbb{R} \tag{2.2}$$

γ parametrizes all possible self-adjoint extensions of $-\Delta_0$ in $L_2(\mathbb{R}^3)$.

Furthermore, $-\dfrac{\gamma}{4\pi} = \mathrm{a}$ is just the scattering length.

Equivalent (weak sense) formulation in terms of a quasipotential.

The initial Hamiltonian h is associated with a generalized Hamiltonian $\widehat{h}$ understood in the distributional sense, say, over $C_0^\infty(\mathbb{R}^3)$. The operator $\widehat{h}$ should be such that for $f \in L_2(\mathbb{R}^3)$, $z \in \mathbb{C}$, the equations

$$(\widehat{h} - z)\psi = f, \qquad \psi \in \widehat{D}, \tag{2.3}$$

and

$$(h - z)\psi = f, \qquad \psi \in \mathscr{D}(h), \tag{2.4}$$

are equivalent.

To describe the generalized Hamiltonians, we use the natural functionals $\mathbf{a}$ and $\mathbf{b}$ on $\widehat{D}$, defined by

$$\mathbf{a} : \psi \mapsto a\,, \qquad \mathbf{a}\psi = \lim_{x\to 0} 4\pi|x|\psi(x), \tag{2.5}$$

$$\mathbf{b} : \psi \mapsto b\,, \qquad \mathbf{b}\psi = \lim_{x\to 0}\left(\psi(x) - \frac{\mathbf{a}\psi}{4\pi|x|}\right). \tag{2.6}$$

In terms of these functionals, the condition (2.2) reads

$$\mathbf{a}\psi = \gamma\mathbf{b}\psi. \tag{2.7}$$

The generalized Laplacian $-\widehat{\Delta}$ acts on $\widehat{D}$ according to the formula

$$-\widehat{\Delta}\psi = -\Delta\psi + \delta(x)\mathbf{a}\psi\,, \tag{2.8}$$

where $-\Delta$ is the classical Laplacian (on $\widetilde{W}_2^2(\mathbb{R}^3\setminus 0)$). It then follows that the condition (2.7) is automatically reproduced if

$$\widehat{h} = -\widehat{\Delta} + \widehat{V}_h,$$

with the generalized potential (quasipotential)

$$\widehat{V}_h\psi = -\gamma\delta(x)\mathbf{b}\psi\,. \tag{2.9}$$

Actually, in this case $(\widehat{h}-z)\psi = f$ for $\psi \in \widehat{D}$, transforms into

$$(-\Delta - z)\psi + \delta(x)\left(\mathbf{a} - \gamma\mathbf{b}\right)\psi = f \tag{2.10}$$

Separately equating regular and singular terms on the both sides of (2.10), one arrives at

$$(h-z)\psi = f, \qquad \psi \in \mathscr{D}(h)$$

and

$$\mathbf{a}\psi = \gamma\mathbf{b}\psi. \tag{2.11}$$

That is, one comes to the original boundary value problem associated with the zero-range interaction. (In other words, the requirement of regularity of the image of the generalized Hamiltonian $\widehat{h}$ is equivalent to condition (2.11)...)

2.2 Point interactions with internal structure

Let A be a (for simplicity) bounded self-adjoint operator on a Hilbert space $\mathfrak{H}^{in}$. Introduce a (generalized) 2×2 matrix Hamiltonian

$$\widehat{\mathbf{h}} = \begin{pmatrix} -\widehat{\Delta}+\widehat{V}_h & B \\ B^+ & A \end{pmatrix}, \tag{2.12}$$

on the orthogonal sum $\mathscr{H} = L_2(\mathbb{R}^3)\oplus\mathfrak{H}^{in}$ of the "external", $L_2(\mathbb{R}^3)$, and "internal", $\mathfrak{H}^{in}$, spaces. Domain: $\widehat{D}\oplus\mathfrak{H}^{in}$. Here

$$\left(\widehat{V}_h\psi\right)(x) = \delta(x)\frac{\mu_{12}}{\mu_{11}}\mathbf{b}\psi, \qquad \psi\in\widehat{D}, \tag{2.13}$$

$$(Bu)(x) = -\delta(x)\frac{1}{\mu_{11}}\langle u\,,\,\theta\rangle, \qquad u\in\mathfrak{H}^{in}, \tag{2.14}$$

$$B^+\psi = \theta\left(\mu_{21}\mathbf{a}+\mu_{22}\mathbf{b}\right)\psi, \tag{2.15}$$

θ is a arbitrary fixed element from $\mathfrak{H}^{in}$, and

$$\mu_{ij}\in\mathbb{C}, \quad i,j=1,2, \quad \mu_{11}\neq 0.$$

The regularity requirement $f^{ex} \in L_2(\mathbb{R}^3)$ of the external component f^{ex} of the vector

$$f = (\widehat{\mathbf{h}} - z)\mathscr{U}\,, \quad f = (f^{ex}, f^{in}), \quad f^{in} \in \mathfrak{H}^{in},$$

for $\mathscr{U} \in \widehat{D} \oplus \mathfrak{H}^{in}$, $\mathscr{U} = (\psi, u)$, yields the following equations

$$\begin{cases} (-\Delta - z)\,\Psi = f^{ex} \\ \theta\,(\mu_{21}\mathbf{a} + \mu_{22}\mathbf{b})\,\psi + (A - z)\,u = f^{in} \end{cases} \tag{2.16}$$

and boundary condition

$$\mu_{11}\mathbf{a}\psi + \mu_{12}\mathbf{b}\psi = \langle u\,,\,\theta\rangle. \tag{2.17}$$

Thus, in this sense the generalized Hamiltonian $\widehat{\mathbf{h}}$ is equivalent to the “regular” operator

$$\mathbf{h}\begin{pmatrix} \psi \\ u \end{pmatrix} = \begin{pmatrix} -\Delta\psi \\ Au + \theta\,(\mu_{21}\mathbf{a} + \mu_{22}\mathbf{b})\,\psi \end{pmatrix} \tag{2.18}$$

on the domain $\mathscr{D}(\mathbf{h}) \subset \widehat{D} \oplus \mathfrak{H}^{in}$ defined by the boundary condition (2.17).

The operator $\mathbf{h}$ is self-adjoint if and only if

$$\det\begin{pmatrix}\mu_{11} & \bar{\mu}_{12}\\ \mu_{21} & \bar{\mu}_{22}\end{pmatrix} = -1, \quad \mu_{11}\bar{\mu}_{21}\in\mathbb{R}, \quad \mu_{12}\bar{\mu}_{22}\in\mathbb{R}. \tag{2.19}$$

In the following, conditions (2.19) will be always assumed.

After excluding the internal component, in the external channel equation we have an energy-dependent quasipotential:

$$\left(-\widehat{\Delta}+\widehat{w}(z)-z\right)\psi = 0, \tag{2.20}$$

$$\widehat{w}(z) = \widehat{V}_h + B\left(zI-A\right)^{-1}B^{+} = \delta(x)w(z) \tag{2.21}$$

where the functional $w(z)$ acts on $\widehat{D}$ and is given by

$$w(z) = \frac{\mu_{12}}{\mu_{11}}\mathbf{b} + \frac{\mu_{21}}{\mu_{11}}\rho(z)\mathbf{a} + \frac{\mu_{22}}{\mu_{11}}\rho(z)\mathbf{b}.$$

Here,

$$\rho(z) = \langle r_A(z)\theta\,,\,\theta\rangle \quad \text{where} \quad r_A(z) = (A-zI)^{-1}.$$

The quasipotential $\widehat{w}(z)$ yields the boundary condition

$$\mathbf{a}\psi = w(z)\psi$$

or, equivalently,

$$\frac{d}{d|x|}\ln\left[|x|\psi(x)\right]\Big|_{x=0} = -4\pi d_0(z),$$

where

$$d_0(z) = \frac{\mu_{11} + \mu_{21}\rho(z)}{\mu_{12} + \mu_{22}\rho(z)}.$$

Notice that if $\dim(\mathfrak{H}^{in}) < \infty$ and A has the eigenvalues ε_1, ε_2, ..., ε_N, then

$$\rho(z) = \sum_{j=1}^{N}\sum_{k=1}^{l_j}\frac{|\beta_{j,k}|^2}{\varepsilon_j - z},$$

where $\beta_{j,k} = \langle\theta, u_{j,k}\rangle$ with $u_{j,k}$ the eigenvectors of A for the eigenvalue ε_j, l_j – multiplicity. Hence, $d_0(z)$ is rational,

$$d_0(z) = \frac{P_N(z)}{Q_N(z)}.$$

Furthermore, d_0 is Herglotz. If $\mu_{12} = 0$, then the degree of Q_N is $N-1$.

2.3 Two classes of point interactions

In the model under consideration, the scattering matrix is given by

$$s(\widehat{k},\widehat{k}',z) = \delta(\widehat{k},\widehat{k}') - \frac{i}{8\pi^2}\frac{1}{d_0(z)+\dfrac{i\sqrt{z}}{4\pi}},$$

$z = E \pm i0$, $E > 0$, $\widehat{k},\widehat{k}' \in S^2$. It differs from the identity operator only in the s-state ($L = 0$). The s-state component reads

$$s(z) = \frac{4\pi d_0(z) - i\sqrt{z}}{4\pi d_0(z) + i\sqrt{z}}.$$

Notice that in the case of the standard zero range interaction

$$s(z) = \frac{-4\pi\gamma^{-1} - i\sqrt{z}}{-4\pi\gamma^{-1} + i\sqrt{z}}.$$

Behavior of $s(E \pm i0)$ as $E \to +\infty$ is determined by the asymptotics of $d_0(z)$.

Two cases

$$A) \qquad \mu_{12} \neq 0, \tag{2.22}$$

$$R) \qquad \mu_{12} = 0. \tag{2.23}$$

In the case (A) the function $d_0(E \pm i0)$ is bounded $\Longrightarrow$ "anomalous" behavior of the scattering matrix,

$$s(E \pm i0) \underset{E \to +\infty}{\to} -1.$$

The class (A) contains the standard zero-range interactions $\widehat{V}_h$ (for $\theta = 0$ and $\gamma = -\mu_{12}/\mu_{11}$).

In the case (R), on the contrary, $d_0(E \pm i0)$ is unbounded as $E \to +\infty$,

$$d_0(E \pm i0) \underset{E \to +\infty}{=} cE + o(E)$$

with some $c > 0$. Hence, we have the "regular" high-energy behavior

$$s(E \pm i0) \underset{E\to+\infty}{\longrightarrow} 1.$$

In other words, only the potential $\widehat{V}_h$ is responsible for the "anomaly". It is the zero-range interaction $\widehat{V}_h$ that leads to the non-semiboundedness of the three-body Hamiltonian and to the "bad" properties of the corresponding version of Faddeev equations (due to Skornyakov–Ter-Martirosyan).

If $\widehat{V}_h = 0$ then none of these two problems arizes [Makarov 1992], [Makarov-Melezhik-M. 1995].

3 Three-particle system with point interactions

3.1 Hamiltonian H_α

Center-of-mass frame; reduced Jacobi variables x_α, y_α, $\alpha = 1,2,3$. For example,

$$x_1 = \left(\frac{2\mathrm{m}_2\mathrm{m}_3}{\mathrm{m}_2+\mathrm{m}_3}\right)^{1/2}(r_2 - r_3)$$

$$y_1 = \left[\frac{2\mathrm{m}_1(\mathrm{m}_2+\mathrm{m}_3)}{\mathrm{m}_1+\mathrm{m}_2+\mathrm{m}_3}\right]^{1/2}\left(r_1 - \frac{\mathrm{m}_2 r_2 + \mathrm{m}_3 r_3}{\mathrm{m}_2+\mathrm{m}_3}\right)$$

Configuration space $\mathbb{R}^6$; six-vectors $X = (x_\alpha, y_\alpha)$. Transition from one to another set of Jacobi variables:

$$\begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix} = \begin{pmatrix} c_{\alpha\beta} & s_{\alpha\beta} \\ -s_{\alpha\beta} & c_{\alpha\beta} \end{pmatrix} \begin{pmatrix} x_\beta \\ y_\beta \end{pmatrix},$$

where $c_{\alpha\beta}$, $s_{\alpha\beta}$ depend only on the particle masses and form an orthogonal (rotation) matrix.

First, the case where only the particle of a pair α interact. Generalized Hamiltonian $\widehat{H}_\alpha$ is build of the two-body Hamiltonian $\widehat{\mathbf{h}}_\alpha$ as

$$\widehat{H}_\alpha = \widehat{\mathbf{h}}_\alpha \otimes I_{y_\alpha} + I_\alpha \otimes (-\Delta_{y_\alpha})$$

Here, I_{y_α} and I_α are the identity operators in $L_2(\mathbb{R}^3_{y_\alpha})$ and $\mathfrak{H}^{in}_\alpha$, resp. The operator $\widehat{H}_\alpha$ acts from

$$\mathscr{G}_\alpha = \mathscr{H}_\alpha \otimes L_2(\mathbb{R}^3_{y_\alpha}) = \mathscr{G}^{ex} \oplus \mathscr{G}^{in}_\alpha,$$

The external and internal channel spaces:

$$\mathscr{G}^{ex} = L_2(\mathbb{R}^6), \quad \mathscr{G}^{in}_\alpha = L_2(\mathbb{R}^3_{y_\alpha}, \mathfrak{H}^{in}_\alpha).$$

$$\mathscr{U} \in \mathscr{G}_\alpha \Leftrightarrow \mathscr{U} = (\Psi, u_\alpha) \quad , \Psi \in \mathscr{G}^{ex}, \ u_\alpha \in \mathscr{G}^{in}_\alpha.$$

The operator $\widehat{H}_\alpha$ is defined on

$$\widehat{\mathbf{D}}_\alpha = \left(\widehat{D}_\alpha \oplus \mathfrak{H}^{in}_\alpha\right) \otimes W^2_2(\mathbb{R}^3_{y_\alpha}) = \widehat{\mathbf{D}}^{ex}_\alpha \oplus \mathbf{D}^{in}_\alpha, \tag{3.1}$$

where

$$\widehat{\mathbf{D}}^{ex}_\alpha = \widehat{D}_\alpha \otimes W^2_2(\mathbb{R}^3_{y_\alpha}) \quad \text{and} \quad \mathbf{D}^{in}_\alpha = \mathfrak{H}^{in}_\alpha \otimes W^2_2(\mathbb{R}^3_{y_\alpha}).$$

Thus, $\widehat{\mathbf{D}}_\alpha$ is formed of the vectors $\mathscr{U} = (\Psi, u_\alpha)$ whose external components Ψ, $\Psi \in \widehat{\mathbf{D}}^{ex}_\alpha$, behave like

$$\Psi(X) \underset{x_\alpha \to 0}{\sim} \frac{a_\alpha(y_\alpha)}{4\pi|x|} + b_\alpha(y_\alpha) + o(1), \tag{3.2}$$

with a_α, $b_\alpha \in W_2^2(\mathbb{R}^3_{y_\alpha})$, and

$$\Psi \in \widetilde{W}_2^2(\mathbb{R}^6 \setminus \mathscr{M}_\alpha), \quad \mathscr{M}_\alpha = \{X \in \mathbb{R}^6 \,|\, x_\alpha = 0\}$$

Internal components: $u_\alpha \in \mathbf{D}^{in}_\alpha = W_2^2(\mathbb{R}^3_{y_\alpha}, \mathfrak{H}^{in}_\alpha)$. One may identify $\mathbf{D}^{in}_\alpha$ with $W_2^2(\mathscr{M}_\alpha, \mathfrak{H}^{in}_\alpha)$.

The Hamiltonian $\widehat{H}_\alpha$ (on $\widehat{\mathbf{D}}_\alpha$) may be viewed as a 2×2 block matrix,

$$\widehat{H}_\alpha = \begin{pmatrix} -\widehat{\Delta}_{x_\alpha} + \widehat{V}_h^{(\alpha)} - \Delta_{y_\alpha} & B_\alpha \\ B_\alpha^+ & A_\alpha - \Delta_{y_\alpha} \end{pmatrix} = \begin{pmatrix} -\widehat{\Delta}_X + \widehat{V}_h^{(\alpha)} & B_\alpha \\ B_\alpha^+ & A_\alpha - \Delta_{y_\alpha} \end{pmatrix}.$$

The Laplacian $-\widehat{\Delta}_X = -\widehat{\Delta}_{x_\alpha} - \Delta_{y_\alpha}$ should be understood in the sense of distributions over $C_0^\infty(\mathbb{R}^6)$.

Then the *generalized Hamiltonian* $\widehat{H}_\alpha$ is equivalent *to the self-adjoint operator*

$$H_\alpha \begin{pmatrix} \Psi \\ u_\alpha \end{pmatrix} = \begin{pmatrix} (-\Delta_X + v_\alpha)\,\Psi \\ (A_\alpha - \Delta_{y_\alpha})\,u_\alpha + \theta_\alpha \left(\mu_{21}^{(\alpha)}\mathbf{a}_\alpha + \mu_{22}^{(\alpha)}\mathbf{b}_\alpha\right)\Psi \end{pmatrix} \tag{3.3}$$

whose domain $\mathscr{D}(H_\alpha)$ *consists of those elements from* $\widehat{\mathbf{D}}_\alpha$ *that satisfy the boundary condition*

$$\left(\left[\mu_{11}^{(\alpha)}\mathbf{a}_\alpha + \mu_{12}^{(\alpha)}\mathbf{b}_\alpha\right]\Psi\right)(y_\alpha) = \langle u_\alpha(y_\alpha)\,,\,\theta_\alpha\rangle. \tag{3.4}$$

3.2 Total Hamiltonian H

If every pair subsystem has an internal channel, the generalized three-body Hamiltonian is introduced as the following operator matrix

$$\widehat{H} = \begin{pmatrix} -\widehat{\Delta}_X + \sum\limits_{\alpha} \widehat{V}_h^{(\alpha)} & B_1 & B_2 & B_3 \\ B_1^+ & A_1 - \Delta_{y_1} & 0 & 0 \\ B_2^+ & 0 & A_2 - \Delta_{y_2} & 0 \\ B_3^+ & 0 & 0 & A_3 - \Delta_{y_3} \end{pmatrix}, \qquad (3.5)$$

considered in the Hilbert space $\mathscr{G} = \mathscr{G}^{ex} \oplus \bigoplus\limits_{\alpha=1}^{3} \mathscr{G}_{\alpha}^{in}$. The operator $\widehat{H}$ acts in $\mathscr{G}$ on the set

$$\widehat{\mathbf{D}} = \widehat{\mathbf{D}}^{ex} \oplus \bigoplus_{\alpha=1}^{3} \mathbf{D}_{\alpha}^{in},$$

where $\mathbf{D}_\alpha^{in} = \mathfrak{H}_\alpha^{in} \otimes \widetilde{W}_2^2(\mathbb{R}^3_{y_\alpha} \setminus \{0\})$. The external component $\widehat{\mathbf{D}}^{ex}$ consists of the functions

$$\Psi \in \widetilde{W}_2^2\left(\mathbb{R}^6 \setminus \bigcup_{\beta=1}^{3} \mathscr{M}_\beta\right),$$

possessing the asymptics (3.2) for any $\alpha = 1,2,3$ with the coefficients

$$a_\alpha, b_\alpha \in \widetilde{W}_2^2(\mathbb{R}^3_{y_\alpha} \setminus \{0\}).$$

The structure of the matrix (3.5) demonstrates by itself the truly pairwise character of the point interactions in $\widehat{H}$ (in contrast to [Pavlov 1988]).

A state of the system is a four-component vector $\mathscr{U} = (\Psi, u_1, u_2, u_3)$, $\Psi \in \mathscr{G}^{ex}$, $u_\alpha \in \mathscr{G}_\alpha^{in}$.

Further, for $\mathscr{U} \in \widehat{\mathbf{D}}$, impose the regularity requirement for its image $\widehat{H}\mathscr{U}$... And obtain the corresponding Hamiltonian H that is understood in the usual sense:

$$H_\alpha \mathscr{U} = \begin{pmatrix} -\Delta_X \Psi \\ \bigoplus_{\alpha=1}^{3} \left[(A_\alpha - \Delta_{y_\alpha}) u_\alpha + \theta_\alpha \left(\mu_{21}^{(\alpha)} \mathbf{a}_\alpha + \mu_{22}^{(\alpha)} \mathbf{b}_\alpha \right) \right] \Psi \end{pmatrix} \qquad \textbf{(3.6)}$$

The domain $\mathscr{D}(H)$ consists of those elements from $\widehat{\mathbf{D}}$ that satisfy the boundary conditions

$$\left(\left[\mu_{11}^{(\alpha)} \mathbf{a}_\alpha + \mu_{12}^{(\alpha)} \mathbf{b}_\alpha \right] \Psi \right) (y_\alpha) = \langle u_\alpha(y_\alpha)\,, \theta_\alpha \rangle, \quad \forall \alpha = 1, 2, 3. \qquad \textbf{(3.7)}$$

By inspection, H is symmetric on $\mathscr{D}(H)$. Furthermore, if $\mu_{12}^{(\alpha)} = 0$, $\forall \alpha = 1, 2, 3$ [class (R)], H is self-adjoint and semibounded from below [Makarov 1992]. This follows, e.g., from the study of the corresponding Faddeev equations (see [Makarov-Melezhik-M. 1995]).

If $\mu_{12}^{(\alpha)} \neq 0$ at least for two of α's [class (A)], one encounters the same problems as in the Skornyakov-Ter-Martirosyan case.

The study of the spectral properties of H is reduced to the study of the resolvent $\mathbf{R}(z) = (H - z)^{-1}$ which is a 4×4 matrix with the components R_{ab} $(\mathrm{a}, \mathrm{b} = 0, 1, 2, 3)$ (0 – external channel; 1,2,3 – internal channels). All the study is reduced to that of $R(z) := R_{00}(z)$.

3.3 Faddeev integral equations

$R(z)$ satisfies the resolvent identities (Lippmann-Schwinger equations)

$$R_\alpha(z) = R_\alpha(z) - R_\alpha(z) \sum_{\beta \neq \alpha} \widehat{W}_\beta(z) R(z) \quad (\alpha = 1,2,3), \tag{3.8}$$

where $R_\alpha(z)$ is the external component the resolvent $(H_\alpha - z)^{-1}$. This equations are non-Fredholm.

Introduce $\widehat{M}_\alpha(z) = \widehat{W}_\alpha(z) R(z)$, $\alpha = 1,2,3$. Clearly,

$$R(z) = R_0(z) - R_0(z) \sum_\alpha \widehat{M}_\alpha(z),$$

and, from (3.8),

$$\widehat{M}_\alpha(z) = \widehat{W}_\alpha(z) R_\alpha(z) - \widehat{W}_\alpha(z) R_\alpha(z) \sum_{\beta \neq \alpha} \widehat{M}_\beta(z) \quad (\alpha = 1,2,3), \tag{3.9}$$

the Faddeev integral equations. Extract δ-factors $\delta(x_\alpha)$ in $\widehat{M}_\alpha$ and pass to the regular kernels (functions) $M_\alpha(y_\alpha, X', z)$, $\widehat{M}_\alpha(z) =$

$\delta(x_\alpha)M_\alpha(z)$. This results in

$$M_\alpha(z) = W_\alpha(z)R_\alpha(z) - W_\alpha(z)R_\alpha(z)\sum_{\beta\neq\alpha}\delta_\beta M_\beta(z), \tag{3.10}$$

where δ_β is multiplication by the δ-function $\delta(x_\beta)$.

If one deals with the (R) case, all further study follows the usual Faddeev procedure: good, improving iterations with a nicer and nicer asymptotic behavior of the iterated kernels. *The fourth iteration gives a compact operator* ($+$ known estimates concerning the behavior with respect to z).

In case (A) one can not prove that the kernel $(W_\alpha(z)R_\alpha)(y_\alpha, X', z)$ is integrable over a domain where $X' \in \mathscr{M}_\beta$, $\beta \neq \alpha$ and $|x'_\alpha|$ and $|y_\alpha - y'_\alpha|$ are both small (this is just the neighborhood of the triple collision point). Details in [Makarov-Melezhik-M. 1995]. [Makarov-Melezhik 1996] used the momentum space representation.

Recall that if $\theta = 0$ (i.e. the standard zero-range interactions), equations (3.10) are nothing but the Skornyakov-Ter-Martirosyan ones.