Spectral properties of Schrödinger operators with singular interactions on Lipschitz surfaces

Jussi Behrndt (TU Graz)

with Pavel Exner and Vladimir Lotoreichik

PART I

δ and δ'-interactions on one smooth compact hypersurface

δ-hypersurface interactions in $\mathbb{R}^n$

Give meaning to $-\Delta - \alpha\delta_{\mathcal{C}}$ with $\mathcal{C}$ hypersurface, $\alpha \in L^\infty(\mathcal{C})$ real

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- ac-parts of $H_{\delta,\alpha}$ and $H_{\delta,0}$ unitarily equivalent

Other points of view on the Hamiltonian $H_{\delta,\alpha}$

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$H_{\delta,\alpha}$ corresponds to closed symmetric form on $H^1(\mathbb{R}^n)$:

$$\mathfrak{a}_\delta[\psi, \phi] := (\nabla\psi, \nabla\phi)_{L^2(\mathbb{R}^n)^n} - (\alpha\psi, \phi)_{L^2(\mathcal{C})}.$$

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$H_{\delta,\alpha}$ norm resolvent limit of $H_\varepsilon = -\Delta - V_\varepsilon$, where supp $V_\varepsilon \to \mathcal{C}$,

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Remark

Assumption $\alpha \in L^\infty(\mathcal{C})$ allows to study non-closed surfaces

Literature

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Some references

- [BrascheExnerKuperinSeba] JMAA 184 (1994), 112-139
- Exner & Fraas, Ichinose, Kondej, Němcová, Yoshitomi
- [AntoineGesztesyShabani'87][Herczyński'89][Shabani'88]
- [Teta'90][BrascheTeta'92][BrascheFigariTeta'98]
- [AlbeverioNizhnik'00][BirmanSuslinaShterenberg'00]
- [Posilicano'01][DerkachHassiSnoo'03][KondejVeselić'07]

and many, many more ....

More recent related work

- [CorreggiDell'AntonioFincoMichelangeliTeta'12]
- [AlbeverioKostenkoMalamudNeidhart'13][ExnerJex'13]
- [DucheneRaymond'14][ExnerPankrashkin'14]

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δ'-hypersurface interactions in $\mathbb{R}^n$

Give meaning to $-\Delta + \beta\delta'_{\mathcal{C}}$ with $\mathcal{C}$ hypersurface, $\beta^{-1} \in L^\infty(\mathcal{C})$

Definition

Decompose $\mathbb{R}^n = \Omega_i \,\dot\cup\, \mathcal{C} \,\dot\cup\, \Omega_e$ in interior and exterior domain

$$H_{\delta',\beta} = -\Delta$$
$$\operatorname{dom} H_{\delta',\beta} = \left\{ \psi \in H^{3/2}_\Delta(\mathbb{R}^n \setminus \mathcal{C}) : \begin{matrix} \partial_{n_i}\psi_i|_\mathcal{C} = -\partial_{n_e}\psi_e|_\mathcal{C} \\ \beta\partial_{n_e}\psi_e|_\mathcal{C} = \psi_e|_\mathcal{C} - \psi_i|_\mathcal{C} \end{matrix} \right\}$$

Theorem [B. Langer Lotoreichik '13]

- $H_{\delta',\beta}$ semibounded selfadjoint operator in $L^2(\mathbb{R}^n)$
- $H_{\delta',0}$ unperturbed Laplacian; $\sigma(H_{\delta',0}) = \sigma_{\mathrm{ess}}(H_{\delta',0}) = [0,\infty)$
- $(H_{\delta',\beta} - \lambda)^{-k} - (H_{\delta',0} - \lambda)^{-k} \in \mathfrak{S}_p$ for all $p > \frac{n-1}{2k}$
- Wave operators for $\{H_{\delta',\beta}, H_{\delta',0}\}$ exist and are complete
- ac-parts of $H_{\delta',\beta}$ and $H_{\delta',0}$ unitarily equivalent

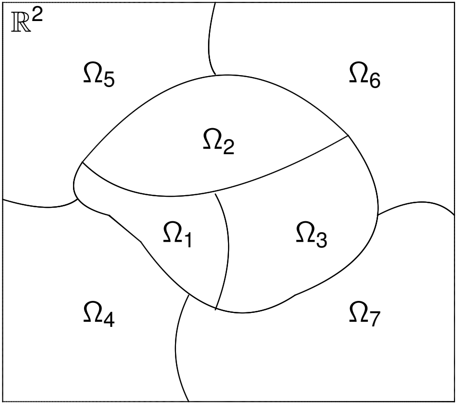
PART II

δ and δ'-interactions on Lipschitz partitions

$H_{\delta,\alpha}$, $H_{\delta',\beta}$ on Lipschitz partitions $\mathcal{P} = \{\Omega_k\}_{k=1}^n$ of $\mathbb{R}^d$

Support of δ: Boundary $\Sigma := \cup_{k=1}^n \partial\Omega_k$ of Lipschitz partition $\mathcal{P}$

Ω_k Lipschitz domains, $\quad \mathbb{R}^d = \bigcup_{k=1}^n \overline{\Omega}_k, \quad \Omega_k \cap \Omega_l = \varnothing.$



Chromatic number of a Lipschitz partition $\mathcal{P} = \{\Omega_k\}_{k=1}^n$

χ = minimal number of colours needed to colour all Ω_k such that any two neighbouring domains have different colours

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The chromatic number of any Lipschitz partition $\mathcal{P}$ of $\mathbb{R}^2$ is ≤ 4.

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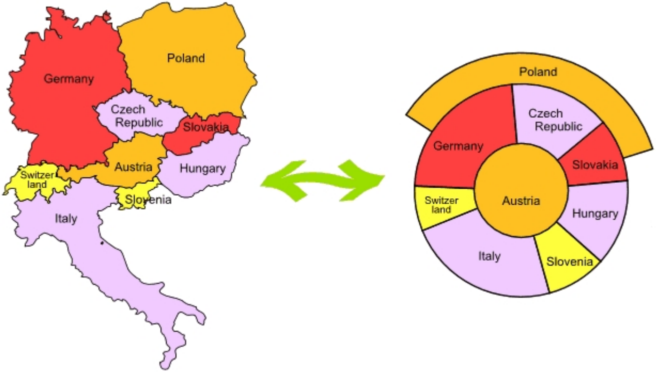
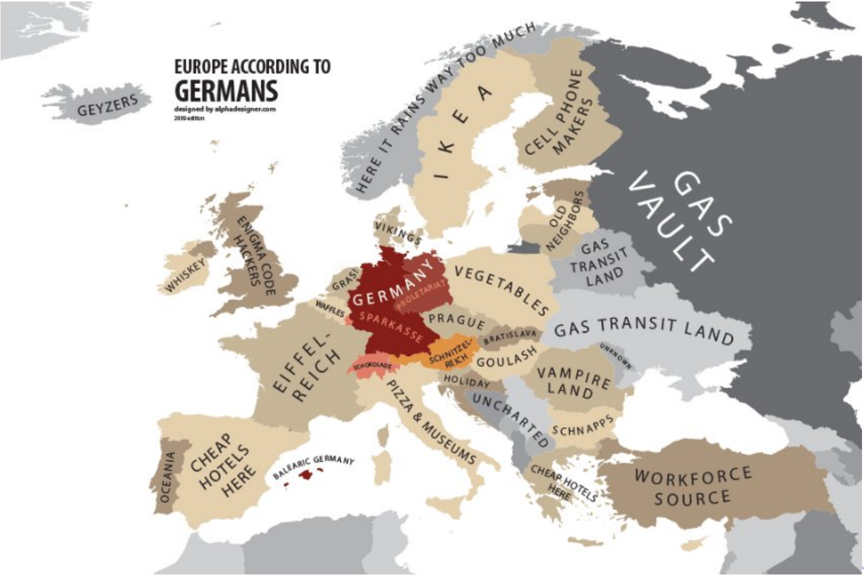


More examples: A german colouring



An operator inequality for $H_{\delta,\alpha}$ and $H_{\delta',\beta}$

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For $\alpha, \beta > 0$ recall $\sigma_p(H_{\delta,\alpha}) = \{-\frac{\alpha^2}{4}\}$ and $\sigma_p(H_{\delta',\beta}) = \{-\frac{4}{\beta^2}\}$

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- If $\min \sigma_{\rm ess}(H_{\delta,\alpha}) = \min \sigma_{\rm ess}(H_{\delta',\beta})$ then $N(H_{\delta,\alpha}) \le N(H_{\delta',\beta})$

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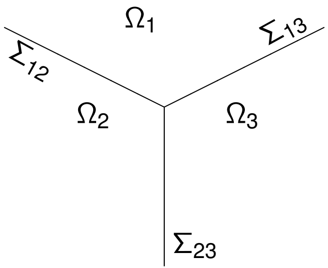
and there is no unitary operator such that $U^{-1}(H_{\delta',\beta})U \leq H_{\delta,\alpha}$.

Example 2: Symmetric star graph with 3 leads in $\mathbb{R}^2$



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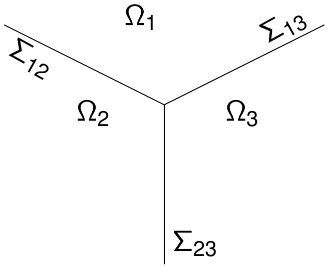

- $\min \sigma(H_{\delta,\alpha}) = -\frac{\alpha^2}{4}$

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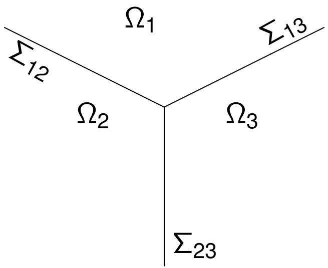

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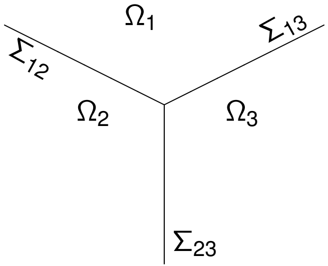

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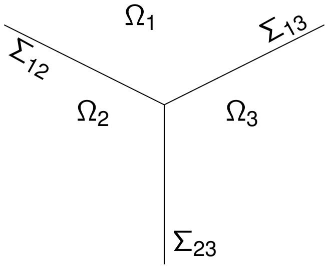

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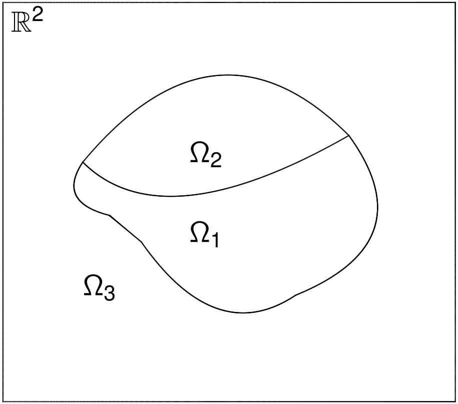
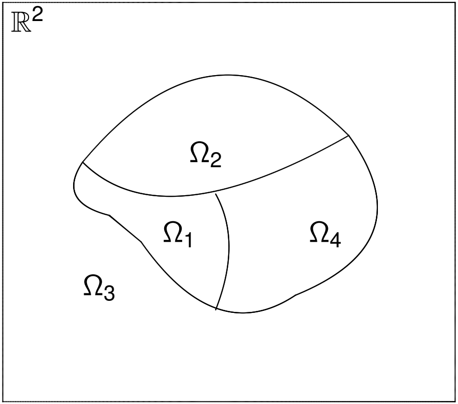
Corollary *'Chromatic number needed'*

If $\chi = 3$ the assumption $0 < \beta \leq \frac{3}{\alpha}$ can NOT be replaced by the weaker assumption $0 < \beta \leq \frac{4}{\alpha}$ (which corresponds to $\chi = 2$)

Example 3: Compact Lipschitz partitions



$$\mathcal{P} = \{\Omega_k\}_{k=1}^3, \quad \chi = 3$$



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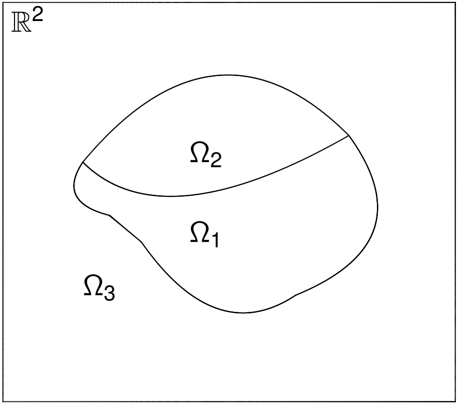

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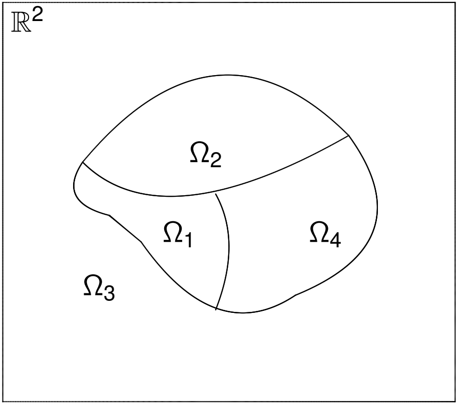
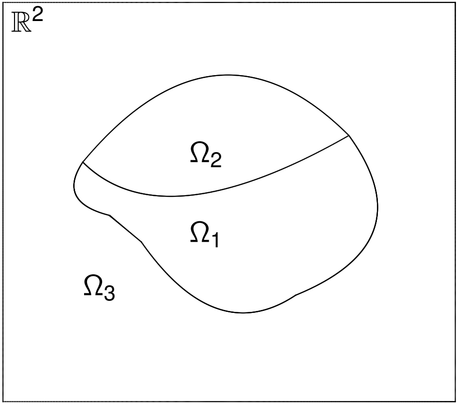


$$\mathcal{P} = \{\Omega_k\}_{k=1}^{4}, \quad \chi = 4$$

Theorem

$$\sigma_{\text{ess}}(H_{\delta,\alpha}) = \sigma_{\text{ess}}(H_{\delta',\beta}) = [0,\infty), \qquad \alpha, \beta^{-1} \in L^\infty(\Sigma, \mathbb{R})$$

Example 3: Compact Lipschitz partitions - $\sigma_p(H_{\delta',\beta})$



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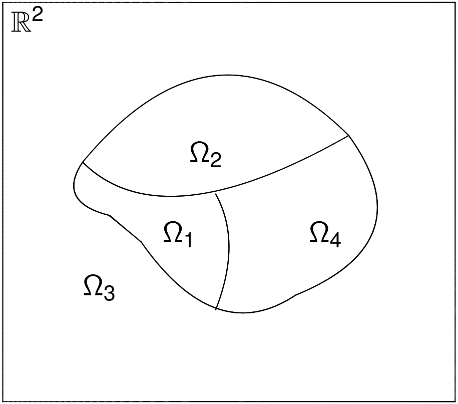

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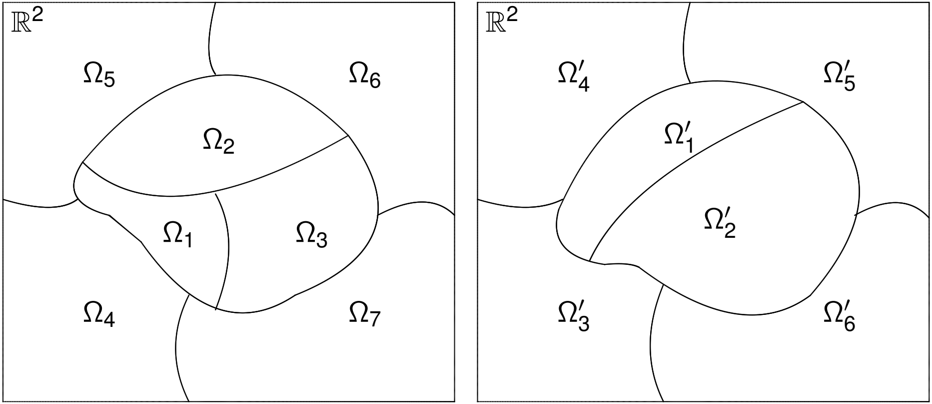
Theorem *'A special δ'-type spectral effect'*

$$\beta > 0 \text{ on some } \partial\Omega_k \quad \Longrightarrow \quad N(H_{\delta',\beta}) \geq 1$$

Example 4: Locally deformed Lipschitz partitions



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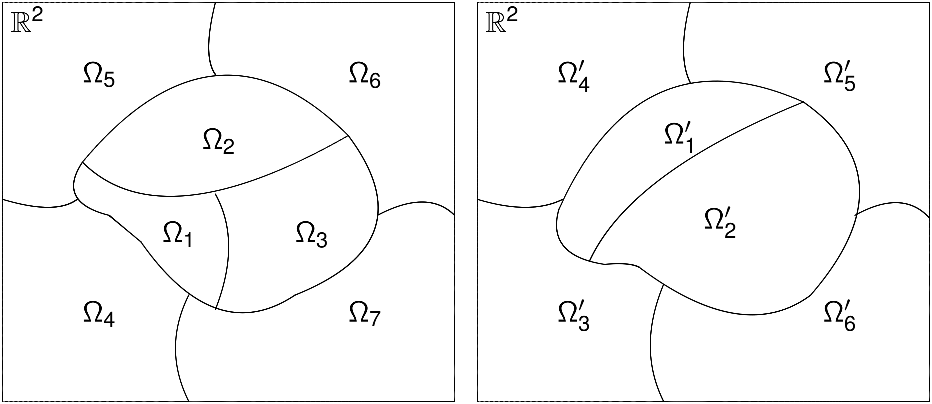

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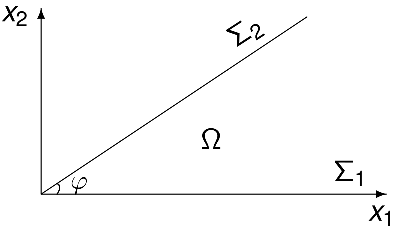
Theorem. Assume $\alpha = \alpha'$ and $\beta = \beta'$ outside compact set.

$$\sigma_{\rm ess}(H_{\delta,\alpha}) = \sigma_{\rm ess}(H'_{\delta,\alpha'}) \qquad \sigma_{\rm ess}(H_{\delta',\beta}) = \sigma_{\rm ess}(H'_{\delta',\beta'})$$

Example 5: Local deformations of a wedge Ω in $\mathbb{R}^2$



- $\alpha, \beta > 0$ constant
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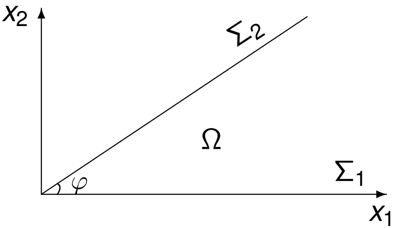

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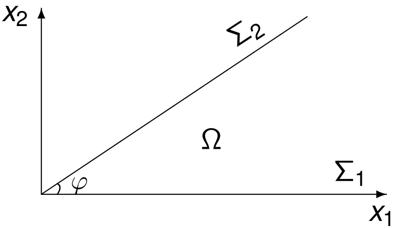

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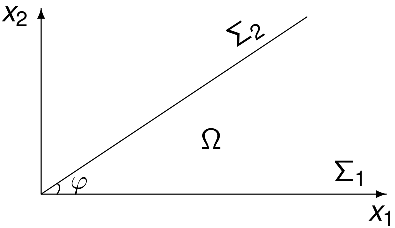

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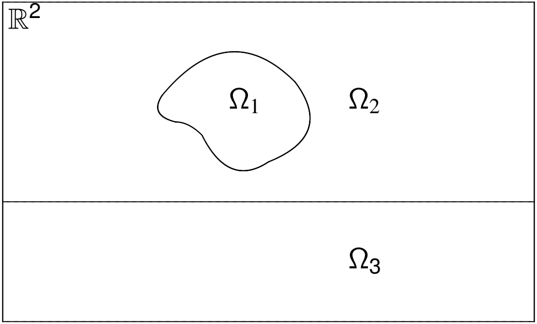
- $\lambda_k(H_{\delta',\beta}) \leq \lambda_k(H_{\delta,\alpha})$ for all $k \in \mathbb{N}$
- $N(H_{\delta,\alpha}) \leq N(H_{\delta',\beta})$

Example 6: Bound states appear



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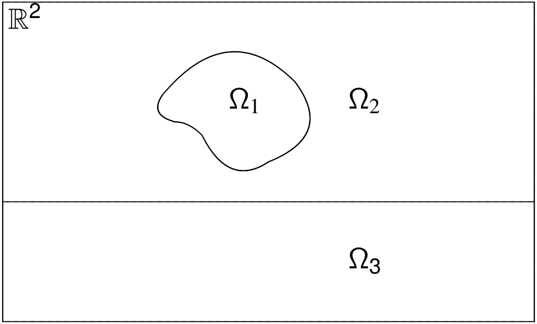

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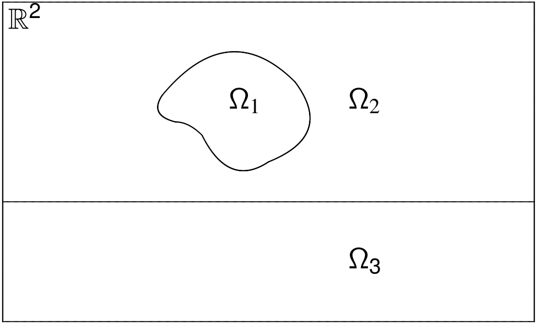

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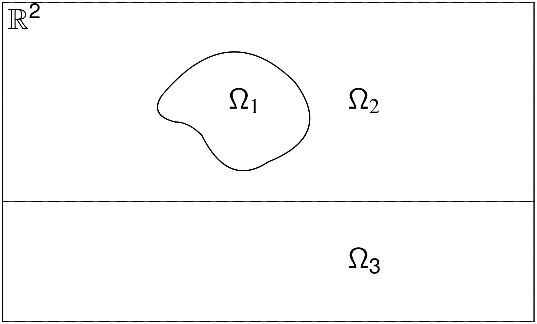

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Theorem '*$H_{\delta,\alpha}$ and $H_{\delta',\beta}$ have at least one eigenvalue*'

$$N(H_{\delta,\alpha}) > 1 \qquad \text{and} \qquad N(H_{\delta',\beta}) > 1$$

Example 7: Recent results for cones in $\mathbb{R}^3$

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...discuss if time allows and audience is still awake

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...Stop now finally, it was too much material anyway !

Thank you for your attention

References

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