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Two fermions and a test particle: a detailed analysis

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Mathematical challenges of zero-range Physics: rigorous results and open problems

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Joint work with M.Correggi, G. Dell'Antonio, A.Michelangeli, A.Teta

- Quadratic forms for the Fermionic Unitary Gas, D.F and A.Teta, *Reports on Mathematical Physics*, **69** (2012)
- Stability for a system of N fermions a different particle with zero-range interactions, M.Correggi, G. Dell'Antonio, D.Finco, A.MIchelangeli, A.Teta, *Reviews in Mathematical Physics*, **24**, (2012).

Many body Hamiltonians

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System of n quantum particles in $\mathbb{R}^3$, interacting via a zero-range, two-body interaction. Formally

$$\mathcal{H} = -\sum_{i=1}^{n} \frac{1}{2m_i} \Delta_{\mathbf{x}_i} + \sum_{\substack{i,j=1 \\ i<j}}^{n} \mu_{ij}\, \delta(\mathbf{x}_i - \mathbf{x}_j),$$

where $\mathbf{x}_i \in \mathbb{R}^3$, $i = 1, \ldots, n$, m_i is the mass, $\Delta_{\mathbf{x}_i}$ is the Laplacian relative to $\mathbf{x}_i$, and $\mu_{ij} \in \mathbb{R}$. We set $\hbar = 1$.

Motivation: Nuclear Physics, ultra-cold quantum gases.

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Mathematical problem: rigorous construction and stability

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Elements in the domain of $\mathcal{H}$ are regular away from $\{\mathbf{x}_i - \mathbf{x}_j = 0\}$ but we must specify a boundary condition at the coincidence planes (Bethe-Peierls contact condition).

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For $\mathbf{n} = \mathbf{2}$, in the relative coordinate $\mathbf{x}$

$$\mathcal{H} = -\frac{1}{2m}\Delta_{\mathbf{x}} + \delta(\mathbf{x})$$

The domain is $\psi \in L^2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3 \setminus \{0\})$ satisfying the b.c. at the origin

$$\psi(\mathbf{x}) = \frac{q}{|\mathbf{x}|} + \alpha q + o(1), \qquad \text{for } |\mathbf{x}| \to 0, \qquad q \in \mathbb{C}, \alpha \in \mathbb{R}$$

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For $\mathbf{n} > 2$, by analogy, one considers the **Skornyakov-Ter-Martirosyan** (STM) Hamiltonian $\mathcal{H}_\alpha$, defined on $L^2(\mathbb{R}^{3n}) \cap H^2(\mathbb{R}^{3n} \setminus \cup_{i<j}\{\mathbf{x}_i = \mathbf{x}_j\})$ and s.t.

$$\psi(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{q_{ij}}{|\mathbf{x}_i - \mathbf{x}_j|} + \alpha q_{ij} + o(1), \qquad \text{for } |\mathbf{x}_i - \mathbf{x}_j| \to 0, \qquad \alpha \in \mathbb{R}$$

q_{ij} functions on $\{\mathbf{x}_i = \mathbf{x}_j\}$ and α parametrizes strength of the interaction

Three body Hamiltonians

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Already for $\mathbf{n = 3}$ problems appears: in many cases the STM Hamiltonian is not s.a. and any s.a. extension is unbounded from below due to the presence of infinitely many eigenvalues E_n accumulating at $-\infty$, i.e. the **Thomas effect**.

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- three identical bosons [Faddeev, Minlos 1961]
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One way to prevent the collapse of the system is to introduce **fermionic symmetry** (kills part of the interaction)

N fermions and a test particle

Consider N fermions of mass 1 and a test particle of mass m

$$\mathcal{H} = -\frac{1}{2m}\Delta_{\mathbf{x}_0} - \sum_{i=1}^{n} \frac{1}{2}\Delta_{\mathbf{x}_i} + \alpha \sum_{i=1}^{n} \delta(\mathbf{x}_0 - \mathbf{x}_i)$$

For some values of the physical parameters m and N it is possible to define this Hamiltonian ad a bounded from below s.a. operator

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Consider 2 fermions of mass 1 and a test particle of mass m

$$\mathcal{H} = -\frac{1}{2m}\Delta_{\mathbf{x}_0} - \frac{1}{2}\Delta_{\mathbf{x}_1} - \frac{1}{2}\Delta_{\mathbf{x}_2} + \alpha\delta(\mathbf{x}_0 - \mathbf{x}_1) + \alpha\delta(\mathbf{x}_0 - \mathbf{x}_2)$$

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Stability for $N = 2$

There is a threshold $m^* = 0.0735 = (13.607)^{-1}$ such that the system is stable for $m > m^*$ and unstable otherwise.

Quadratic forms

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Theorem (Representation Theorem)

The set of self adjoint semi bounded Hamiltonians is in 1 to 1 correspondence with semi bounded closed quadratic forms.

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Advantages

- simpler than searching for all s.a. extensions of a symmetric operators
- construction is quicker

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One has to *guess* a quadratic form and then has to prove that it is closed and bounded from below

We shall consider the following quadratic form $\mathcal{F}_\alpha$ defined on $L^2(\mathbb{R}^6)$, (we can subtract the center of mass motion)

Quadratic form $\mathcal{F}_\alpha$

$$\mathscr{D}(\mathcal{F}_\alpha) = \left\{ \psi \in L^2_f(\mathbb{R}^6)\,\text{s.t.}\,\psi = \phi^\lambda + \mathcal{G}^\lambda\xi,\ \phi^\lambda \in H^1_f(\mathbb{R}^6),\ \xi \in H^{1/2}(\mathbb{R}^3) \right\}$$

$$\mathcal{G}^\lambda\xi(\mathbf{k}_1, \mathbf{k}_2) = \frac{\xi(\mathbf{k}_1) - \xi(\mathbf{k}_2)}{k^2 + k'^2 + \frac{2}{m+1}\mathbf{k}\cdot\mathbf{k}' + \lambda}$$

$$\mathcal{F}_\alpha[\psi] + \lambda\|\psi\|^2_{L^2(\mathbb{R}^6)} = \mathcal{F}_0[\phi^\lambda] + \lambda\|\phi^\lambda\|^2_{L^2(\mathbb{R}^6)} + \boldsymbol{\Phi}^{\lambda,\alpha}[\xi]$$

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$$\Phi^{\lambda,\alpha}[\xi] = \Phi^\lambda_d[\xi] + \Phi^\lambda_o[\xi] + \alpha\|\xi\|^2$$

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$$\mathbf{\Phi}^\lambda_d[\xi] = 2\pi^2 \int \sqrt{\frac{m(m+2)}{(m+1)^2} k^2 + \lambda}\, |\xi(\mathbf{k})|^2 \, d\mathbf{k}$$

$$\mathbf{\Phi}^\lambda_o[\xi] = \int \frac{\overline{\xi(\mathbf{k})}\xi(\mathbf{k}')}{k^2 + k'^2 + \frac{2}{m+1}\mathbf{k} \cdot \mathbf{k}' + \lambda} \, d\mathbf{k}\, d\mathbf{k}'$$

Remarks on $\mathcal{F}_\alpha$

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Proofs

Some remarks are in order

- $\lambda > 0$ is a free parameter which regularizes the behavior at infinity of $\frac{1}{|\mathbf{x}|}$
- decomposition is meaningful
- heuristic argument to justify $\mathcal{F}_\alpha$: renormalization of the energy through a coupling constant renormalization (Γ-limit of regularized functionals)
- if $\psi \in \mathscr{D}(\mathcal{H}_\alpha)$ then $\langle \psi | \mathcal{H}_\alpha | \psi \rangle = \mathcal{F}_\alpha[\psi]$
- all the interaction is concentrated in $\Phi^{\lambda,\alpha}$

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Theorem

If there exists λ such that $\Phi^{\lambda,\alpha}[\xi] \geq c\|\xi\|^2_{H^{1/2}(\mathbb{R}^3)}$ then $\mathcal{F}_\alpha$ is closed and bounded from below.

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Partial wave decomposition on Φ^λ

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We exploit rotational invariance and reduce to the subspace of angular momentum l

$$\Phi_d^\lambda[f] = 2\pi^2 \int_0^\infty \sqrt{\frac{m(m+2)}{(m+1)^2} k^2 + \lambda} \, |f(k)|^2 \, k^2 \, dk$$

$$\Phi_{o,l}^\lambda[f] = 2\pi \int_0^\infty dk \, dk' \int_{-1}^1 dy \, P_l(y) \, \frac{k^2 \, k'^2}{k^2 + k'^2 + \frac{2y}{m+1} k \, k' + \lambda} \, \overline{f(k)} f(k')$$

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Proposition

The off diagonal term has definite sign depending on the parity of l and it is monotone w.r.t. to λ that is

- $0 \le \Phi_{o,l}^\lambda[f] \le \Phi_{o,l}[f]$ for even l
- $\Phi_{o,l}[f] \le \Phi_{o,l}^\lambda[f] \le 0$ for odd l

Diagonalization

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The previous proposition suggests that we carefully analyze the case $\lambda = 0$. We can get optimal results since $\Phi[f]$ can be diagonalized.

$$\Phi_d[f] = 2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}} \int_0^\infty |f(k)|^2 \, k^3 \, dk$$

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Define

$$f^\sharp(z) = \frac{1}{\sqrt{2\pi}} \int dk \, e^{-ikz} \, e^{2k} \, f(e^k)$$

then

$$\Phi_l[f] = \int_{-\infty}^\infty dz \, S_l(z) \, |f^\sharp(z)|^2 = \int_{-\infty}^\infty dz \, \left(S_d + S_{o,l}(z)\right) \, |f^\sharp(z)|^2$$

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We have

$$\Phi_l[f] = \int_{-\infty}^{\infty} dz\, S_l(z) \left|f^\sharp(z)\right|^2 = \int_{-\infty}^{\infty} dz\, \left(S_d + S_{o,l}(z)\right) \left|f^\sharp(z)\right|^2$$

$$S_d = 2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}}$$

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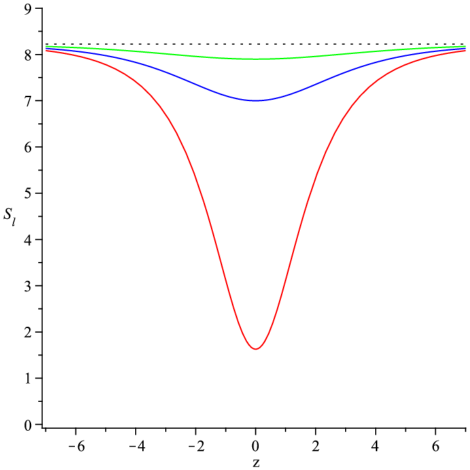
We have to find the infimum of $S_l(z)$ over l and z

Diagonalization



Plot of $S_1(z)$, $S_3(z)$, $S_5(z)$ for $m = 0.1$.

Diagonalization

From the picture it is clear that the infimum is achieved by $S_1(0)$.

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Proposition

For fixed z, $S_l(z)$ is an increasing function function of l. Moreover $S_1(z)$ has an absolute minimum for $z = 0$.

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Proposition

For fixed z, $S_l(z)$ is an increasing function function of l. Moreover $S_1(z)$ has an absolute minimum for $z = 0$.

It is sufficient to search for which m

$$F_1^*(m) = S_1(0) = 2\pi^2\sqrt{\frac{m(m+2)}{(m+1)^2}} + \pi\int_{-1}^{1} dy\, y \int dk\, \frac{1}{\cosh(k) + \frac{y}{m+1}}$$

The plot of $F_1^*(m)$ is simple

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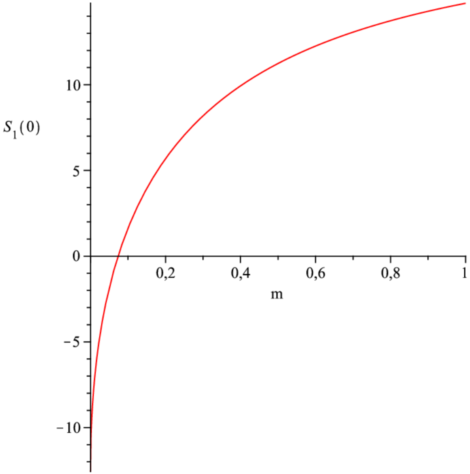

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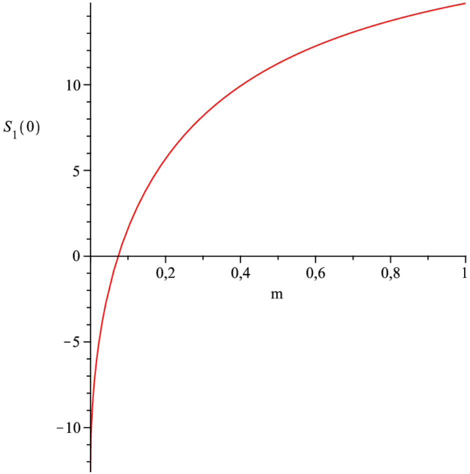

The condition $F_1^*(m) > 0$ is equivalent to $m > m^*$

Conclusions from partial wave analysis

Let us introduce

$$\Lambda = |S_1(0)|/S_d$$

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The condition $m > m^*$ implies

- $0 < \Lambda < 1$
- the negative part of Φ_o^λ is small in the sense of quadratic forms compared to Φ_d^λ
- Φ^λ is coercive and $\Phi^\lambda \geq (1 - \Lambda)\Phi_d^\lambda$

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Stability

For $m > m^*$ the quadratic form $\mathcal{F}_\alpha$ defines a s.a. and bounded from below operator that we identify with $\mathcal{H}_\alpha$

Instability

Take ψ_n such that $\phi_n^\lambda = 0$ and ξ_n has non trivial components only for $l = 1$ given by

$$f_n(k) = \frac{1}{n} f\left(\frac{k}{n}\right)$$

With this scaling $\|\mathcal{G}^\lambda \xi_n\| < c$

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If $m < m^*$ then $S_1(0) < 0$ and we can find f such that $\Phi[f] < 0$.

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Theorem

The quadratic form $\mathcal{F}_\alpha$ is closed and bounded from below iff $m > m^*$

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Recently Minlos, analyzing the case $l = 1$, pointed that there is a richer structure and there is not a unique Hamiltonian for $m > m^*$.

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$$T_l = T_d + T_{o,l} \qquad \mathscr{D}(T_l) = \mathscr{D}(T_d)$$

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$$T_{o,l}[f] = 2\pi \int_0^\infty dk' \int_{-1}^1 dy\, P_l(y) \,\frac{k'^2}{k^2 + k'^2 + \frac{2y}{m+1} k\, k'}\, f(k')$$

Minlos

There is a second threshold m^{**} such that

- for $m^* < m < m^{**}$, T_1 is not essentially s.a. and there is a one parameter family of s.a. extensions
- for $m > m^{**}, T_1$ is essentially s.a.

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A big part of this picture can be easily carried to any subspace with odd l.

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Theorem

There are two sequences of thresholds m_l^* , m_l^{**} with $m_l^* < m_l^{**}$, $m_1^* > m_3^* > m_5^* > \dots$ and $m_1^{**} > m_3^{**} > m_5^{**} > \dots$ such that

- *for $m < m_l^*$, the form Φ_l^λ is unbounded from below*
- *for $m_l^* < m < m_l^{**}$, T_l is not essentially s.a.*
- *for $m > m_l^*$, T_l is essentially s.a. and positive*

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Define

$$F_l^*(m) \equiv S_l(0) = 2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}} + \pi \int_{-1}^{1} dy\, P_l(y) \int dk\, \frac{1}{\cosh(k) + \frac{y}{m+1}}$$

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Then m_l^* is defined by

$$F_l^*(m) = 0$$

$m < m_l^*$

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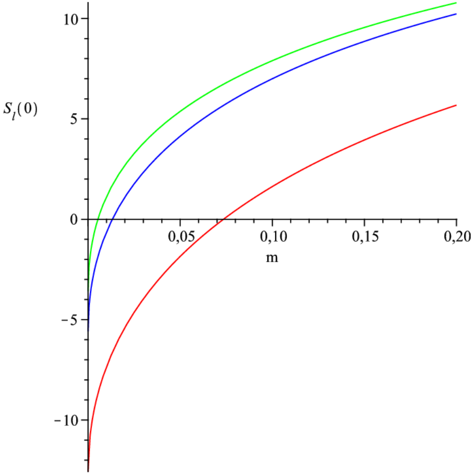

Plot of F_1^*, F_3^*, F_5^*

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$m_l^* < m < m_l^{**}$

In order to prove that T_l is not s.a. it is sufficient to prove that $\mathscr{D}(T_l) \subsetneq \mathscr{D}(T_l^*)$.

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In order to prove that T_l is not s.a. it is sufficient to prove that $\mathscr{D}(T_l) \subsetneq \mathscr{D}(T_l^*)$. Consider

$$f_\gamma(k) = \chi_{\{k>1\}} \frac{1}{k^{2-\gamma}} \qquad 0 < \gamma < \frac{1}{2}$$

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then

$$f_\gamma \notin \mathscr{D}(T_l) \qquad f_\gamma \in \mathscr{D}(T_l^*)$$

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then

$$f_\gamma \notin \mathscr{D}(T_l) \qquad f_\gamma \in \mathscr{D}(T_l^*)$$

Notice that $\gamma(m)$ is a monotone increasing function of m and (m_l^*, m_l^{**}) is mapped onto $(0, 1/2)$.

$m_l^* < m < m_l^{**}$

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Notice that $\gamma(m)$ is a monotone increasing function of m and (m_l^*, m_l^{**}) is mapped onto $(0, 1/2)$.
The picture is incomplete: at the moment we do not know the quadratic form of the new family of Hamiltonians.

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If we prove that $T_{o,l}$ is Kato-small w.r.t. T_d then T_l^λ is positive and essentially s.a.

$m > m_l^{**}$

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$$\| T_{o,l} f \| \leq \Gamma \| T_d f \|$$

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$$\Gamma = \frac{\left| \pi \int_{-1}^{1} dy \, P_l(y) \int dx \, \frac{e^{x/2}}{\cosh(x) + \frac{y}{m+1}} \right|}{2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}}}$$

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The condtion $\Gamma < 1$ translates into

$$F_l^{**}(m) = 2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}} + \pi \int_{-1}^{1} dy \, P_l(y) \int dx \, \frac{e^{x/2}}{\cosh(x) + \frac{y}{m+1}} > 0$$

which is equivalent to $m > m_l^{**}$

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We can summarize the situation in the following way:

Smallness properties

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We can summarize the situation in the following way:

Smallness Properties

- If the negative part of T_o is small compared to T_d in quadratic form sense then the system is stable
- If the negative part of T_o is small compared to T_d in Kato sense then the system is essentially s.a.

The same statement holds true in each subspace of fixed angular momentum

Numerical values of thresholds

Numerical values of the first thresholds

$$m_1{}^* = 0.0735 = (13.607)^{-1} \qquad m_1{}^{**} = 0.0812 = (12.31)^{-1}$$

$$m_3{}^* = 0.01316 = (75.99)^{-1} \qquad m_3{}^{**} = 0.013415 = (74.54)^{-1}$$

$$m_5{}^* = 0.00532 = (187.97)^{-1} \qquad m_5{}^{**} = 0.00536 = (186.57)^{-1}$$

Stability for N fermions

The previous results can be used also in the case of N fermions.

Stability for N fermions

The previous results can be used also in the case of N fermions.
New ingredient: the charge $\xi(\mathbf{k}_1, \ldots, \mathbf{k}_{N-1})$ is antisymmetric under exchange

$$\Phi_d^\lambda[\xi] = 2\pi^2 \int \sqrt{\frac{m(m+2)}{(m+1)^2} \sum_{i=1}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{i<j} \mathbf{k}_i \cdot \mathbf{k}_j + \lambda} \; |\xi(\mathbf{k}_1, \ldots, \mathbf{k}_{N-1})|^2 \, d\mathbf{k}_1$$

$$\Phi_o^\lambda[\xi] = (N-1) \int \frac{\overline{\xi(\mathbf{k}_0, \mathbf{k}_2, \ldots, \mathbf{k}_N)} \xi(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N)}{\frac{m(m+2)}{(m+1)^2} \sum_{i=0}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{i<j} \mathbf{k}_i \cdot \mathbf{k}_j + \lambda} \, d\mathbf{k}_0 \ldots d\mathbf{k}_{N-1}$$

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With some change of variables we can reduce to the previous case

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Define

$$\boldsymbol{\sigma} = \mathbf{k}_0 + \frac{1}{m+2}\sum_{i=2}^{N-1}\mathbf{k}_i \quad \boldsymbol{\tau} = \mathbf{k}_1 + \frac{1}{m+2}\sum_{i=2}^{N-1}\mathbf{k}_i$$

$$\widetilde{\xi}(\boldsymbol{\sigma}, \mathbf{K}) = \xi\left(\boldsymbol{\sigma} - \frac{1}{m+2}\sum_{i=2}^{N-1}\mathbf{k}_i, \mathbf{K}\right) \qquad \mathbf{K} = \mathbf{k}_2, \ldots, \mathbf{k}_{N-1}$$

$$D(\mathbf{K}) = \frac{m}{(m+1)(m+2)}\left((m+3)\sum_{i=2}^{N-1}k_i^2 + 2\sum_{i<j}\mathbf{k}_i\cdot\mathbf{k}_j\right)$$

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$$\Phi_d^\lambda[\xi] = 2\pi^2\int\sqrt{\frac{m(m+2)}{(m+1)^2}\sigma^2 + D(\mathbf{K}) + \lambda}\,\left|\widetilde{\xi}(\boldsymbol{\sigma}, \mathbf{K})\right|^2 d\boldsymbol{\sigma} d\mathbf{K}$$

$$\Phi_o^\lambda[\xi] = (N-1)\int\frac{\overline{\widetilde{\xi}(\boldsymbol{\sigma}, \mathbf{K})}\widetilde{\xi}(\boldsymbol{\tau}, \mathbf{K})}{\sigma^2 + \tau^2 + \frac{2}{m+1}\boldsymbol{\tau}\cdot\boldsymbol{\sigma} + D(\mathbf{K}) + \lambda}\, d\boldsymbol{\tau} d\boldsymbol{\sigma} d\mathbf{K}$$

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Define $m^*(N)$ as the solution of

$$2\pi^2\sqrt{\frac{m(m+2)}{(m+1)^2}} + (N-1)\pi\int_{-1}^{1} dy\, y \int dk\, \frac{1}{\cosh(k) + \frac{y}{m+1}} = 0$$

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Theorem

The quadratic form $\Phi^{\lambda,\alpha}$ is closed and bounded from below for $m > m^*(N)$ and it is unbounded from below for $m < m^*(2)$

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Final remarks and perspectives

- The partial wave analysis can be applied also to other variants of the three body problems: for instance three bosons are stable outside $l = 0$

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- We want to understand the new family of Hamiltonians for $m_l^* < m < m_l^{**}$

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- The partial wave analysis can be applied also to other variants of the three body problems: for instance three bosons are stable outside $l = 0$
- We want to understand the new family of Hamiltonians for $m_l^* < m < m_l^{**}$
- Construction of the 2+2 fermion model

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- The partial wave analysis can be applied also to other variants of the three body problems: for instance three bosons are stable outside $l = 0$
- We want to understand the new family of Hamiltonians for $m_l^* < m < m_l^{**}$
- Construction of the 2+2 fermion model
- Improvement of the analysis of N+1 model

Representation Theorems

Definition (Closed Form)

A quadratic form Q on an Hilbert space is said to be closed if for any $\{u_n\} \subset \mathscr{D}(Q)$ such that $u_n \to u$ and $Q[u_n - u_m] \to 0$ then $u \in \mathscr{D}(Q)$ and $Q[u_n - u] \to 0$

Theorem (First representation Theorem)

Let Q be closed and bdd from below then there is a unique s.a. and bdd from below operator T such that $\mathscr{D}(T) \subset \mathscr{D}(Q)$ and

$$Q[u, v] = (u, Tv) \qquad u \in \mathscr{D}(Q),\ v \in \mathscr{D}(T)$$

The domain $\mathscr{D}(T)$ are the vectors v such that $Q[\cdot, v]$ is continuous.

Theorem (Second representation Theorem)

Let Q be a positive and closed quadratic form and let T be the associated s.a. operator, then $\mathscr{D}(Q) = \mathscr{D}(\sqrt{T})$ and

$$Q[u, v] = (\sqrt{T}u, \sqrt{T}v) \qquad u, v \in \mathscr{D}(\sqrt{T})$$

Definite sign of $\Phi^{\lambda}_{o,l}$

Remember

$$P_l(y) = \frac{1}{2^l\, l!} \frac{d^l}{dy^l}(y^2 - 1)$$

$\Phi^{\lambda}_{o,l}[f] =$

$$\pi \int_0^\infty dk\, dk' \int_{-1}^{1} dy\, P_l(y)\, \frac{k^2\, k'^2}{k^2 + k'^2 + \frac{2y}{m+1} k\, k' + \lambda}\, \overline{f(k)} f(k')$$

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$$\int_0^{\infty} dk\, dk'\, \frac{k^{2+n}\overline{f(k)}\, k'^{2+n} f(k')}{(k^2+k'^2+\lambda)^{n+1}} =$$

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$$\int_0^{\infty}dk\, dk'\,\frac{k^{2+n}\overline{f(k)}\, k'^{2+n} f(k')}{(k^2+k'^2+\lambda)^{n+1}} = \int_0^{\infty}dk\, dk'\, k^{2+n}\overline{f(k)}\, k'^{2+n} f(k')\,\frac{1}{n!}\int_0^{\infty}\nu^n\, e^{-\nu(k^2+k'^2+\lambda)}$$

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$$\int_0^{\infty} dk\, dk' \frac{k^{2+n}\overline{f(k)}\, k'^{2+n} f(k')}{(k^2 + k'^2 + \lambda)^{n+1}} = \frac{1}{n!} \int_0^{\infty} d\nu\, \nu^n\, e^{-\nu\lambda} \left| \int_0^{\infty} dk\, k^{2+n} f(k) e^{-\nu k^2} \right|^2$$

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Diagonalization of Φ

Remember

$$f^\sharp(z) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{-ikz}\, e^{2k}\, f(e^k)$$

$$\Phi_d[f] = 2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}} \int_0^\infty |f(k)|^2\, k^3\, dk$$

$$\Phi_{o,l}[f] = 2\pi \int_0^\infty dk\, dk' \int_{-1}^1 dy\, P_l(y)\, \frac{k^2\, k'^2}{k^2 + k'^2 + \frac{2y}{m+1} k\, k'}\, \overline{f(k)} f(k')$$

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$$\Phi_{o,l}[f] = 2\pi \int_0^\infty dx\, dx' \int_{-1}^{1} dy\, P_l(y)\, \frac{e^{x+x'}}{e^{2x} + e^{2x'} + \frac{2y}{m+1} e^{x+x'}}\, \overline{f(e^x)} e^{2x}\, f(e^{x'}) e^{2x'}$$

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$$f^\sharp(z) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{-ikz}\, e^{2k}\, f(e^k)$$

$$\Phi_d[f] = 2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}} \int_{-\infty}^{\infty} |f^\sharp(z)|^2\, dz$$

$$\Phi_{o,l}[f] = \pi \int_0^\infty dx\, dx' \int_{-1}^{1} dy\, P_l(y)\, \frac{1}{\cosh(x-x') + \frac{2y}{m+1}}\, \overline{f(e^x)} e^{2x}\, f(e^{x'}) e^{2x'}$$

Diagonalization of Φ

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Remember

$$f^\sharp(z) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{-ikz}\, e^{2k}\, f(e^k)$$

$$\Phi_d[f] = 2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}} \int_{-\infty}^{\infty} |f^\sharp(z)|^2\, dz$$

$$\Phi_{o,l}[f] = \int_{-\infty}^{\infty} dz\, S_l(z) |f^\sharp(z)|^2$$

$$S_l(z) = \pi \int_{-1}^{1} dy\, P_l(y) \int dx\, e^{-ikz} \frac{1}{\cosh(x) + \frac{2y}{m+1}}$$

Monotonicity of $S_l(z)$

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For odd l

$$S_l(z) = 2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}} + \pi \int_{-1}^{1} dy\, P_l(y) \int dx\, e^{-ikz} \frac{1}{\cosh(x) + \frac{2y}{m+1}}$$

Monotonicity of $S_l(z)$

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Proofs

For odd l

$$
\begin{aligned}
S_l(z) =& 2\pi^2 \sqrt{1 - \frac{1}{(m+1)^2}} \\
& + \pi \sum_{j=0}^{\infty} \left(-\frac{2}{m+1} \right)^j \int_{-1}^{1} dy\, P_l(y)\, y^n \int dx\, e^{-ikz} \frac{1}{\cosh^{j+1}(x)}
\end{aligned}
$$

This representation allows to derive all the monotonicity properties of F_l^* and F_l^{**}.

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For odd l

$$S_l(z) = 2\pi^2\sqrt{1-\frac{1}{(m+1)^2}} - \frac{1}{2^l}\sum_{k=0}\frac{1}{(m+1)^{l+2k}}\binom{l+2k}{2k}\int_{-1}^{1}(1-y^2)^l\,y^{2k}\int\frac{e^{-izx}}{(\cosh(x))^{l+1+2k}}$$

This representation allows to derive all the monotonicity properties of F_l^* and F_l^{**}.
Notice that

$$\int\frac{e^{-izx}}{(\cosh(x))^2} > 0 \Longrightarrow \int\frac{e^{-izx}}{(\cosh(x))^{l+1+2k}} > 0$$

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We can estimate Γ^2 by the the norm of $\mathcal{O} : L^2(\mathbb{R}^+, dk') \to L^2(\mathbb{R}^+, dk'')$

$$\mathcal{O}(k', k'') = \left(2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}}\right)^{-1} 4\pi^2 \int_{-1}^{1} dy'\, P_l(y') \int_{-1}^{1} dy''\, P_l(y'') \int_0^\infty dk \frac{k^2}{(k^2 + k'^2 + \frac{2y}{m+1} k\, k')(k^2 + k''^2 + \frac{2y}{m+1} k\, k'')} \tag{1}$$

Kato smallness

We can estimate Γ^2 by the the norm of $\mathcal{O} : L^2(\mathbb{R}^+, dk') \to L^2(\mathbb{R}^+, dk'')$

$$\mathcal{O}(k', k'') = \left(2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}}\right)^{-1} 4\pi^2 \int_{-1}^{1} dy'\, P_l(y') \int_{-1}^{1} dy''\, P_l(y'') \int_0^\infty dk \frac{k^2}{(k^2 + k'^2 + \frac{2y}{m+1} k\, k')(k^2 + k''^2 + \frac{2y}{m+1} k\, k'')} \quad (1)$$

Generalized Schur's test with $1/\sqrt{k}$ as test function. Notice the pointwise positivity of the kernel $\mathcal{O}$.