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Lineare Algebra II – Tutoriumsblatt 2

Aufgabe 1.

1. Let p be a prime number, $n \in \mathbb{Z}$ such that $p \nmid n$. Prove that the natural map

$$\mathbb{Z}/(pn)\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

sending a class $a + (pn)\mathbb{Z}$ to the pair $(a + p\mathbb{Z}, a + n\mathbb{Z})$, is in fact an isomorphism.

Hint. Compute the number of elements in both rings.

2. (Chinese remainder theorem) Let $p_1, \dots, p_m > 0$ be pairwise distinct prime numbers. Prove that

$$\mathbb{Z}/(p_1 \cdot \dots \cdot p_m)\mathbb{Z} \cong \mathbb{Z}/p_1\mathbb{Z} \times \dots \times \mathbb{Z}/p_m\mathbb{Z}.$$

Aufgabe 2.

1. For $x \in \mathbb{Z}$ assume that

$$\begin{cases} x \equiv 3 \pmod{5}, \\ x \equiv 2 \pmod{7}. \end{cases}$$

Find all possible x .

2. (5th-century book Sunzi Suanjing) There are certain things whose number is unknown. If we count them by threes, we have two left over; by fives, we have three left over; and by sevens, two are left over. How many things are there?

Aufgabe 3.

Let K be a field and $P(X) \in K[X]$ a polynomial of degree n . Recall that $(P(X))$ denotes the ideal generated by $P(X)$:

$$(P(X)) = P(X)K[X] = \{P(X) \cdot F(X) \mid F(X) \in K[X]\},$$

and the quotient group $K[X]/(P(X))$ has an induced structure of a ring.

1. Prove that $K[X]/(P(X))$ is a K -vector space with the basis $\bar{1}, \dots, \bar{X}^{n-1}$.
2. Prove that for $P(X) = X - c$, $c \in K$, the ring $K[X]/(P(X))$ is isomorphic to K .
3. Prove that for $K = \mathbb{R}$, $P(X) = X^2 + 1$, the ring $\mathbb{R}[X]/(X^2 + 1)$ is isomorphic to $\mathbb{C}$.

Aufgabe 4.

1. Let K be a field and $P_1(X), \dots, P_m(X) \in K[X]$ be pairwise co-prime polynomials. Prove that

$$K[X]/(P_1(X) \cdot \dots \cdot P_m(X)) \cong K[X]/(P_1(X)) \times \dots \times K[X]/(P_m(X)).$$

2. Prove that $\mathbb{R}[X]/(X^2 - 1) \cong \mathbb{R} \times \mathbb{R}$.