Atomicity, coherence of information, and point-free structures

Basil A. Karádais

Mathematisches Institut, Ludwig-Maximilians-Universität
Theresienstraße 39, 80333 München, Germany
karadais@math.lmu.de *

Abstract

We prove basic facts about the properties of atomicity and coherence for Scott information systems, and we establish direct connections between coherent information systems and well-known point-free structures.

1 Introduction

Domain theory has been a well-established branch of mathematics for several years now, one that exhibits a wide array of applications [1, 5]. In particular it bears great significance regarding the denotational semantics of programming languages, which was historically one of the reasons that the theory emerged in the first place.

In [24], Dana Scott represents *Scott domains*, that is, pointed complete partial orders (cpo's) which are additionally consistently complete and algebraic, by *information systems*. These are supposed to structure atomic tokens of information according to their *consistency* and *entailment*, where entailment models deduction of information and preorders the carrier, so that the actual objects of the domain are then recovered as ideals. Scott's information systems have served as a natural approach to the domain-theoretic treatment of semantics, at least from the computer scientist's viewpoint, since they provide the means to discuss higher-type algorithms in a tangible way, namely in terms of their *finite approximations*: their tokens and the finite consistent sets of tokens that they consist of. In the context of information systems, the principle of finite support for computation finds one of its uttermost formulations: an algorithm is but a consistent and deductively closed collection of concrete, finite pieces of information. But there are also theoretical merits, as information systems are more basic than the domains they induce. In particular, properties of ideals reduce to properties of tokens and consistent sets, thus providing the possibility of elementary methods of argumentation. A prime example of this is the solution of domain equations up to identity [24, 29, 28], rather than up to isomorphism, as previous arguments could already show [25].

In our case, favoring the tangible nature of information systems is tied to the development of a constructive formal theory of partial computable functionals, one that should lend itself as naturally and intuitively as possible to an implementation in a proof assistant [7]. This objective motivates an in-depth study of

*Current address: Dipartimento di Matematica, Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy. Email: basilio@math.unipd.it

information systems in their own sake, and leads to a bottom-up, constructive, and implementable redevelopment of domain theory for higher-type computability.

Helmut Schwichtenberg [22] started this redevelopment by employing a cartesian-closed class of information systems which feature *coherence* as well as *atomicity*, which technically reduce consistency and entailment, respectively, to binary predicates. Both of these properties, in one version or another, have proved crucial for various studies in denotational semantics. Already in the formative period of domain semantics, coherence came to the attention of Gordon Plotkin [16], when he was arguing for using cpo's instead of lattices, and noticed that it is a quite omnipresent property in the usual domains of study; later, it became one of the key features of the standard model of Jean-Yves Girard's *linear logic* [6]. As for atomicity (also known as "linearity"), it is notably needed for the representation of stable domains [30], which are important in the study of the notorious notion of *sequentiality* [2], but also appears, again, in particular models of linear logic [4].

In this work we retain a top-down approach and present results [9] which relate, in a direct way, some of the point-free structures that have been put to successful use in the past by the community, to *coherent* information systems. The work has a rather cartographic flavor which we deem necessary in order to clearly understand the nature of information systems we have used in practice from a point-free viewpoint.

We begin in section 2 by recalling basic facts and observations concerning information systems. In section 3 we define the notions of atomicity and coherence and we show that atomic and coherent versions of information systems feature more ideals than the generic version. In section 4 we concentrate on point-free versions of coherence (atomicity, of apparently limited use in comparison to coherence, would be a subject for another text): we consider well-known point-free structures, namely domains, precusl's, and formal topologies, and impose appropriate coherence properties on them to show that they correspond to coherent information systems. Such correspondences naturally imply certain categorical equivalences, which we make explicit in the case of formal topologies, the less covered case of the three in the literature. In section 5 we gather some relevant notes and an outlook on future work.

2 Scott information systems

A *(Scott) information system* is a triple $\rho = (\mathsf{Tok}_\rho, \mathsf{Con}_\rho, \vdash_\rho)$, where Tok_ρ is a countable set of *tokens*, $\mathsf{Con}_\rho \subseteq \mathscr{P}_f(\mathsf{Tok}_\rho)$ is a collection of *consistent sets*, also called *(formal) neighborhoods* and $\vdash_\rho \subseteq \mathsf{Con}_\rho \times \mathsf{Tok}_\rho$ is an *entailment* relation, such that: consistency is reflexive and closed under subsets; entailment is reflexive and transitive; consistency propagates through entailment. Formally we have[1]

$$
\begin{aligned}
&\{a\} \in \mathsf{Con}, \\
&U \in \mathsf{Con} \wedge V \subseteq U \to V \in \mathsf{Con}, \\
&a \in U \to U \vdash a, \\
&U \vdash V \wedge V \vdash c \to U \vdash c, \\
&U \in \mathsf{Con} \wedge U \vdash b \to U \cup \{b\} \in \mathsf{Con},
\end{aligned}
$$

where $U \vdash V$ is a shorthand for $\forall_{b \in V} U \vdash b$. Among the properties that follow directly from the definition we have

$$U \vdash V \wedge U \vdash V' \to U \vdash V \cup V'. \tag{1}$$

[1]In general, we may drop the subscripts when we can afford it.

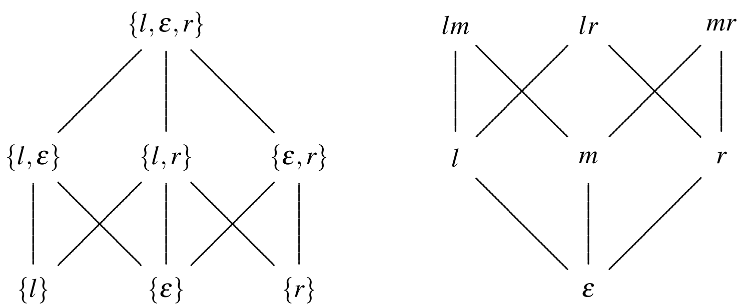

Figure 1: Entailments in $\mathsf{Con}_{\mathscr{C}}\backslash\varnothing$ and $\mathsf{Tok}_{\mathscr{L}}$.

An *ideal* in ρ is a set $u \subseteq \mathsf{Tok}$ which is *consistent* and *closed under entailment*, in the sense that

$$U \in \mathsf{Con} \wedge (\, U \vdash a \to a \in u)$$

for all $U \subseteq^f u$ and $a \in \mathsf{Tok}$. Denote the empty ideal by $\perp$ and the collection of all ideals of ρ by Ide_ρ. Define the *deductive closure* of a neighborhood $U \in \mathsf{Con}_\rho$ by

$$\mathrm{cl}_\rho(U) := \left\{a \in \mathsf{Tok} \mid U \vdash_\rho a\right\}.$$

When ρ is clear from the context we just write $\overline{U}$, and we also write $\overline{\mathsf{Con}_\rho}$ for the collection of all such closures. It is clear that $\overline{U} \in \mathsf{Ide}_\rho$ for any $U \in \mathsf{Con}_\rho$. Write $U \sim_\rho V$ if both $U \vdash_\rho V$ and $V \vdash_\rho U$.

Lemma 1. *It is $U \sim V$ if and only if $\overline{U} = \overline{V}$.*

Proof. The right direction follows from transitivity and the left one from reflexivity and transitivity of entailment. ☐

Examples. Consider the strings over the alphabet $\{l,r,m\}$, the empty one denoted by ε. Define $\mathscr{C}$ by $\mathsf{Tok}_{\mathscr{C}} := \{\varepsilon,l,r\}$, $\mathsf{Con}_{\mathscr{C}} := \mathscr{P}_f(\mathsf{Tok}_{\mathscr{C}})$, and

$$\begin{array}{ccc} \{l\} \vdash_{\mathscr{C}} l, & \{r\} \vdash_{\mathscr{C}} r, & \{\varepsilon\} \vdash_{\mathscr{C}} \varepsilon, \\ \{l,r\} \vdash_{\mathscr{C}} l,r,\varepsilon, & \{r,\varepsilon\} \vdash_{\mathscr{C}} r,\varepsilon, & \{l,\varepsilon\} \vdash_{\mathscr{C}} l,\varepsilon, \\ & \{l,r,\varepsilon\} \vdash_{\mathscr{C}} l,r,\varepsilon. & \end{array}$$

Once we see that $\{l,r\} \vdash_{\mathscr{C}} \varepsilon$ is the only nontrivial entailment at hand, it is direct to also see that $\mathsf{Ide}_{\mathscr{C}} = \mathscr{P}_f(\mathsf{Tok}_{\mathscr{C}})\backslash\{l,r\}$.

Further define $\mathscr{L}$ by $\mathsf{Tok}_{\mathscr{L}} := \{\varepsilon,l,m,r,lm,lr,mr\}$, $U \in \mathsf{Con}_{\mathscr{L}}$ when there is a token $a \in \mathsf{Tok}_{\mathscr{L}}$ which is a superword of all $b \in U$, and $U \vdash_{\mathscr{L}} b$ when there exists an $a \in U$ which is a superword of b. Its ideals, $\mathsf{Ide}_{\mathscr{L}}$, are

$$\begin{array}{l} \perp,\ \{\varepsilon\}, \\ \{l,\varepsilon\},\ \{m,\varepsilon\},\ \{r,\varepsilon\}, \\ \{l,m,\varepsilon\},\ \{l,r,\varepsilon\},\ \{m,r,\varepsilon\}, \\ \{lm,l,m,\varepsilon\},\ \{lr,l,r,\varepsilon\},\ \{mr,m,r,\varepsilon\}. \end{array}$$

We will use these finite systems in section 3 to illustrate the notions of atomicity and coherence. ☐

Let ρ and σ be information systems. Define their *function space* $\rho \to \sigma$ by the following clauses.

- If $U \in \mathsf{Con}_\rho$ and $b \in \mathsf{Tok}_\sigma$ then $(U,b) \in \mathsf{Tok}_{\rho\to\sigma}$.
- Let $U_1,\dots,U_l \in \mathsf{Con}_\rho$, $b_1,\dots,b_l \in \mathsf{Tok}_\sigma$, and $J := \{1,\dots,l\}$; if for all $I \subseteq J$, $\bigcup_{i\in I} U_i \in \mathsf{Con}_\rho$ implies $\bigcup_{i\in I}\{b_i\} \in \mathsf{Con}_\sigma$, then $\{(U_j,b_j) \mid j \in J\} \in \mathsf{Con}_{\rho\to\sigma}$.
- Let $U_1,\dots,U_l,U \in \mathsf{Con}_\rho$, $b_1,\dots,b_l,b \in \mathsf{Tok}_\sigma$, and $J := \{1,\dots,l\}$; if for some $I \subseteq J$, it is both $U \vdash_\rho U_i$ for all $i \in I$ and $\{b_i \mid i \in I\} \vdash_\sigma b$, then $\{(U_j,b_j) \mid j \in J\} \vdash_{\rho\to\sigma} (U,b)$.

The definition of entailment can be formulated in terms of *application* between neighborhoods: $\{(U_1,b_1),\dots,(U_l,b_l)\}U := \{b_i \mid U \vdash_\rho U_i, i \in J\}$; so

- $\{(U_1,b_1),\dots,(U_l,b_l)\}U \vdash_\sigma b$ implies $\{(U_1,b_1),\dots,(U_l,b_l)\} \vdash_{\rho\to\sigma} (U,b)$.

These make $\rho \to \sigma$ an information system.

A relation $r \subseteq \mathsf{Con}_\rho \times \mathsf{Tok}_\sigma$ is called an *approximable map* from ρ to σ if it is consistently defined and deductively closed:

$$\forall_{b\in V}(U,b) \in r \to V \in \mathsf{Con}_\sigma,$$
$$U' \vdash_\rho U \wedge \forall_{b\in V}(U,b) \in r \wedge V \vdash_\sigma b' \to (U',b') \in r.$$

Write $\mathsf{Apx}_{\rho\to\sigma}$ for these relations. Approximable maps provide an alternative description of ideals in a function space; in particular, it is $\mathsf{Apx}_{\rho\to\sigma} = \mathsf{Ide}_{\rho\to\sigma}$.

As already remarked by Scott in [24, §5] (where actually the converse route was taken), any approximable map r from ρ to σ induces a relation $\hat{r} \subseteq \mathsf{Con}_\rho \times \mathsf{Con}_\sigma$ by letting $(U,V) \in \hat{r}$ if and only if $(U,b) \in r$ for all $b \in V$.

Fact 2. *Let r be an approximable map from ρ to σ. For the relation $\hat{r}$ it is*

$$(\varnothing,\varnothing) \in \hat{r},$$
$$(U,V) \in \hat{r} \wedge (U,V') \in \hat{r} \to (U, V \cup V') \in \hat{r},$$
$$U' \vdash_\rho U \wedge (U,V) \in \hat{r} \wedge V \vdash_\sigma V' \to (U',V') \in \hat{r}.$$

Conversely, if $R \subseteq \mathsf{Con}_\rho \times \mathsf{Con}_\sigma$ satisfies the above, then the relation $\check{R} \subseteq \mathsf{Con}_\rho \times \mathsf{Tok}_\sigma$, defined by

$$(U,b) \in \check{R} := (U,\{b\}) \in R,$$

is an approximable map from ρ to σ.

In what follows we will identify r with $\hat{r}$ and R with $\check{R}$.

3 Atomicity and coherence of information

Let $\rho = (\mathsf{Tok},\mathsf{Con},\vdash)$ be an arbitrary information system. Define its *atomic entailment* by $U \vdash^A_\rho b$ when $\{a\} \vdash_\rho b$ for some $a \in U$; define its *coherent neighborhoods* by $U \in H\mathsf{Con}_\rho$ when $\{a,a'\} \in \mathsf{Con}_\rho$ for all $a,a' \in U$. Further, for each coherent neighborhood U and token b define the *coherent entailment* $U \vdash^H_\rho b$ by $U \vdash_\rho b$ or $b \in U$. Write $A\rho$ and $H\rho$ for the triples $(\mathsf{Tok}_\rho,\mathsf{Con}_\rho,\vdash^A_\rho)$ and $(\mathsf{Tok}_\rho,H\mathsf{Con}_\rho,\vdash^H_\rho)$ respectively. If $A\rho = \rho$, call ρ *atomic*, and if $H\rho = \rho$, call it *coherent*; the system $\mathscr{L}$ in Figure 1 is atomic but incoherent, since $\{l,m\},\{l,r\},\{m,r\} \in \mathsf{Con}_{\mathscr{L}}$ but $\{l,m,r\} \notin \mathsf{Con}_{\mathscr{L}}$, whereas $\mathscr{C}$ is coherent but nonatomic because of $\{l,r\} \vdash_{\mathscr{C}} \varepsilon$.[2]

[2] The information system $\mathscr{C}$ is based on a remark by Thierry Coquand at the *Mathematics, Algorithms, and Proofs* summer school in Genoa, August 2006, against the choice of atomicity for information systems which are induced by algebras given by superunary constructors. The system $\mathscr{L}$ stems from [16] by Gordon Plotkin, where he uses the entailment graph of $\mathscr{L}$ as an example of a "consistently complete" but not "coherent" complete partial order.

As we mentioned in section 1, these two notions have each proven important for program semantics in several relevant works. In conjunction, they were used in [22], where various fundamental results were established, like density, preservation of values, and adequacy, while in [7] definability was established as well[3]. For the general setting of [23, Chapter 6], atomic-coherent systems turn out to be particularly appropriate when modeling type systems over data types like the natural or the boolean numbers, whose tokens are built by at most unary constructors [9, §1.4], while in general it would seem more reasonable to work within coherent systems.

Back on an abstract level, there are some direct observations to make, like that it is in general $\vdash_\rho^A \subseteq \vdash_\rho$ and $\mathsf{Con}_\rho \subseteq H\mathsf{Con}_\rho$, or that for any $U \in \mathsf{Con}_\rho$ it is $\mathrm{cl}_{A\rho}(U) \subseteq \mathrm{cl}_\rho(U)$ as well as $\mathrm{cl}_{H\rho}(U) = \mathrm{cl}_\rho(U)$. The important and most basic observation is the following.

Proposition 3. *Let ρ be an information system. The triples $A\rho$ and $H\rho$ are both information systems. Furthermore, let σ be an information system; if σ is atomic then $\rho \to \sigma$ is also atomic; if σ is coherent, then $\rho \to \sigma$ is also coherent.*

Proof. In order to show that $A\rho$ is an information system we have to check the laws of the definition concerning entailment. For reflexivity, if $U \in \mathsf{Con}$ then $\{a\} \vdash a$ for all $a \in U$, so $U \vdash^A a$ for all $a \in U$. For transitivity, let $U \vdash^A V \wedge V \vdash^A c$; by the atomicity we get an $a_b \in U$ for each $b \in V$, such that $\{a_b\} \vdash b$, as well as a $b_c \in V$ such that $\{b_c\} \vdash c$; it follows that there is an $a_{b_c} \in U$, such that $\left\{a_{b_c}\right\} \vdash c$. Finally, consistency propagates through atomic entailment, since it does so through entailment in general.

We now show that $H\rho$ is an information system. The reflexivity for coherent consistency is immediate from the definition, since all singletons are already in Con. For the closure of coherent consistency under subsets, let $U \in H\mathsf{Con}$ and $V \subseteq U$; for all $a, a' \in V$ it is $a, a' \in U$, so $\left\{a, a'\right\} \in \mathsf{Con}$ by the coherence of U. The reflexivity of coherent entailment follows directly from the definition. For the transitivity of coherent entailment, let $U \vdash^H V$ and $V \vdash^H c$; then, for each $b \in V$ it is either $U \vdash b$ or $b \in U$, and similarly $V \vdash c$ or $c \in V$; it follows from the transitivity of entailment (and elementary set theory) that $U \vdash c$ or $c \in U$, so $U \vdash^H c$ by the definition. To show that coherent consistency propagates by coherent entailment, consider a $U \in H\mathsf{Con}$ and a $b \in \mathsf{Tok}$, such that $U \vdash^H b$. It suffices to show that $\{a, b\} \in \mathsf{Con}$ for an arbitrary $a \in U$: by the definition of coherent entailment it is either $U \vdash b$ or $b \in U$; in both cases it is indeed $\{a, b\} \in \mathsf{Con}$ for all $a \in U$, so $U \cup \{b\} \in H\mathsf{Con}$.

Now assume that σ is atomic. Let $\left\{(U_j, b_j) \mid j \in J\right\} \in \mathsf{Con}_{\rho\to\sigma}$ and $(U, b) \in \mathsf{Tok}_{\rho\to\sigma}$. By definition, $\left\{(U_j, b_j) \mid j \in J\right\}$ entails (U, b) at $\rho \to \sigma$ if $\left\{(U_j, b_j) \mid j \in J\right\} U$ entails b at σ. Let J_0 be the subset of these indices $j \in J$ for which $U \vdash_\rho U_j$; by the atomicity in σ, $\left\{b_j \mid j \in J_0\right\} \vdash_\sigma b$ implies that $\left\{b_j\right\} \vdash_\sigma b$ for a specific $j \in J_0$; then $\left\{(U_j, b_j)\right\} \vdash_{\rho\to\sigma} (U, b)$ for the same j. The case of coherence is treated in an equally straightforward way.[4] □

Both atomicity and coherence may provide extra ideals. For example, for the information systems of Figure 1, it is direct to check that $\{l, r\}$ is an ideal in $A\mathscr{C}$ but not in $\mathscr{C}$, and that $\{l, m, r, \varepsilon\}$ is an ideal in $H\mathscr{L}$ but not in $\mathscr{L}$. So atomic and coherent information systems may be ideal-wise richer. We now show that this is the only direction that we get richer in ideals and also that this is the richest we can

[3] Along the same lines, see also the independent work of Fritz Müller [13], which came to our attention only recently.

[4] A proof can be found in [23, Chapter 6]. In [22], the preservation of both atomicity and coherence by the formation of the function space is shown simultaneously for atomic-coherent systems.

get in this manner. Write $\rho \hookrightarrow \sigma$ if there is an embedding of Ide_ρ into Ide_σ, and $\rho \simeq \sigma$ if Ide_ρ and Ide_σ are isomorphic.

Theorem 4. *Let ρ be an information system.*

1. *It is $\rho \hookrightarrow A\rho$ and $\rho \hookrightarrow H\rho$.*
2. *Atomicity and coherence are idempotent, in the sense that $A(A\rho) \simeq A\rho$ and $H(H\rho) \simeq H\rho$.*

Proof. We show that $\mathsf{Ide}_\rho \subseteq \mathsf{Ide}_{A\rho}$: let $u \in \mathsf{Ide}_\rho$; if $U \subseteq^f u$, then $U \in \mathsf{Con}$ by the consistency in ρ; if further $U \vdash^A b$, then $U \vdash b$ since $\vdash^A \subseteq \vdash$, so $b \in u$ by the deductive closure in ρ. We now show that $\mathsf{Ide}_\rho \subseteq \mathsf{Ide}_{H\rho}$: let $u \in \mathsf{Ide}_\rho$; if $U \subseteq^f u$, then $U \in \mathsf{Con} \subseteq H\mathsf{Con}$ by the consistency in ρ; if further $U \vdash^H b$, then $U \vdash b$ since $U \in \mathsf{Con}$ again, so $b \in u$ by the deductive closure in ρ.

Now we show idempotence. For atomicity, the only thing we have to show is that $\vdash^{A^A} = \vdash^A$. By definition, it is $U \vdash^{A^A} b$ if and only if there is an $a \in U$ such that $\{a\} \vdash^A b$; this in turn means that there is an $a' \in \{a\}$ such that $\{a'\} \vdash b$; clearly it must be $a = a'$, so we've found an $a \in U$ such that $\{a\} \vdash b$. So it is $A(A\rho) \simeq A\rho$—actually with the trivial isomorphism. For the idempotence of coherence, let $U \in H(H\mathsf{Con}_\rho)$; then

$$\forall_{a,a' \in U} \{a, a'\} \in H\mathsf{Con}_\rho \overset{(\star)}{\Leftrightarrow} \forall_{a,a' \in U} \{a, a'\} \in \mathsf{Con}_\rho,$$

which means that $U \in H\mathsf{Con}_\rho$. The step $(\star)$ holds since two-element sets that are consistent are also coherently consistent and vice versa. Finally, let $U \vdash^{H^H} b$; by the definition, it is either $U \vdash^H b$, so we're done, or $b \in U$, whereby $U \in H\mathsf{Con}$, and again $U \vdash^H b$ by the reflexivity of coherent entailment. $\square$

4 Coherent point-free structures

Links between domain theory and point-free topology have been studied by several people already [21, 18, 14, 15, 26]. Our main objective here is to find direct correspondences between the information systems that we find useful in practice and respective point-free structures, domains included.

The basic problem that we face in such an endeavor lies, not surprisingly, in the very nature of atomicity and coherence, which are both defined in terms of *tokens*; but tokens are not observable in a point-free setting, where everything starts with a formal version of neighborhoods. With atomicity in particular, the problem seems more involved, since it is a property which fully exploits the presence of atomic pieces of data. Indeed, in order to express atomicity in a point-free setting, it seems that we can not avoid talking about "atomic" or "prime neighborhoods" (which would correspond intuitively to singletons of tokens); regarding the relatively limited status of atomic systems in our pursued applications, this has not proved a convincing task up to now.[5]

On the other hand, *coherent cpo's* appear already in [16] by Gordon Plotkin, where he attributes the notion to George Markowsky and Barry Rosen from [11]. In the handbook chapter [1] by Samson Abramsky and Achim Jung, coherence is studied in the more general setting of *continuous domains*, while Viggo Stoltenberg-Hansen *et al.* introduce the notion too as an exercise [26, Exercise 8.5.19]. Also, as already mentioned, Jean-Yves Girard's *coherence spaces* [6]

[5] However, in strict regard to domain theory, atomic information systems are shown in [4] to represent *prime algebraic domains*. See also section 5.

are used instead of Scott–Ershov domains to model linear logic. Moreover, coherence has been considered in point-free topology as well, at least since Peter Johnstone discussed *coherent locales* [8].

On the technical side, unlike atomicity, coherence lends itself to a characterization on the level of consistent sets in a trivial way.

Lemma 5. *A finite set of tokens is a coherent neighborhood if and only if every two of its subsets have a coherent union: it is $U \in H\mathsf{Con}$ if and only if $U \in \mathscr{P}_f(\mathsf{Tok})$ and $U_1 \cup U_2 \in H\mathsf{Con}$ for any $U_1, U_2 \subseteq U$.*

By this we stress that the issue of acceptance of a neighborhood in a coherent system is raised from *comparisons of its tokens*, to *comparisons of its subsets*. The coherence conditions (2), (3), and (5), that we pose in the following are all modeled after Lemma 5.

In this section we restrict ourselves to the case where we have countable carrier sets.

4.1 Domains

We start with the known correspondence of arbitrary Scott information systems and domains (of countable base), which we quickly recount here without proofs, to set the mood for what comes next—for details we refer to [26].

Let $D = (D, \sqsubseteq, \bot)$ be a domain and define $I_D(D) := (\mathsf{Tok}, \mathsf{Con}, \vdash)$ by

$$\begin{aligned} \mathsf{Tok} &:= D_c, \\ \{u_i\}_{i\in I} \in \mathsf{Con} &:= \{u_i\}_{i\in I} \subseteq^f D_c \wedge \operatorname{lub}\{u_i\}_{i\in I} \in D, \\ \{u_i\}_{i\in I} \vdash u &:= u \sqsubseteq \operatorname{lub}\{u_i\}_{i\in I}. \end{aligned}$$

Note that, by the well-known fact that a bounded finite set of compact elements has a *compact* least upper bound, if $\{u_i\}_{i\in I} \in \mathsf{Con}$ then $\operatorname{lub}\{u_i\}_{i\in I} \in D_c$. Conversely, for an information system $\rho = (\mathsf{Tok}, \mathsf{Con}, \vdash)$, define $D(\rho) := (\mathsf{Ide}_\rho, \subseteq, \bot)$.

Fact 6. *If D is a domain and ρ an information system, then $I_D(D)$ is an information system and $D(\rho)$ a domain, where $D(\rho)_c = \overline{\mathsf{Con}}_\rho$, and $\operatorname{lub}\{\overline{U}, \overline{V}\} := \overline{U \cup V}$. Furthermore, if D is a domain then $\mathsf{Ide}_{I_D(D)} \simeq D$, through the isomorphism pair $u \mapsto \operatorname{lub} u$ and $u \mapsto \{v \in D_c \mid v \sqsubseteq u\}$ (the set of the compact approximations of u).*

Let now r be an approximable map from ρ to σ. Define a mapping $D(r) : D(\rho) \to D(\sigma)$ by

$$D(r)(u) := \bigcup \left\{ V \in \mathsf{Con}_\sigma \mid \exists_{U \in \mathsf{Con}_\rho} (U \subseteq^f u \wedge (U,V) \in r) \right\}.$$

Conversely, let f be a continuous mapping from a domain D to a domain E. Define a relation $I_D(f) \subseteq \mathsf{Con}_{I_D(D)} \times \mathsf{Con}_{I_D(E)}$ by

$$(\{u_i\}_{i\in I}, \{v_i\}_{i\in I}) \in I_D(f) := \operatorname{lub}\{v_i\}_{i\in I} \sqsubseteq f(\operatorname{lub}\{u_i\}_{i\in I}).$$

These establish a bijective correspondence, as the following statement expresses.

Fact 7. *If r is an approximable map from ρ to σ then $D(r) : D(\rho) \to D(\sigma)$ is a continuous mapping. Conversely, if $f : D \to E$ is a continuous mapping then $I_D(f)$ is an approximable map from $I_D(D)$ to $I_D(E)$. Furthermore, the collection of continuous mappings from D to E is in a bijective correspondence with the collection of approximable maps between $I_D(D)$ and $I_D(E)$.*

Coherent domains

Let $D = (D, \sqsubseteq, \bot)$ be a domain and $\{u_i\}_{i\in I} \subseteq^f D_c$ an arbitrary *finite* set of compact elements. Call D a *coherent domain* if

$$\operatorname{lub}\{u_i\}_{i\in I} \in D_c \leftrightarrow \forall_{i,j\in I} \operatorname{lub}\{u_i, u_j\} \in D_c. \tag{2}$$

Note that the choice of D_c instead of the more modest D is justified again by the basic fact that bounded finite sets of compacts have a compact least upper bound.

Theorem 8. *Let D be a coherent domain and ρ a coherent information system. Then $I_D(D)$ is a coherent information system and $D(\rho)$ is a coherent domain.*

Proof. Let D be a coherent domain and $\{u_i\}_{i\in I} \subseteq^f \mathsf{Tok}_{I_D(D)}$ be such that $\{u_i, u_j\} \in \mathsf{Con}_{I_D(D)}$ for every pair i, j in I. By the definition of $I_D(D)$, $\operatorname{lub}\{u_i, u_j\} \in D$. Since bounded finite sets of compacts have a compact least upper bound, it is $\operatorname{lub}\{u_i, u_j\} \in D_c$ for every such $i, j \in I$, so by (2) we get $\operatorname{lub}\{u_i\}_{i\in I} \in D_c$, and therefore $\{u_i\}_{i\in I} \in \mathsf{Con}_{I_D(D)}$.

Let now ρ be a coherent information system. By Fact 6, it is $u \in D(\rho)_c$ if there is a $U \in \mathsf{Con}$ with $\overline{U} = u$, and $\operatorname{lub}\{\overline{U}, \overline{V}\} = \overline{U \cup V}$. So let $\{\overline{U_i}\}_{i\in I} \subseteq^f D(\rho)_c$ and set $U := \bigcup_{i\in I} U_i$. We show that (2) holds for $D(\rho)$. Assuming that $\overline{U} \in \overline{\mathsf{Con}}$, and since $U \vdash U_i$ for all i, we have $U_i \cup U_j \in \mathsf{Con}$ for each $i, j \in I$, by propagation; it follows that $\overline{U_i \cup U_j} \in \overline{\mathsf{Con}}$. For the other direction, assume that $\overline{U_i \cup U_j} \in \overline{\mathsf{Con}}$ for each $i, j \in I$; by the coherence of ρ and Lemma 5, we get that $U \in \mathsf{Con}$, so $\overline{U} \in \overline{\mathsf{Con}}$, and we're done. $\square$

4.2 Precusl's

Precusl's are structures that provide yet another representation of domains, this time with a more order-theoretic emphasis. Stoltenberg-Hansen *et al.* [26, Chapter 6] have used precusl's as an alternative to information systems to solve domain equations up to identity. In their textbook one can also find a correspondence between precusl's and information systems, which we recall here before moving to the issue of coherence.

A *preordered conditional upper semilattice with a distinguished least element*, or just *precusl*, is a consistently complete preordered set with a distinguished least element, that is, a quadruple $P = (N, \sqsubseteq, \sqcup, \bot)$, where $\sqsubseteq$ is a preorder on N, $\bot$ is a (distinguished) least element and $\sqcup$ is a partial binary operation on N which is defined only on *consistent* pairs, that is, on pairs having an upper bound, and then yields a (distinguished) least upper bound:

$$\begin{aligned} &U \sqcup V \in N := \exists_{W\in N} (U \sqsubseteq W \wedge V \sqsubseteq W), \\ &U \sqcup V \in N \to U \sqsubseteq U \sqcup V \wedge V \sqsubseteq U \sqcup V \\ &\qquad\qquad\qquad\qquad \wedge \forall_{W\in N} (U \sqsubseteq W \wedge V \sqsubseteq W \to U \sqcup V \sqsubseteq W). \end{aligned}$$

We think of N as "a set of formal basic opens", $\sqsubseteq$ as "formal inclusion", $\bot$ as "a formal empty set" and $\sqcup$ as "a formal union". Call a subset $u \subseteq N$ a *precusl ideal* when it satisfies

$$\bot \in u \ \wedge\ \forall_{U,V\in u} U \sqcup V \in u \ \wedge\ \forall_{U\in u} (V \sqsubseteq U \to V \in u).$$

Write Ide_P for the class of all precusl ideas of P. Observe that the second of the three requirements for a precusl ideal expresses the property of being upward directed, so it follows that *any* finite subset in a precusl ideal will have a least upper bound in the ideal.

Let now $P = (N, \sqsubseteq, \sqcup, \perp)$ be a precusl and define $I_P(P) = (\mathsf{Tok}, \mathsf{Con}, \vdash)$ by

$$\begin{aligned} \mathsf{Tok} &:= N, \\ \mathscr{U} \in \mathsf{Con} &:= \mathscr{U} \subseteq^f N \wedge \bigsqcup \mathscr{U} \in N, \\ \mathscr{U} \vdash U &:= U \sqsubseteq \bigsqcup \mathscr{U}. \end{aligned}$$

Conversely, let $\rho = (\mathsf{Tok}, \mathsf{Con}, \vdash)$ be an information system and define $P(\rho) = (N, \sqsubseteq, \perp, \sqcup)$ by

$$\begin{aligned} N &:= \mathsf{Con}, \\ U \sqsubseteq V &:= V \vdash U, \\ \perp &:= \varnothing, \\ U \sqcup V &:= U \cup V \text{ if } U \cup V \in \mathsf{Con}. \end{aligned}$$

The following is Theorem 6.3.4 of [26].

Fact 9. *If P is a precusl and ρ an information system, then $I_P(P)$ is an information system and $P(\rho)$ is a precusl. Furthermore, it is $\mathsf{Ide}_P = \mathsf{Ide}_{I_P(P)}$ and $\mathsf{Ide}_\rho \simeq \mathsf{Ide}_{P(\rho)}$.*

A *precusl approximable map* from P to P' is a relation $\mathscr{R} \subseteq N \times N'$, such that

- $(\perp, \perp') \in \mathscr{R}$,
- $(U, V) \in \mathscr{R} \wedge (U, V') \in \mathscr{R} \rightarrow (U, V \sqcup V') \in \mathscr{R}$,
- $U \sqsubseteq U' \wedge (U, V) \in \mathscr{R} \wedge V' \sqsubseteq V \rightarrow (U', V') \in \mathscr{R}$,

where $(U, V \sqcup V') \in \mathscr{R}$ naturally implies that $V \sqcup V'$ is defined. Write $\mathsf{Apx}_{P \rightarrow P'}$ for all precusl approximable maps from P to P'. For every $\mathscr{R} \in \mathsf{Apx}_{P \rightarrow P'}$ define a relation $I_P(\mathscr{R}) \subseteq \mathsf{Con}_{I_P(P)} \times \mathsf{Con}_{I_P(P')}$ by

$$(\mathscr{U}, \mathscr{V}) \in I_P(\mathscr{R}) := \left(\bigsqcup \mathscr{U}, \bigsqcup \mathscr{V} \right) \in \mathscr{R}.$$

Conversely, let r be an approximable map from ρ to σ. Define a relation $P(r) \subseteq N_{P(\rho)} \times N_{P(\sigma)}$ by

$$(U, V) \in P(r) := (U, V) \in r.$$

One can show [26, pp. 151–2] that these establish a bijective correspondence.

Fact 10. *If r is an approximable map from ρ to σ then $P(r)$ is a precusl approximable map from $P(\rho)$ to $P(\sigma)$. Conversely, if $\mathscr{R}$ is a precusl approximable map from P to P' then $I_P(\mathscr{R})$ is an approximable map from $I_P(P)$ to $I_P(P')$. Furthermore, it is $\mathsf{Apx}_{\rho \rightarrow \sigma} \simeq \mathsf{Apx}_{P(\rho) \rightarrow P(\sigma)}$ and $\mathsf{Apx}_{P \rightarrow P'} \simeq \mathsf{Apx}_{I_P(P) \rightarrow I_P(P')}$.*

Coherent precusl's

Call a precusl *coherent* if, for a finite collection $\mathscr{U} \subseteq^f N$,

$$\bigsqcup \mathscr{U} \in N \leftrightarrow \forall_{U, V \in \mathscr{U}} U \sqcup V \in N. \tag{3}$$

Theorem 11. *If P is a coherent precusl then $I_P(P)$ is a coherent information system. Conversely, if ρ is a coherent information system then $P(\rho)$ is a coherent precusl.*

Proof. Suppose first that P is coherent. Let $\mathscr{U} \in \mathsf{Con}_{I_P(P)}$, which by definition means $\bigsqcup \mathscr{U} \in N$; by (3), this is equivalent to $U \sqcup V \in N$ for all $U, V \in \mathscr{U}$, which is equivalent to $\{U, V\} \in \mathsf{Con}_{I_P(P)}$, again by definition. So $I_P(P)$ is a coherent information system.

Now suppose that ρ is a coherent information system, that is, such that

$$U \in \mathsf{Con} \leftrightarrow \forall_{a,b \in U} \{a,b\} \in \mathsf{Con} \tag{4}$$

for all $U \subseteq^f \mathsf{Tok}$. Let $\mathscr{U} \subseteq^f N_{P(\rho)}$, that is, $\mathscr{U} \subseteq^f \mathsf{Con}$, and assume that $\bigsqcup \mathscr{U} \in N_{P(\rho)}$. By definition it is $\bigcup \mathscr{U} \in \mathsf{Con}$; by (4) and Lemma 5, we get that $\bigcup \mathscr{U} \subseteq^f \mathsf{Tok}$ and $U \cup V \in \mathsf{Con}$, that is, $U \sqcup V \in N_{P(\rho)}$ for all $U, V \subseteq \bigcup \mathscr{U}$; a fortiori it is $U \sqcup V \in N_{P(\rho)}$ for all $U, V \in \mathscr{U}$. Conversely, assume that $\mathscr{U} \subseteq^f N_{P(\rho)}$ and $U \sqcup V \in N_{P(\rho)}$ for all $U, V \in \mathscr{U}$; by definition it is $\mathscr{U} \subseteq^f \mathsf{Con}$ and $U \sqcup V \in \mathsf{Con}$ for all $U, V \in \mathscr{U}$; by (4), it must hold that $\{a,b\} \in \mathsf{Con}$ for every $a, b \in U \cup V$, where $U, V \in \mathscr{U}$; it follows that the same must hold for every $a, b \in \bigcup \mathscr{U}$; by (4) again we get $\bigcup \mathscr{U} \in \mathsf{Con}$, so $\bigsqcup \mathscr{U} \in N_{P(\rho)}$, by definition. So $P(\rho)$ is indeed a coherent precusl. □

4.3 Scott–Ershov formal topologies

The *structure* of a "formal topology" was defined by Giovanni Sambin [17] as early as 1987, and as the area has developed a number of alternative definitions has appeared. Suitable for our purposes is a version of the definition in [14], whose main difference from Sambin's original is the disposal of the "positivity predicate". In fact, we depart a bit from this definition as well, in that we require the presence of a top element among the formal basic opens.[6]

We will use order-theoretic notions which are *dual* to notions appearing before, namely a *greatest* or *top element* and *greatest lower bounds* of sets of elements; all of these are to be understood in the usual order-theoretic way.

Define a *formal topology* as a triple $\mathscr{T} = (N, \sqsubseteq, \prec)$, where N is the collection of *formal basic opens*, $\sqsubseteq \subseteq N \times N$ is a preorder with a top element $\top$, which formalizes inclusion between basic opens, and $\prec \subseteq N \times \mathscr{P}(N)$, called *covering*, formalizes inclusion between opens (not just the basic ones), and is reflexive, transitive, *localizing*, and extends formal inclusion between formal basic opens, that is,

$$\begin{aligned}
&U \in \mathscr{U} \to U \prec \mathscr{U}, \\
&U \prec \mathscr{U} \wedge \mathscr{U} \prec \mathscr{V} \to U \prec \mathscr{V}, \\
&U \prec \mathscr{U} \wedge U \prec \mathscr{V} \to U \prec \mathscr{U} \downarrow \mathscr{V}, \\
&V \sqsubseteq U \wedge U \prec \mathscr{U} \to V \prec \mathscr{U},
\end{aligned}$$

where we write $\mathscr{U} \prec \mathscr{V}$ for $\forall_{U \in \mathscr{U}} U \prec \mathscr{V}$ and $\mathscr{U} \downarrow \mathscr{V}$ for the collection

$$\{W \in N \mid \exists_{U \in \mathscr{U}} \exists_{V \in \mathscr{V}} (W \sqsubseteq U \wedge W \sqsubseteq V)\}.$$

A *formal point* in $\mathscr{T}$ is a subset $u \subseteq N$ such that

$$\top \in u \wedge \forall_{U,V \in u} \exists_{W \in u} (W \sqsubseteq U \wedge W \sqsubseteq V) \wedge \forall_{U \in u} (U \prec \mathscr{U} \to \exists_{V \in \mathscr{U}} V \in u).$$

Denote the collection of formal points of $\mathscr{T}$ by $\mathsf{Pt}_{\mathscr{T}}$. Call a formal topology $\mathscr{T}$ *unary* if

$$U \prec \mathscr{U} \to \exists_{V \in \mathscr{U}} U \prec V,$$

where we write $U \prec V$ for $U \prec \{V\}$, and *consistently complete* if

$$\forall_{U,V \in N} (\exists_{W \in N} (W \sqsubseteq U \wedge W \sqsubseteq V) \to U \sqcap V \in N),$$

[6] Concerning the positivity predicate see [14, §2.4] or [19, footnote 13] for relevant discussions (see also section 5). For the issue of a top element see [26, Exercise 6.5.21].

where $\bigsqcap \mathscr{U}$ denotes the greatest lower bound of $\mathscr{U}$. Finally, call $\mathscr{T}$ a *Scott–Ershov formal topology* if it is both unary and consistently complete.

Note that in the case of a consistently complete formal topology, dually to the case of precusl ideals, the last two of the requirements for a formal point yield the property of being *downward* directed (that is, a *filter*): if $U,V \in u$, then they are bounded below by some $W \in u$; by consistent completeness it is $U \sqcap V \in N$, and since $W \prec \{U \sqcap V\}$, it is $U \sqcap V \in u$. It follows that *any* finite subset in a formal point will have a greatest lower bound in the formal point.

One can prove that every domain can be represented by the collection of formal points of a certain Scott–Ershov formal topology—see [14, Theorem 4.35] and [26, Theorem 6.2.15]. Here we proceed to link formal topologies *directly* to information systems, before we discuss coherence.

Let $\mathscr{T} = (N, \sqsubseteq, \prec)$ be a Scott–Ershov formal topology. Define $I_F(\mathscr{T}) = (\mathsf{Tok}, \mathsf{Con}, \vdash)$ by

$$\begin{aligned} \mathsf{Tok} &:= N, \\ \mathscr{U} \in \mathsf{Con} &:= \mathscr{U} \subseteq^f N \wedge \bigsqcap \mathscr{U} \in N, \\ \mathscr{U} \vdash U &:= \bigsqcap \mathscr{U} \sqsubseteq U. \end{aligned}$$

Conversely, let $\rho = (\mathsf{Tok}, \mathsf{Con}, \vdash)$ be an information system. Define $F(\rho) = (N, \sqsubseteq, \prec)$ by

$$\begin{aligned} N &:= \overline{\mathsf{Con}}, \\ \overline{U} \sqsubseteq \overline{V} &:= U \vdash V, \\ \overline{U} \prec \mathscr{U} &:= \exists_{\overline{V} \in \mathscr{U}} U \vdash V. \end{aligned}$$

Note that the definition is independent from the choice of representatives by Lemma 1.

Proposition 12. *If $\mathscr{T}$ is a Scott–Ershov formal topology and ρ an information system, then $I_F(\mathscr{T})$ is an information system and $F(\rho)$ is a Scott–Ershov formal topology. Furthermore, it is $\mathsf{Pt}_{\mathscr{T}} = \mathsf{Ide}_{I_F(\mathscr{T})}$ and $\mathsf{Ide}_\rho \simeq \mathsf{Pt}_{F(\rho)}$.*

Proof. First let $\mathscr{T}$ be a Scott–Ershov formal topology. We check the defining properties of an information system for $I_F(\mathscr{T})$. For reflexivity of consistency, let $U \in N$; it is $U \sqsubseteq U$, so $\bigsqcap\{U\} \in N$ and $\{U\} \in \mathsf{Con}$ by definition. For closure under subsets, let $\mathscr{U} \in \mathsf{Con}$ and $\mathscr{V} \subseteq \mathscr{U}$; then $\bigsqcap \mathscr{U} \in N$ and $\bigsqcap \mathscr{U} \sqsubseteq V$ for all $V \in \mathscr{V}$, so $\bigsqcap \mathscr{V} \in N$ and $\mathscr{V} \in \mathsf{Con}$ by definition. For reflexivity of entailment, let $\mathscr{U} \in \mathsf{Con}$ and $U \in \mathscr{U}$; then $\bigsqcap \mathscr{U} \sqsubseteq U$, so $\mathscr{U} \vdash U$ by definition. For transitivity of entailment, let $\mathscr{U} \vdash \mathscr{V}$ and $\mathscr{V} \vdash W$; then $\bigsqcap \mathscr{U} \sqsubseteq \bigsqcap \mathscr{V}$ and $\bigsqcap \mathscr{V} \sqsubseteq W$; by transitivity we get $\bigsqcap \mathscr{U} \sqsubseteq W$, so $\mathscr{U} \vdash W$ by definition. Finally, for propagation of consistency through entailment, let $\mathscr{U} \in \mathsf{Con}$ and $\mathscr{U} \vdash V$; by definition, $\bigsqcap \mathscr{U} \in N$ and $\bigsqcap \mathscr{U} \sqsubseteq V$, so $\bigsqcap(\mathscr{U} \cup \{V\}) \in N$ and $\mathscr{U} \cup \{V\} \in \mathsf{Con}$ by definition.

Now let ρ be an information system. We check the defining properties of a Scott–Ershov formal topology for $F(\rho)$. That $\sqsubseteq$ is a preorder with $\top := \overline{\varnothing}$ is direct to see. For reflexivity of covering, let $\overline{U} \in \mathscr{U}$; since $U \vdash U$, it is $\overline{U} \prec \mathscr{U}$ by definition. For transitivity of covering, let $\overline{W} \prec \mathscr{U}$ and $\mathscr{U} \prec \mathscr{V}$; then there exists a $\overline{U} \in \mathscr{U}$ such that $W \vdash U$ and $U \vdash V$ for some $\overline{V} \in \mathscr{V}$; by transitivity we get $W \vdash V$, so $\overline{W} \prec \mathscr{V}$ by definition. For localization, let $\overline{W} \prec \mathscr{U}$ and $\overline{W} \prec \mathscr{V}$; then there exists a $\overline{U} \in \mathscr{U}$ such that $W \vdash U$ and $W \vdash V$ for some $\overline{V} \in \mathscr{V}$; by definition, this means that $\overline{W} \sqsubseteq \overline{U}$ and $\overline{W} \sqsubseteq \overline{V}$ respectively, so we get $\overline{W} \in \mathscr{U} \downarrow \mathscr{V}$, which gives $\overline{W} \prec \mathscr{U} \downarrow \mathscr{V}$ by reflexivity. To show that the covering extends formal inclusion between formal basic opens, let $\overline{W} \sqsubseteq \overline{U}$ and $\overline{U} \prec \mathscr{V}$; then $W \vdash U$ and $U \vdash V$ for some $\overline{V} \in \mathscr{V}$; by transitivity we get $W \vdash V$, so $\overline{W} \prec \mathscr{V}$, by definition.

So $F(\rho)$ is indeed a formal topology. To show that it is unary is easy: let $\overline{U} \prec \mathscr{U}$; by definition there is a $\overline{V} \in \mathscr{U}$, for which $U \vdash V$, that is, $\overline{U} \sqsubseteq \overline{V}$; by reflexivity and extension, we get $\overline{U} \prec \{\overline{V}\}$. To show, finally, that it is consistently complete, let $\overline{U}, \overline{V}, \overline{W} \in N$, with $\overline{W} \sqsubseteq \overline{U}$ and $\overline{W} \sqsubseteq \overline{U}$, that is, $W \vdash U$ and $W \vdash V$; by (1), we get $W \vdash U \cup V$, and so, $\overline{W} \sqsubseteq \overline{U \cup V}$; let $\overline{U} \sqcap \overline{V} := \overline{U \cup V}$; that this does the job is direct to see.

We now show the bijective correspondence between *information system ideals* and *formal points*. For $\mathsf{Pt}_{\mathscr{T}} \subseteq \mathsf{Ide}_{I_F(\mathscr{T})}$, let $u \in \mathsf{Pt}_{\mathscr{T}}$ and $\mathscr{U} \subseteq^f u$. Since, as we noted, u is downward directed in $\mathscr{T}$ by consistent completeness, it is $\bigsqcap \mathscr{U} \in u \subseteq N$, and so $\mathscr{U} \in \mathsf{Con}_{I_F(\mathscr{T})}$ by definition. If further $\mathscr{U} \vdash_{I_F(\mathscr{T})} U$, it is $\bigsqcap \mathscr{U} \sqsubseteq U$ by definition, and $\bigsqcap \mathscr{U} \prec \{U\}$; hence $U \in u$ by the third formal point property.

For $\mathsf{Ide}_{I_F(\mathscr{T})} \subseteq \mathsf{Pt}_{\mathscr{T}}$, let $u \in \mathsf{Ide}_{I_F(\mathscr{T})}$. That $\top = \overline{\varnothing} \in u$, follows from downward closure in $I_F(\mathscr{T})$. Let $U, V \in u$; by the consistency in $I_F(\mathscr{T})$, $\{U, V\} \in \mathsf{Con}_{I_F(\mathscr{T})}$, and then $U \sqcap V \in N$ by definition; since $\{U, V\} \vdash_{I_F(\mathscr{T})} U \sqcap V$, it is $U \sqcap V \in u$ by the deductive closure in $I_F(\mathscr{T})$. If now $U \in u$ and $U \prec \mathscr{U}$, then, since $\mathscr{T}$ is unary, there exists a $V \in \mathscr{U}$ such that $U \sqsubseteq V$, which by definition means that $\{U\} \vdash_{I_F(\mathscr{T})} V$; by the deductive closure in $I_F(P)$, it follows that $V \in u$, and we're done.

For $\mathsf{Ide}_\rho \simeq \mathsf{Pt}_{F(\rho)}$, take the isomorphism pair

$$u \mapsto \mathscr{P}_c(u) : \mathsf{Ide}_\rho \to \mathsf{Pt}_{F(\rho)},$$
$$u \mapsto \bigcup u : \mathsf{Pt}_{F(\rho)} \to \mathsf{Ide}_\rho,$$

where $\mathscr{P}_c(u) := \{\overline{U}\}_{U \subseteq^f u}$ contains the closures of subsets of u.

Indeed, for the right embedding, since $\varnothing \subseteq^f u$, it is $\top \in \mathscr{P}_c(u)$; for every $U, V \subseteq^f u$, since $U \cup V \subseteq^f u$, it is also $\overline{U} \sqcap \overline{V} \in \mathscr{P}_c(u)$; if $U \subseteq^f u$ and $\overline{U} \prec \mathscr{U}$, then there exists a $\overline{V} \in \mathscr{U}$ such that $\overline{U} \sqsubseteq \overline{V}$, that is, such that $U \vdash V$ by definition; then, for this $\overline{V} \in \mathscr{U}$, it is $V \subseteq^f u$ by the deductive closure in ρ.

For the left embedding, if $\{a_i\}_{i<n} \subseteq^f \bigcup u$, then for each $i < n$ there exists a $\overline{U_i} \in u$ such that $U_i \vdash a_i$; since u is downward directed in $F(\rho)$, there exists a $\overline{W} \in u$ such that $\overline{W} \sqsubseteq \overline{U_i}$ for each i; the latter gives by definition $W \vdash U_i \vdash a_i$, so $\{a_i\}_{i<n} \in \mathsf{Con}$ by the transitivity of entailment and (1). If now $U \subseteq^f \bigcup u$ and $U \vdash a$, then, by definition, $\overline{U} \sqsubseteq \overline{a}$, that is $\overline{U} \prec \{\overline{a}\}$; by the third formal point property in $F(\rho)$, we have $\overline{a} \in u$, so $a \in \bigcup u$.

That the two embeddings are mutually inverse is also quite direct. Indeed, let $u \in \mathsf{Ide}_\rho$. We have $a \in \bigcup \mathscr{P}_c(u)$ if and only if there exists a $U \subseteq^f u$ such that $U \vdash a$; but this is equivalent to $a \in u$ (rightwards by deductive closure of u and leftwards by setting $U := \{a\}$). Also, let $\overline{U} \in \mathscr{P}_c\left(\bigcup u\right)$, which by definition means that $U \subseteq^f \bigcup u$; writing $U = \{a_i\}_{i<n}$, we have on the one hand

$$\begin{aligned}
\{a_i\}_{i<n} \subseteq^f \bigcup u &\Leftrightarrow \forall_{i<n} \exists_{\overline{U_i} \in u} U_i \vdash a_i \\
&\Leftrightarrow \forall_{i<n} \exists_{\overline{U_i} \in u} \overline{U_i} \sqsubseteq \overline{a_i} \\
&\Rightarrow \exists_{\overline{W} \in u} \forall_{i<n} \overline{W} \sqsubseteq \overline{a_i} \quad \text{(by the second formal point property)} \\
&\Rightarrow \exists_{\overline{W} \in u} \forall_{i<n} \overline{W} \prec \{\overline{a_i}\} \\
&\Rightarrow \forall_{i<n} \overline{a_i} \in u \quad \text{(by the third formal point property),}
\end{aligned}$$

and on the other hand $\overline{a_i} \in u$ implies $\{a_i\}_{i<n} \subseteq^f \bigcup u$ trivially; so we have shown that the latter is equivalent to $\{\overline{a_i}\}_{i<n} \subseteq u$; since u, as we noted, is downward directed, it is $\bigsqcap \{\overline{a_i}\}_{i<n} = \overline{\{a_i\}_{i<n}} = \overline{U} \in u$, and since $\overline{U} \in u$ implies $\{\overline{a_i}\}_{i<n} \subseteq u$ (because $\overline{U} \prec \{\overline{a_i}\}$ for each i), we're done. $\square$

An *approximable map of Scott–Ershov formal topologies* from $\mathscr{T}$ to $\mathscr{T}'$ is a relation $\mathscr{R} \subseteq N \times N'$, such that

- $(\top, \top') \in \mathscr{R}$,
- $(U,V) \in \mathscr{R} \wedge (U,V') \in \mathscr{R} \to (U, V \sqcap V') \in \mathscr{R}$,
- $U' \sqsubseteq U \wedge (U,V) \in \mathscr{R} \wedge V \sqsubseteq V' \to (U',V') \in \mathscr{R}$.

Write $\mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'}$ for all approximable maps of Scott–Ershov formal topologies from $\mathscr{T}$ to $\mathscr{T}'$. For every $\mathscr{R} \in \mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'}$ define a relation $I_F(\mathscr{R}) \subseteq \mathsf{Con}_{I_F(\mathscr{T})} \times \mathsf{Con}_{I_F(\mathscr{T}')}$ by

$$(\mathscr{U},\mathscr{V}) \in I_F(\mathscr{R}) := \left(\bigsqcap \mathscr{U}, \bigsqcap \mathscr{V}\right) \in \mathscr{R}.$$

Conversely, let r be an approximable map from ρ to σ. Define a relation $F(r) \subseteq N_{F(\rho)} \times N_{F(\sigma)}$ by

$$(\overline{U},\overline{V}) \in F(r) := (U,V) \in r.$$

Again, it is easy to see that the definition does not rely on the choice of the representatives, due to deductive closure of r.

We show that these establish a bijective correspondence.

Proposition 13. *If r is an approximable map from ρ to σ then $F(r)$ is an approximable map of Scott–Ershov formal topologies from $F(\rho)$ to $F(\sigma)$. Conversely, if $\mathscr{R}$ is an approximable map of Scott–Ershov formal topologies from $\mathscr{T}$ to $\mathscr{T}'$ then $I_F(\mathscr{R})$ is an approximable map from $I_F(\mathscr{T})$ to $I_F(\mathscr{T}')$. Furthermore, it is $\mathsf{Apx}_{\rho\to\sigma} \simeq \mathsf{Apx}_{F(\rho)\to F(\sigma)}$ and $\mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'} \simeq \mathsf{Apx}_{I_F(\mathscr{T})\to I_F(\mathscr{T}')}$.*

Proof. Let r be an approximable map from ρ to σ. Since, by Fact 2, $(\varnothing,\varnothing) \in r$, it is $(\top,\top') \in F(r)$. If $(\overline{U},\overline{V}), (\overline{U},\overline{V'}) \in F(r)$, then, by the definitions, $(U, V \cup V') \in r$, so $(\overline{U}, \overline{V \cup V'}) \in F(r)$, and $(\overline{U}, \overline{V} \sqcap \overline{V'}) \in F(r)$. If $\overline{U'} \sqsubseteq \overline{U}$, $(\overline{U},\overline{V}) \in F(r)$ and $\overline{V} \sqsubseteq \overline{V'}$, then, by the definitions, $U' \vdash U$, $(U,V) \in r$ and $V \vdash V'$ respectively, so, $(U',V') \in r$ and $(\overline{U'},\overline{V'}) \in F(r)$.

Conversely, let $\mathscr{R}$ be an approximable map of Scott–Ershov formal topologies from $\mathscr{T}$ to $\mathscr{T}'$. Since $(\top,\top') \in \mathscr{R}$, it is $(\varnothing,\varnothing) \in I_F(\mathscr{R})$. If $(\mathscr{U},\mathscr{V}), (\mathscr{U},\mathscr{V}') \in I_F(\mathscr{R})$, then, by definition, $(\bigsqcap\mathscr{U}, \bigsqcap\mathscr{V}), (\bigsqcap\mathscr{U}, \bigsqcap\mathscr{V}') \in \mathscr{R}$; since $\mathscr{R}$ is an approximable map of Scott–Ershov formal topologies, $\left(\bigsqcap\mathscr{U}, (\bigsqcap\mathscr{V}) \sqcap (\bigsqcap\mathscr{V}')\right) \in \mathscr{R}$, or, $\left(\bigsqcap\mathscr{U}, \bigsqcap(\mathscr{V} \cup \mathscr{V}')\right) \in \mathscr{R}$, that is, $(\mathscr{U}, \mathscr{V} \cup \mathscr{V}') \in I_F(\mathscr{R})$. If now $\mathscr{U}' \vdash \mathscr{U}$, $(\mathscr{U},\mathscr{V}) \in I_F(\mathscr{R})$ and $\mathscr{V} \vdash \mathscr{V}'$, by definition we obtain $\bigsqcap\mathscr{U}' \sqsubseteq \bigsqcap\mathscr{U}$, $(\bigsqcap\mathscr{U}, \bigsqcap\mathscr{V}) \in \mathscr{R}$ and $\bigsqcap\mathscr{V} \sqsubseteq \bigsqcap\mathscr{V}'$; then $\left(\bigsqcap\mathscr{U}', \bigsqcap\mathscr{V}'\right) \in \mathscr{R}$, so $(\mathscr{U}',\mathscr{V}') \in I_F(\mathscr{R})$.

We show that $F : \mathsf{Apx}_{\rho\to\sigma} \to \mathsf{Apx}_{F(\rho)\to F(\sigma)}$ is bijective. To show injectivity, let $F(r) = F(r')$; it is $(U,V) \in r$ if and only if $(\overline{U},\overline{V}) \in F(r)$ (by the definition of F), if and only if $(\overline{U},\overline{V}) \in F(r')$, if and only if $(U,V) \in r'$ (by the definition again), so $r = r'$. To show surjectivity, let $\mathscr{R} \in \mathsf{Apx}_{F(\rho)\to F(\sigma)}$; set

$$(U,V) \in r := (\overline{U},\overline{V}) \in \mathscr{R};$$

it is straightforward to check that $r \in \mathsf{Apx}_{\rho\to\sigma}$ and $F(r) = \mathscr{R}$.

We show finally that $I_F : \mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'} \to \mathsf{Apx}_{I_F(\mathscr{T})\to I_F(\mathscr{T}')}$ is bijective. To show injectivity, let $I_F(\mathscr{R}) = I_F(\mathscr{R}')$ and $(U,V) \in \mathscr{R}$; since $U = \bigsqcap\{U\}$ and $V = \bigsqcap\{V\}$, it is $(\{U\},\{V\}) \in I_F(\mathscr{R})$ by the definition of I_F; by the assumption we get equivalently that $(\{U\},\{V\}) \in I_F(\mathscr{R}')$, so $(U,V) \in \mathscr{R}'$, and $\mathscr{R} = \mathscr{R}'$. To show surjectivity, let $r \in \mathsf{Apx}_{I_F(\mathscr{T})\to I_F(\mathscr{T}')}$ and set

$$(U,V) \in \mathscr{R} := \exists_{\mathscr{U}\in\mathsf{Con}_{I_F(\mathscr{T})}} \exists_{\mathscr{V}\in\mathsf{Con}_{I_F(\mathscr{T}')}} \left(U = \bigsqcap\mathscr{U} \wedge V = \bigsqcap\mathscr{V} \wedge (\mathscr{U},\mathscr{V}) \in r\right).$$

It is $\mathscr{R} \in \mathsf{Apx}_{\mathscr{T}\to\mathscr{T}'}$:

- by $r \in \mathsf{Apx}_{I_F(\mathscr{T})\to I_F(\mathscr{T}')}$ we get $(\varnothing,\varnothing) \in r$, which yields $(\top,\top') \in \mathscr{R}$;

- let $(U,V_1),(U,V_2) \in \mathscr{R}$; then there exist $\mathscr{U}_i$ and $\mathscr{V}_i$ such that, for each $i = 1,2$, $U = \bigsqcap \mathscr{U}_i$, $V_i = \bigsqcap \mathscr{V}_i$, and $(\mathscr{U}_i, \mathscr{V}_i) \in r$; since $\mathscr{U}_1 \cup \mathscr{U}_2 \in \mathsf{Con}_{I_F(\mathscr{T})}$ because $\bigsqcap(\mathscr{U}_1 \cup \mathscr{U}_2) = \bigsqcap \mathscr{U}_1 \sqcap \bigsqcap \mathscr{U}_2 = U \in N_{\mathscr{T}}$, and since $\mathscr{U}_1 \cup \mathscr{U}_2 \vdash \mathscr{U}_i$, we get that $(\mathscr{U}_1 \cup \mathscr{U}_2, \mathscr{V}_i) \in r$; this in turn implies that $(\mathscr{U}_1 \cup \mathscr{U}_2, \mathscr{V}_1 \cup \mathscr{V}_2) \in r$; since, as we saw, $\bigsqcap(\mathscr{U}_1 \cup \mathscr{U}_2) = U$, and similarly $\bigsqcap(\mathscr{V}_1 \cup \mathscr{V}_2) = \bigsqcap \mathscr{V}_1 \sqcap \bigsqcap \mathscr{V}_2 = V_1 \sqcap V_2$, by the definition of $\mathscr{R}$ we get $(U, V_1 \sqcap V_2) \in \mathscr{R}$;
- let $U' \sqsubseteq U$, $(U,V) \in \mathscr{R}$, and $V \sqsubseteq V'$; then there exist $\mathscr{U}$, $\mathscr{V}$ such that $U = \bigsqcap \mathscr{U}$, $V = \bigsqcap \mathscr{V}$, and $(\mathscr{U}, \mathscr{V}) \in r$; on the one hand $U' \sqsubseteq U$ implies $\{U'\} \vdash_{I_F(\mathscr{T})} U$, so $\{U'\} \vdash_{I_F(\mathscr{T})} \mathscr{U}$ by transitivity in $I_F(\mathscr{T})$, and on the other hand $V \sqsubseteq V'$ implies $\{V\} \vdash_{I_F(\mathscr{T}')} V'$, so $\mathscr{V} \vdash_{I_F(\mathscr{T}')} V'$ by transitivity in $I_F(\mathscr{T}')$; it follows that $(\{U'\},\{V'\}) \in r$, so $(U',V') \in \mathscr{R}$, by the definition.

Finally, by the definitions we have that $(\mathscr{U}, \mathscr{V}) \in I_F(\mathscr{R})$ if and only if $(\bigsqcap \mathscr{U}, \bigsqcap \mathscr{V}) \in \mathscr{R}$, if and only if $(\mathscr{U}, \mathscr{V}) \in r$, which means that $I_F(\mathscr{R}) = r$. □

The categories ISys and SEFTop

Category theory is arguably the most appropriate framework to support the discussion of statements like "information systems and Scott–Eshov formal topologies are essentially the same", and our presentation certainly would suggest a reformulation in this direction. We show here how to do this.[7]

Write **ISys** for the category of information systems with approximable maps and **SEFTop** for the category of Scott–Ershov formal topologies with approximable maps. We use the well-known characterization of equivalence via fullness, faithfulness, and "essential" surjectivity of a functor (as is found for example in [10, section IV.4]).

Theorem 14. *The categories* **ISys** *and* **SEFTop** *are equivalent.*

Proof. The mapping $I_F : \mathbf{SEFTop} \to \mathbf{ISys}$, consisting of the object-mapping and the synonymous arrow-mapping defined above, is easily seen to be a *functor*, that is, to preserve identities and compositions. In the proof of Proposition 13, it is shown that this functor is *full* and *faithful* (the surjectivity and the injectivity property respectively for arbitrary topologies $\mathscr{T}$ and $\mathscr{T}'$). Finally, the functor is *"essentially" surjective*, in the sense that for each information system $\rho \in \mathbf{ISys}_{obj}$ there exists a Scott–Ershov formal topology $\mathscr{T}_\rho \in \mathbf{SEFTop}_{obj}$, such that $I_F(\mathscr{T}_\rho)$ and ρ are *isomorphic*: indeed, set $\mathscr{T}_\rho := F(\rho)$; the information systems ρ and $I_F(F(\rho))$ are isomorphic, since for $r \in \mathsf{Apx}_{\rho \to I_F(F(\rho))}$ with

$$(U, \{\overline{V_1}, \ldots, \overline{V_n}\}) \in r \text{ if and only if } U \vdash_\rho \bigcup_{j=1}^{n} V_j,$$

and $s \in \mathsf{Apx}_{I_F(F(\rho)) \to \rho}$ with

$$(\{\overline{U_1}, \ldots, \overline{U_m}\}, V) \in s \text{ if and only if } \bigcup_{i=1}^{m} U_i \vdash_\rho V,$$

it is $r \circ s = \mathrm{id}_{I_F(F(\rho))}$ and $s \circ r = \mathrm{id}_\rho$; using the easy fact that $\overline{\overline{U} \cup \overline{V}} = \overline{U \cup V}$, the above are direct to check, and the claim follows. □

[7] The categorical equivalences between information systems and domains or precusl's are well known and covered in the literature, a textbook exposition of both being found for example in [26].

Coherent Scott–Ershov formal topologies

Call a Scott–Ershov formal topology *coherent* if, for a finite collection $\mathscr{U} \subseteq^f N$,

$$\bigsqcap \mathscr{U} \in N \leftrightarrow \forall_{U,V \in \mathscr{U}} U \sqcap V \in N. \tag{5}$$

Theorem 15. *If $\mathscr{T}$ is a coherent Scott–Ershov formal topology then $I_F(\mathscr{T})$ is a coherent information system. Conversely, if ρ is a coherent information system then $F(\rho)$ is a coherent Scott–Ershov formal topology.*

Proof. Suppose first that $\mathscr{T}$ is coherent, and let $\mathscr{U} \in \mathsf{Con}_{I_F(\mathscr{T})}$. By definition it is $\mathscr{U} \subseteq^f N$ and $\bigsqcap \mathscr{U} \in N$; the latter, by (5), holds if and only if for all $U, V \in \mathscr{U}$ it is $U \sqcap V \in N$, which is by definition $\{U, V\} \in \mathsf{Con}_{I_F(\mathscr{T})}$, so $I_F(\mathscr{T})$ is a coherent information system.

Now suppose that ρ is a coherent information system, that is, such that

$$U \in \mathsf{Con} \leftrightarrow \forall_{a,b \in U} \{a, b\} \in \mathsf{Con} \tag{4}$$

for all $U \subseteq^f \mathsf{Tok}$. Let $\mathscr{U} \subseteq^f N_{F(\rho)}$, that is, $\mathscr{U} \subseteq^f \overline{\mathsf{Con}}$. Assume that $\bigsqcap \mathscr{U} \in N_{F(\rho)}$, that is, $\bigcup \mathscr{U} \in \overline{\mathsf{Con}}$, and let $\overline{U}, \overline{V} \in \mathscr{U}$; the assumption implies $\overline{U} \cup \overline{V} \in \mathsf{Con}$, so $U \cup V \in \mathsf{Con}$, and then $\overline{U \cup V} \in \overline{\mathsf{Con}}$, hence $\overline{U} \sqcap \overline{V} \in N_{F(\rho)}$ by definition. Conversely, assume that $\mathscr{U} \subseteq^f N_{F(\rho)}$ and that for all $\overline{U}, \overline{V} \in \mathscr{U}$ it is $\overline{U} \sqcap \overline{V} \in N_{F(\rho)}$, that is, $\overline{U} \sqcap \overline{V} \in \overline{\mathsf{Con}}$; now, for arbitrary U and V, $\overline{U} \sqcap \overline{V} \in \overline{\mathsf{Con}}$ is equivalent to $\{a, b\} \in \mathsf{Con}$ for all $a, b \in \overline{U \cup V}$ by (4), so we get that $\overline{\{a, b\}} \in \overline{\mathsf{Con}}$ for all $a, b \in \bigcup \mathscr{U}$; by (4) again, we have that $\bigcup \mathscr{U} \in \overline{\mathsf{Con}}$, which is what $\bigsqcap \mathscr{U} \in N_{F(\rho)}$ means by definition. So $F(\rho)$ is indeed a coherent Scott–Ershov formal topology. □

5 Notes

On the notion of atomicity

The defining property of a *unary* formal topology in section 4.3 looks similar to the atomicity property for an information system in section 3—in fact, unary formal topologies are called "atomic" by Erik Palmgren in a preliminary version of [15]—but the two are not essentially related from our viewpoint. The property of being unary for a formal topology expresses *atomicity of compact covering*, whereas in information systems we have *atomicity of information flow* (that is, of *entailment*): in the first case, an "atom" would be a formal basic open while in the second case, an atom (that is, a token) represents a simple piece of data. In order to avoid confusions, one should notice how the transition from an information system to a point-free structure—domains included—involves jumping from the level of atomic pieces of data to (finitely determined) *sets* of atomic pieces of data: atomicity of information appears in the presence of atomic pieces of data, which become indiscernible when one moves to a point-free setting (see however the last note).

As already mentioned, atomicity of entailment has been studied by various authors in the past. Guo-Qiang Zhang [30] studies a special case of atomic information systems which he calls *prime information systems*, to represent dI-domains; he seems to have adopted atomicity from Glynn Winskel's *prime event structures* [27]. More generally, atomic information systems (in a different but equivalent axiomatization) are used by Antonio Bucciarelli *et al.* [4] to provide a model for intuitionistic linear logic.

On the absence of positivity

In section 4.3 we associated information systems to a version of formal topologies, where the positivity predicate is absent. The intuitive meaning of positivity in this context should be taken as *inhabitation*: a neighborhood is positive when it contains a point, that is, when it can be extended to an ideal. This is clearly an important and necessary concept to raise in a general constructive setting. Our restricted setting though fulfills inhabitation by design, since $\overline{U}$, defined for every consistent set U, is an ideal; to suppress positivity is merely an Occam's razor choice, reflecting our predilection for the barest possible setting within which to achieve the wanted connection *starting from information systems*. However, as Giovanni Sambin has suggested (private communication [20]), starting from formal topologies with positivity we may seek to establish similar connections, in particular with information systems as well as with appropriate *positivity semilattices* (that is, semilattices equipped with a monotone positivity predicate) replacing precusl's.

Nomenclature discrepancies

Antonio Bucciarelli *et al.* [4] call atomic information systems *linear information systems*. Giovanni Sambin *et al.* [21] call consistently complete ordered sets *coherent*. Sara Negri [14] says *Scott formal topology* for a unary formal topology and *consistently complete Scott formal topology* for a Scott–Ershov formal topology (modulo the presence of a top formal basic open). Viggo Stoltenberg-Hansen *et al.* [26] say *formal space* for a consistently complete formal topology and *Scott space* for a Scott–Ershov formal topology.

Outlook

The issue of linking the theory of information systems and formal topology has many facets, at least as many as the various point-free structures that are currently studied by the community. Apart from the ones that we have covered in this chapter, further correspondences may be asked of various other settings, from the *event structures* of [27] to the *apartness spaces* of [3], and also with other versions of formal topologies, ones which accommodate the positivity predicate for instance (see note above).

From a categorical perspective, in particular, we spelled out the equivalence of information systems and Scott–Ershov formal topologies in Theorem 14. One could carry this viewpoint further, and interpret Theorem 15 as expressing the intuitive fact that *the functors I_F and F between* **ISys** *and* **SEFTop** *preserve coherence.* One way to make this precise could be to delineate an appropriate sort of *bicategory* of general "coherent structures", among whose objects we should have **ISys** and **SEFTop** (as well as the corresponding categories of domains and precusl's), and whose arrows would be coherence-preserving functors like I_F and F.

Apart from coherence, there still remains the issue of a point-free formulation of atomicity. While atomic information systems have been studied domain-theoretically to some extend (see the first note), to the knowledge of the author they have not yet been studied in a pure formal topological setting. Further study in this direction would make our cartography more complete.

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