Intermittency in the Parabolic Anderson Model with Catalyst

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Chapter 1

Introduction

1.1 The parabolic Anderson problem and its interpretation

The main object of our investigation is the solution $u\colon \mathbb{R}^+ \times \mathbb{Z}^d \to \mathbb{R}^+$ to the Cauchy problem for the heat equation with random time-dependent potential

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \kappa\Delta u(t,x) + \xi(t,x)\, u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d; \\ u(0,x) = 1, & x \in \mathbb{Z}^d. \end{cases} \tag{1.1}$$

Here, $\kappa \in \mathbb{R}^+$ is a diffusion constant and Δ is the discrete Laplacian, acting on $f\colon \mathbb{Z}^d \to \mathbb{R}$ as

$$\Delta f(x) = \sum_{y\sim x} \left[f(y) - f(x)\right],$$

while

$$\xi = \left\{\xi(t,x)_{t\geq 0} \,|\, x \in \mathbb{Z}^d\right\}$$

is an $\mathbb{R}$-valued random field evolving over time that "drives" the equation. Problem (1.1) is referred to as the *parabolic Anderson model*. It is the parabolic analogue of the Schrödinger equation with a random potential.

A popular interpretation arises from population dynamics. The function $u(t,x)$ describes the mean number of particles present at x at time t when starting with one particle per site. Particles perform independent random walks on $\mathbb{Z}^d$ with jump rate $2d\kappa$ and split into two at rate ξ if $\xi > 0$ (source) or die at rate $-\xi$ if $\xi < 0$ (sink). We outline this interpretation in the next Chapter 2.

If ξ is a nonnegative field, we can also interpret problem (1.1) as a linearized model of chemical reactions. In this case, the solution of the equation describes the evolution of reactants u under the influence of a catalyst medium ξ.

The third interpretation of the model stems from evolution theory. Denote by $\mathbb{Z}^d$ the space of phenotypes and by $v(t,x)$ the number of individuals of phenotype x at time t. Then

Δ represents mutation, whereas $\xi(t,x)$ is the fitness of individuals with phenotype x. We impose the constraint that the total number of individuals is finite and constant over time. Within this context, the *Fisher-Eigen equation of population genetics* reads

$$\frac{\partial v}{\partial t}(t,x) = \kappa\Delta v(t,x) + \left(\xi(t,x) - \langle \xi(t,\cdot)\rangle_v\right) v(t,x),$$

where

$$\langle \xi(t,\cdot)\rangle_v := \frac{\sum_x \xi(t,x)v(t,x)}{\sum_x v(t,x)}$$

is the average fitness of the population $v(t,\cdot)$.

Historically, the Anderson model is derived from solid states physics. The movement of an electron u in a disordered environment ξ can be described by the discrete Schrödinger equation

$$i\hbar\frac{\partial u}{\partial t}(t,x) = \left(-\hbar^2\Delta + \xi(x)\right) u(t,x).$$

This is the original *time-independent* Anderson model. The evolution of the potential $\xi(x)$ is independent of time.

Characteristically for the parabolic Anderson model, the two terms on the right hand side of equation (1.1) compete with each other. The diffusion induced by Δ tends to make u flat whereas ξ tends to make u bumpy. In the context of population dynamics, there is a competition between particles spreading out by diffusion and particles clumping around sources.

Studying problem (1.1), we distinguish between the *quenched* setting which describes the almost sure behaviour of u conditioned on ξ, and the *annealed* setting, where we average over ξ.

The theory currently available for the time-dependent model covers various forms of the potential ξ. In the present paper we consider the case where ξ has the form

$$\xi(t,x) = \gamma\delta_{Y_t}(x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d, \tag{1.2}$$

and $(Y_t)_{t\geq 0}$ is a random walk with generator $\rho\Delta$ starting at the origin 0. The corresponding expectation will be denoted by $\langle\cdot\rangle$. The parameter $\rho \in [0,\infty)$ is the diffusion constant of the catalyst, whereas $\gamma \in (0,\infty)$ indicates the rate of splitting. In the context of chemical reactions, we can interpret ξ as a single catalyst particle, performing a random walk in $\mathbb{Z}^d$ with jump rate $2d\rho$. Reactants split into two at a rate $\gamma > 0$ if they are at the same lattice site as the catalyst.

This overview is inspired by Gärtner and den Hollander [4] and König [8]. For a general discussion of the parabolic Anderson model with applications, the reader is referred to the survey paper by Gärtner and König [5]. The interpretation as a linearized model of chemical kinetics is outlined in Gärtner and Molchanov [6], Section 1.2.

1.2 The Feynman-Kac formula

Our main tool for the analysis of the solution to the parabolic Anderson problem (1.1) is the Feynman-Kac formula.

Proposition 1.1 (Feynman-Kac formula)
Let $\xi\colon [0,\infty) \times \mathbb{Z}^d \to \mathbb{R}$ be a bounded function which is piecewise constant in the first variable and let $u_0\colon \mathbb{Z}^d \to \mathbb{R}^+$ be an arbitrary bounded and nonnegative function. Then the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \kappa\Delta u(t,x) + \xi(t,x)\, u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d, \\ u(0,x) = u_0, & x \in \mathbb{Z}^d, \end{cases}$$

has a unique nonnegative solution, given by

$$u(t,x) = \mathbb{E}_x^X \exp\left\{ \int_0^t \xi(t-s, X_s)\, ds \right\} u_0(X_t), \tag{1.3}$$

where $(X_s)_{s\geq 0}$ is a random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$ and corresponding expectation $\mathbb{E}^X$.

This is a standard result for a time-independent random medium $\xi(x)$. In Appendix A we derive the statement for our case, where the potential $\xi(t,x)$ has the form (1.2).

1.3 Lyapunov exponents and intermittency

The aim of the present thesis is to study the annealed asymptotics of the solution u, i.e. the asymptotics of the moments of the solution. More precisely, we study the exponential growth rates of the moments $\langle u(t,0)^p \rangle$. Therefore, we introduce the Lyapunov exponents corresponding to problem (1.1).

Definition 1.2 (Lyapunov exponent)
For $p \in \mathbb{N}$, the limit

$$\lambda_p := \lim_{t\to\infty} \frac{1}{t} \log\langle u(t,x)^p \rangle \tag{1.4}$$

is called the *p-th moment Lyapunov exponent* of the solution u to the parabolic Anderson problem (1.1).

We will see that the solution u of problem (1.1) with potential (1.2) possesses Lyapunov exponents of all orders. It will be shown in Theorem 4.1 that the limit (1.4) exists and that it is independent of x.

With the help of Lyapunov exponents we can give a definition for intermittency behaviour of the system (1.1).

Definition 1.3 (Intermittency)
For $p \in \mathbb{N} \setminus \{1\}$, we call the parabolic Anderson problem (1.1) to be *p-intermittent*, if the Lyapunov exponents satisfy

$$\frac{\lambda_{p-1}}{p-1} < \frac{\lambda_p}{p}. \tag{1.5}$$

We say the system shows *full intermittency*, if the system is p-intermittent for all $p \in \mathbb{N}\setminus\{1\}$.

So far there exists no rigorous mathematical definition for intermittency. The above definition is very much in the spirit of [1] and [6]. Generally, intermittency corresponds to a very irregular behaviour of the solution u. However, in the case of a stationary random field ξ, intermittency corresponds to the fact that there are some small, but more and more widely spaced *peaks* absorbing the total mass of the solution u. See Gärtner and Molchanov [6], Section 1.1, for a detailed interpretation of intermittency in this case.

For our model in (1.1), we will see that always $\lambda_{p-1}/(p-1) \leq \lambda_p/p$ and p-intermittency implies q-intermittency for all $q > p$. We will find qualitatively different intermittency behaviour in dimension $d = 1, 2$ on the one hand and $d \geq 3$ on the other hand.

1.4 Summary of the results

Given $p \in \mathbb{N}$, let B^p denote the operator in $\ell^2(\mathbb{Z}^{pd})$ given by

$$B^p f(x_1, \dots, x_p) = \sum_{\substack{v \in \mathbb{Z}^d \\ |v|=1}} [f(x_1+v, \dots, x_p+v) - f(x_1, \dots, x_p)], \quad f \in \ell^2(\mathbb{Z}^{pd}),\ x_1, \dots, x_p \in \mathbb{Z}^d,$$

and introduce the Hamilton operator

$$\mathcal{H}^p := \kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho B^p + \gamma\delta_0^{(1)} + \cdots + \gamma\delta_0^{(p)} \tag{1.6}$$

on $\ell^2(\mathbb{Z}^{pd})$. Here, Δ_i is the discrete Laplacian acting on the i-th argument $(i = 1, \dots, p)$ and $\delta_0^{(i)}(x_1, \dots, x_p) = 1$ if $x_i = 0$ and 0 else. Note that $B^1 = \Delta$.

We consider the model in (1.1). Our main result in the first part of this paper is Theorem 4.1. It states that the p-th moment Lyapunov exponent exists and is equal to the upper boundary of the ℓ^2-spectrum of the operator $\mathcal{H}^p$,

$$\lambda_p = \lim_{t\to\infty} \frac{1}{t} \log\langle u(t,x)^p\rangle = \sup \mathrm{Sp}(\mathcal{H}^p). \tag{1.7}$$

In the second part of the present paper, we analyse the spectrum $\mathrm{Sp}(\mathcal{H}^p)$ and derive properties of λ_p as a function of the parameters κ, ρ and γ. The case $p = 1$ can be solved completely by computing the spectrum of

$$\mathcal{H}^1 = (\kappa + \rho)\Delta + \gamma\delta_0. \tag{1.8}$$

In Theorem 5.2 we show that in dimension $d = 1, 2$ always $\lambda_1 > 0$, whereas in dimension $d \geq 3$, there exists a critical constant $r_d > 0$ such that $\lambda_1 > 0$ if and only if $\gamma/(\kappa+\rho) > r_d$. We obtain r_d as the reciprocal $G_d(0)^{-1}$ of the Green's function associated with Δ in the origin with exponential stopping parameter 0.

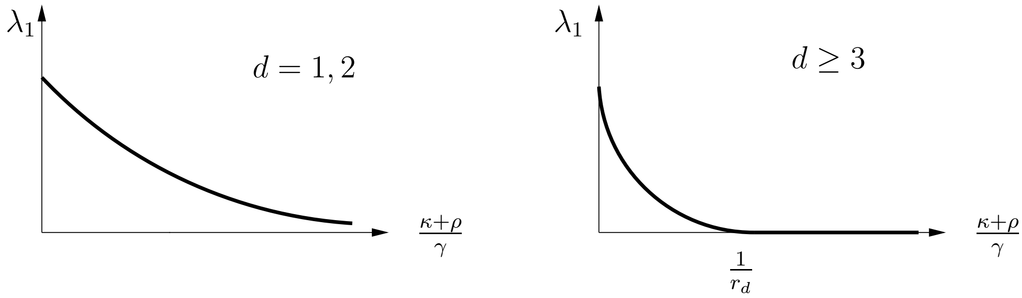

Figure 1.1: The qualitative behaviour of λ_1

We will see that, for $\lambda_1 > 0$ and $\rho > 0$, the system shows full intermittency, see Lemma 6.8. Therefore, in dimension $d = 1, 2$, the system shows full intermittency for all $\kappa \in [0, \infty)$, $\rho, \gamma \in (0, \infty)$.

In dimension $d \geq 3$, the system behaves differently. It will be shown in Theorem 6.5 that the function $\lambda_p = \lambda_p(\kappa, \rho)$, $\kappa, \rho \in [0, \infty)$, is continuous, convex, monotonically decreasing in both arguments and vanishes for $\kappa \geq \gamma/r_d$.

Consider the case $\rho = 0$, i.e., the catalyst is fixed to its starting position 0. Then the random field ξ is time-independent. We show in Lemma 6.1 that, for all $p \in \mathbb{N}$,

$$\frac{\lambda_p(\kappa, 0)}{p} = \lambda_1(\kappa, 0), \qquad \kappa \in [0, \infty), \tag{1.9}$$

and define

$$\bar{\lambda}(\kappa) := \lambda_1(\kappa, 0). \tag{1.10}$$

Consequently, for $\rho = 0$, the system is not intermittent for any $p \in \mathbb{N}$.

Furthermore, we show in Theorem 6.4 that, for $\rho > 0$, $\bar{\lambda}$ is an upper bound for λ_p/p and, for each $\kappa, \rho \in [0, \infty)$,

$$\frac{\lambda_p(\kappa, \rho)}{p} \nearrow \bar{\lambda}(\kappa) \qquad \text{as } p \nearrow \infty. \tag{1.11}$$

This will be used in Theorem 6.7 in order to show that, for $\kappa r_d < \gamma$ and $\rho > 0$, the system is p-intermittent for some $p \in \mathbb{N} \setminus \{1\}$.

Finally, consider the case $\kappa = 0$. In the context of chemical kinetics, we can interpret this constraint as fixed reactants, waiting for the catalyst passing by. Writing $\lambda_p = \lambda_p(\kappa, \rho, \gamma)$, we prove in Lemma 6.6 the identity

$$\lambda_p(0, \rho, \gamma) = \lambda_1(0, \rho, p\gamma).$$

Hence, $\lambda_p(0, \rho, \gamma) > 0$ if and only if $p\gamma > r_d\rho$.

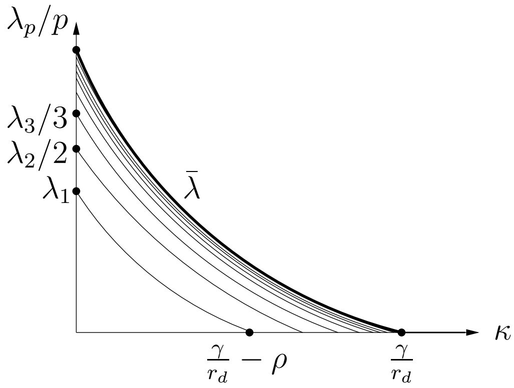

Figure 1.2: The asymptotic behaviour of λ_p as $p \to \infty$ in dimension $d \geq 3$

1.5 Structure of the present work

The present paper is structured as follows.

In Chapter 2 we develop a heuristic approach to the parabolic Anderson model justifying the population dynamics interpretation.

In Chapter 3 we prove some statements on random walks. These cover properties of simple symmetric random walks on $\mathbb{Z}^d$ as well as the transformation of random walks. They form a basis for the proofs in Chapter 4.

In the next Chapter 4 we prove the existence of the Lyapunov exponent λ_p and show that it is equal to the upper boundary of the ℓ^2-spectrum of the Hamiltonian $\mathcal{H}^p$.

The analysis of the spectrum of the operator $\mathcal{H}^1$ is the content of Chapter 5. This yields a representation for the first order Lyapunov exponent λ_1.

In Chapter 6 we examine the asymptotic behaviour of higher moments of the solution to the parabolic Anderson model and prove statements on the intermittency of the system (1.1).

Finally, the appendix contains a derivation of the Feynman-Kac formula for the time-dependent model and lists some theorems cited from other works.

1.6 Related work

There exists a wide variety of papers on the parabolic Anderson model with *time-independent* random field ξ, see Gärtner and König [5]. The theory for the *time-dependent* parabolic Anderson model is less developed, [4] and [1] have been specifically important for the present work.

In the article [4], Gärtner and den Hollander consider the situation where the potential ξ

is given by

$$\xi(t,x) = \gamma \sum_k \delta_0(x - Y_k(t)), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d,$$

with $\{Y_k(t); t \geq 0, k \in \mathbb{N}\}$ being a collection of independent random walks with generator $\rho\Delta$ starting from a Poisson random field with intensity $\nu \in \mathbb{R}^+$. The system describes the interaction of two types of particles. Catalyst particles have diffusion constant ρ and initial intensity ν, whereas reactant particles have diffusion constant κ and initial intensity 1. Then $u(t,x)$ is the average number of reactant particles at site x at time t conditional on the evolution of the catalyst particles.

Gärtner and den Hollander consider the quantity

$$\Lambda_p(t) = \frac{1}{t} \log \left(\mathrm{e}^{-\nu\gamma t} \langle u(0,t)^p \rangle^{1/p} \right)$$

and define the Lyapunov exponents

$$\lambda_p^* = \lim_{t\to\infty} \frac{1}{t} \log \Lambda_p(t),$$

$$\lambda_p = \lim_{t\to\infty} \Lambda_p(t).$$

They say that, for $p \in \mathbb{N}$, the system is
(a) *strongly catalytic* if $\lambda_p^* > 0$ and
(b) *weakly catalytic* if $\lambda_p^* = 0$.
Furthermore, for $p \in \mathbb{N} \setminus \{1\}$, the system is
(a) *strongly p-intermittent* if $\lambda_p > \lambda_{p-1}$ and
(b) *weakly p-intermittent* if $\lambda_p = \lambda_{p-1}$.
They show that λ_p^* equals the upper boundary of the ℓ^2-spectrum of the operator

$$\rho\Delta + p\gamma\delta_0.$$

Consequently, the system is always strongly p-catalytic in dimension $d = 1, 2$, while in dimension $d \geq 3$ it is strongly p-catalytic if and only if $p\gamma/\rho > r_d$. If the system is strongly p-catalytic, then it is also p-intermittent and $\lambda_p = \infty$.
Further results cover the weakly p-catalytic regime and the κ-dependence of λ_p. If $d \geq 3$ and $0 < p\gamma/\rho = r_d$, then the limit λ_p exists and is infinite for any choice of parameters. If $d \geq 3$ and $0 < p\gamma/\rho < r_d$, then the limit λ_p exists, is finite and satisfies that, on $[0, \infty)$, the mapping $\kappa \mapsto \lambda_p(\kappa)$ is continuous, strictly decreasing, convex and

$$\lambda_p(0) = \nu\gamma \frac{1}{\frac{\rho r_d}{p\gamma} - 1}.$$

Finally, they show that, in dimension $d \geq 3$, $\kappa\lambda_p(\kappa)$ converges to a finite positive value. In contrast to λ^*, it was not possible to identify λ_p with the spectrum of an operator. This leads to rather complicated expressions when analysing the κ-dependence of $\lambda_p(\kappa)$.

The monograph by Carmona and Molchanov [1] provides a complete analysis of the growth rates of successive moments of $u(t,0)$ averaged over ξ in the case of a white noise potential, where

$$\xi(t,x) = \gamma \dot{W}_x(t), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d,$$

with $\left\{(W_x(t))_{t\geq 0} \,|\, x \in \mathbb{Z}^d\right\}$ being a collection of independent Brownian motions. They consider the function

$$m_p(t, x_1, \ldots, x_p) := \langle u(t,x_1) \cdots u(t,x_p) \rangle$$

and define the *Lyapunov exponent*

$$\lambda_p := \lim_{t\to\infty} \frac{1}{t} \log m_p(t).$$

Obviously, λ_p is independent of the spatial variable x. They show that $m_p(t)$ satisfies the differential equation

$$\frac{\partial m_p}{\partial t} = \left[\kappa(\Delta_1 + \cdots + \Delta_p) + V_p\right] m_p$$

on $\ell^2(\mathbb{Z}^{pd})$ with

$$V_p(x_1, \ldots, x_p) = \gamma \sum_{1\leq j<k\leq p} \delta_0(x_j - x_k), \qquad x_1, \ldots, x_p \in \mathbb{Z}^d,$$

and derive that the limit λ_p exists for all $p \in \mathbb{N}$ and

$$\lambda_p = \sup \operatorname{Sp}\left(\kappa(\Delta_1 + \cdots + \Delta_p) + V_p\right).$$

In contrast to our work, they always have $\lambda_1 = 0$, because $V_1 \equiv 0$. Thus, they observe intermittent behaviour (in the sense of Definition 1.3) for all p with $\lambda_p > 0$. Carmona and Molchanov proceed by analysing the κ-dependence of λ_p. They show that $\lambda_p(\kappa)$ is convex and monotonically decreasing for all p. If κ is small, then

$$\frac{\lambda_p}{p} = \frac{p-1}{2} - 2d\kappa + \mathcal{O}(\kappa^2) \qquad \text{as } \kappa \searrow 0.$$

The asymptotics for large values of κ depends on the dimension d. If $p \geq 2$ and $\kappa \to \infty$, then there exist positive constants c_p and d_p with

$$\lambda_p(\kappa) \asymp \frac{c_p}{\kappa} \qquad \text{for } d = 1,$$

$$\log \lambda_p(\kappa) \asymp \frac{d_p}{\kappa} \qquad \text{for } d = 2$$

and, for $d \geq 3$, there exists a constant $\bar{\kappa}_p$, such that

$$\lambda_p(\kappa) = 0 \qquad \text{for all } \kappa \geq \bar{\kappa}_p.$$

The notation $a_p \asymp b_p$ means that $0 < c_1 < a_p/b_p < c_2 < \infty$ for some constants c_1 and c_2 and p large enough.

Our model in (1.1) is similar to the situation discussed by Gärtner and den Hollander, but the methods we use are more related to the approach of Carmona and Molchanov. Their analysis is triggered by the disturbed potential V_p, whereas in our model, we have disturbances of the jump term caused by B^p.

1.7 Open problems and extensions of the model

In this section we list some questions and open problems not covered in the present paper.

1° For $p \in \mathbb{N}$, let

$$\kappa_{p,\mathrm{cr}} := \inf\left\{\kappa \geq 0 \middle| \lambda_p(\kappa) = 0\right\}$$

denote the critical value for κ, where $\lambda_p(\kappa)$ hits the horizontal axis. It is clear from our results in Chapter 6 that

$$\kappa_{p,\mathrm{cr}} \nearrow \gamma/r_d \qquad \text{as } p \nearrow \infty.$$

In the present thesis we were not able to show that $\kappa_{p,\mathrm{cr}}$ is *strictly* monotonically increasing in p. But we conjecture $\kappa_{1,\mathrm{cr}} \neq \kappa_{2,\mathrm{cr}}$, which, by Lemma 6.1, is satisfied if there exists a corresponding Eigenfunction for $\mathcal{H}^1$ at $(\kappa + \rho)\gamma = r_d$.

2° Next, after having studied the time evolution of the moments of the solution u, it seems natural to ask about the almost sure behaviour of u. More precisely, it is an open problem whether the *almost sure* Lyapunov exponent

$$\lim_{t\to\infty} \frac{1}{t} \log u(t,x)^p$$

exists and what his properties are as a function of the parameters κ, ρ and γ.

3° In the present work we investigate the asymptotic behaviour of the *mean* number of particles u. As a next step one can ask about the evolution of the *exact* number of particles η. In this case the following problem arises: Instead of a single differential equation, we get a system of infinitely many connected differential equations.

4° Finally, one can extend the setting to a multiple catalyst model. Let us assume that the system has a finite number n of catalyst particles and the potential ξ has the form

$$\xi(t,x) = \gamma \sum_{i=1}^{n} \delta_0\left(x - Y_t^{(i)}\right), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d,$$

with $Y^{(1)}, \ldots, Y^{(n)}$ being a collection of n independent random walks with generator $\rho\Delta$. The degenerate cases $\kappa = 0$ and $\rho = 0$ can be solved easily, but the general case is more complex than the single catalyst setting. However, with the Feynman-Kac formula one can see that the first order Lyapunov exponent with n catalysts $\lambda_1^{(n)}$ satisfies the equation

$$\lambda_1^{(n)}(\kappa,\rho,\gamma) = \lambda_n^{(1)}(\rho,\kappa,\gamma).$$

Note that the roles of κ and ρ are exchanged. We conjecture that there exists again an operator replacing the role of $\mathcal{H}^p$ in our work.

Chapter 2

Branching Processes in Time-Dependent Random Medium

In this chapter we develop an intuition for the correctness of the population dynamics interpretation of the parabolic Anderson model. The reader may skip this chapter and proceed to Chapter 3, if he is only interested in the analysis of the solution.

We denote by $\eta(t,x)$ the number of particles in $x \in \mathbb{Z}^d$ at time $t \in \mathbb{R}^+$. In our model, the time evolution of $\eta(t,\cdot)$ is governed by the following rules:

- at time $t = 0$, each lattice site is occupied by one particle;
- particles act independently of each other;
- particles jump from x to y at rate $2d\kappa$, if $x \sim y$, i.e., $\|x-y\| = 1$;
- particles split into two at rate $\xi(t,x)$.

Then $(\eta(t,\cdot), \mathbb{P}_{\eta_0})$ describes a Markov process on $\mathbb{N}^{\mathbb{Z}^d}$, where $\mathbb{P}_{\eta_0}$ is the corresponding probability measure. Set

$$u(t,x) := \mathbb{E}\,\eta(t,x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d, \tag{2.1}$$

where $\mathbb{E} = \mathbb{E}_{\eta_0}$ for $\eta_0(\cdot) \equiv 1$. We consider the initial condition

$$u(0,x) = \mathbb{E}\,\eta(0,x) = 1, \qquad x \in \mathbb{Z}^d. \tag{2.2}$$

In other words, we start with one particle per site.

Our aim is to derive (on a heuristic level) a differential equation for u. More precisely, we will show that, together with the initial condition (2.2), u solves the parabolic Anderson problem (1.1). To this end, we use the *log-Laplace method.* By $(\cdot,\cdot)$ we denote the inner product on $\ell^2(\mathbb{Z}^d)$. We fix a time horizon $t > 0$, an initial site $z \in \mathbb{Z}^d$ and a parameter $\lambda > 0$ and consider the solution $v = v_{(z,t,\lambda)}(s,y)\colon [0,t] \times \mathbb{Z}^d \to \mathbb{R}^+$ of

$$\mathbb{E}\,\mathrm{e}^{-(\eta(s,\cdot),v(0,\cdot))} = \mathbb{E}\,\mathrm{e}^{-(\eta(0,\cdot),v(s,\cdot))}, \qquad s \in [0,t], \tag{2.3}$$

and the initial condition

$$v(0,y) = \lambda\delta_z(y), \qquad y \in \mathbb{Z}^d. \tag{2.4}$$

This is a natural approach for modelling spatial branching processes. The descendants of every single particle at time $t = 0$ produce a branching tree (see Figure 2.1). We obtain the total number of particles at site x at time s by summing up the corresponding values of all branching trees. The trees are independent of each other, hence we can multiply the values for distinct trees to justify the ansatz (2.3).

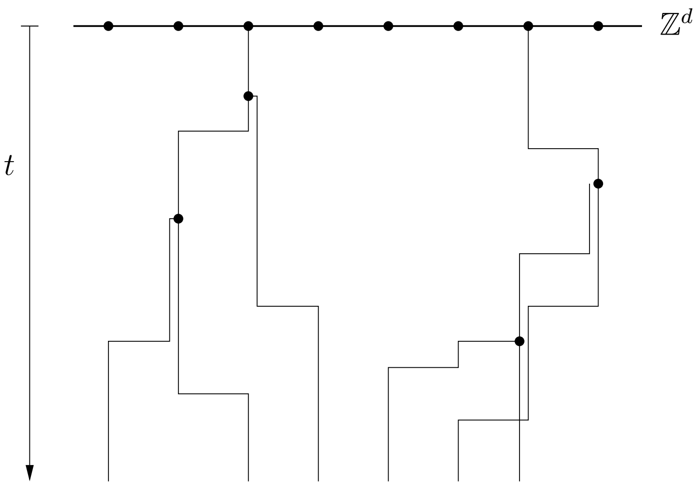

Figure 2.1: Branching trees for single particles

We can derive $u(t,z)$ from $v_{(z,t,\lambda)}$ as follows. Define $w_{(z,t)}(s,y)$ by

$$w_{(z,t)}(s,y) := \frac{\partial}{\partial\lambda} v_{(z,t,\lambda)}(s,y)\bigg|_{\lambda=0}, \tag{2.5}$$

assuming that the limit exists. Differentiation of equation (2.3) with respect to λ yields

$$\mathbb{E}\,\left(\eta(t,\cdot),\delta_z(\cdot)\right)\mathrm{e}^{-(\eta(t,\cdot),\lambda\delta_z(\cdot))} = \mathbb{E}\left(\eta(0,\cdot),\frac{\partial}{\partial\lambda}v(t,\cdot)\right)\mathrm{e}^{-(\eta(0,\cdot),v(t,\cdot))}. \tag{2.6}$$

For $\lambda = 0$, this reduces to the formula

$$\mathbb{E}\,\eta(t,z) = \left(\eta_0, w_{(z,t)}(t,\cdot)\right), \qquad (t,z) \in \mathbb{R}^+ \times \mathbb{Z}^d. \tag{2.7}$$

Combining (2.7), (2.1) and the initial condition (2.2), we get a representation for u:

$$u(t,z) = \sum_{y\in\mathbb{Z}^d} w_{(z,t)}(t,y), \qquad (t,z) \in \mathbb{R}^+ \times \mathbb{Z}^d. \tag{2.8}$$

Our next step is to derive an explicit formula for the function v determined by (2.3). Equation (2.3) is obviously satisfied if

$$\frac{\partial}{\partial s}\mathbb{E}\,\mathrm{e}^{-(\eta(s,\cdot),v(t-s,\cdot))} = 0, \qquad 0 \le s \le t. \tag{2.9}$$

Therefore we try to find a function v satisfying (2.9). In the following we abbreviate $\eta(s)$ and $v(s)$ for $\eta(s,\cdot)$ and $v(s,\cdot)$. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the underlying filtration of the process η. We use the Markov property for η to derive for every function $\varphi \in \mathbb{R}^{\mathbb{Z}^d}$ and all $s \in \mathbb{R}^+$ that

$$\begin{aligned}
\mathbb{E}\,\mathrm{e}^{-(\eta(s+h),\varphi)} &= \mathbb{E}\Big[\mathbb{E}\left[\,\mathrm{e}^{-(\eta(s+h),\varphi)}\middle|\,\mathcal{F}_s\right]\Big] \\
&= \mathbb{E}\Bigg[\sum_{x\in\mathbb{Z}^d} h\,\kappa\,\eta(s,x)\sum_{y\sim x}\mathrm{e}^{-(\eta(s)+\delta_y-\delta_x,\varphi)} \\
&\quad + \sum_{x\in\mathbb{Z}^d} h\,\xi(s,x)\,\eta(s,x)\,\mathrm{e}^{-(\eta(s)+\delta_x,\varphi)} \\
&\quad + \left(1-\sum_{x\in\mathbb{Z}^d} h\,\kappa\,\eta(s,x)-\sum_{x\in\mathbb{Z}^d} h\,\xi(s,x)\,\eta(s,x)\right)\mathrm{e}^{-(\eta(s),\varphi)} \\
&\quad + o(h)\Bigg],
\end{aligned}$$

as $h \searrow 0$. The first line on the right hand side represents a jump from x to y, the second line represents splitting, the third line indicates that nothing happens and the last line represents the occurrence of more than one of these events during the time interval $[s, s+h]$. This yields

$$\begin{aligned}
&\lim_{h\searrow 0}\frac{1}{h}\left(\mathbb{E}\,\mathrm{e}^{-(\eta(s+h),\varphi)}-\mathbb{E}\,\mathrm{e}^{-(\eta(s),\varphi)}\right) \qquad (2.10)\\
=\;& \mathbb{E}\,\mathrm{e}^{-(\eta(s),\varphi)}\Bigg[\underbrace{\sum_{x\in\mathbb{Z}^d}\kappa\,\eta(s,x)\sum_{y\sim x}\left(\mathrm{e}^{-(\varphi(y)-\varphi(x))}-1\right)}_{\text{diffusion}}+\underbrace{\sum_{x\in\mathbb{Z}^d}\eta(s,x)\,\xi(s,x)\,\left(\mathrm{e}^{-\varphi(x)}-1\right)}_{\text{branching}}\Bigg].
\end{aligned}$$

Next we compute

$$\begin{aligned}
\frac{\partial}{\partial s}\,\mathbb{E}\,\mathrm{e}^{-(\eta(s),v(t-s))} &= \lim_{h\searrow 0}\frac{1}{h}\left[\mathbb{E}\,\mathrm{e}^{-(\eta(s+h),v(t-s-h))}-\mathbb{E}\,\mathrm{e}^{-(\eta(s),v(t-s))}\right] \\
&= \lim_{h\searrow 0}\frac{1}{h}\Big[\mathbb{E}\,\mathrm{e}^{-(\eta(s+h),v(t-s-h))}-\mathbb{E}\,\mathrm{e}^{-(\eta(s),v(t-s-h))} \\
&\qquad\quad +\mathbb{E}\,\mathrm{e}^{-(\eta(s),v(t-s-h))}-\mathbb{E}\,\mathrm{e}^{-(\eta(s),v(t-s))}\Big] \\
&= \lim_{h\searrow 0}\frac{1}{h}\left[\mathbb{E}\,\mathrm{e}^{-(\eta(s+h),v(t-s-h))}-\mathbb{E}\,\mathrm{e}^{(\eta(s),v(t-s-h))})\right] \\
&\quad +\left(\eta(s),\frac{\partial v}{\partial s}(t-s)\right)\mathbb{E}\,\mathrm{e}^{-(\eta(s),v(t-s))}.
\end{aligned}$$

We substitute $\varphi = v(t-s)$ into (2.10) and obtain

$$\frac{\partial}{\partial s}\,\mathbb{E}\,\mathrm{e}^{-(\eta(s),v(t-s))} = \mathbb{E}\,\mathrm{e}^{-(\eta(s),v(t-s))}\sum_{x\in\mathbb{Z}^d}\eta(s,x)\Bigg[\kappa\sum_{y\sim x}\left(\mathrm{e}^{-(v(t-s,y)-v(t-s,x))}-1\right) + \xi(s,x)\,\left(\mathrm{e}^{-v(t-s,x)}-1\right)+\frac{\partial v}{\partial s}(t-s,x)\Bigg].$$

But this vanishes if

$$\frac{\partial v}{\partial s}(t-s,x) = -\kappa\sum_{y\sim x}\left(\mathrm{e}^{-(v(t-s,y)-v(t-s,x))}-1\right)-\xi(s,x)\,\left(\mathrm{e}^{-v(t-s,x)}-1\right)$$

for all $x\in\mathbb{Z}^d$ and all $s\in[0,t]$. Hence, we have shown that v solves the Cauchy problem

$$\begin{cases}\frac{\partial v}{\partial s}(s,x) = \kappa\sum_{y\sim x}\left(1-\mathrm{e}^{-(v(s,y)-v(s,x))}\right)+\xi(t-s,x)\,\left(1-\mathrm{e}^{-v(s,x)}\right), & (s,x)\in[0,t]\times\mathbb{Z}^d;\\ v(0,x)=\lambda\delta_z(x), & x\in\mathbb{Z}^d.\end{cases} \tag{2.11}$$

Differentiation of (2.11) with respect to λ yields

$$\frac{\partial}{\partial s}\frac{\partial}{\partial\lambda}v(s,x) = \kappa\sum_{y\sim x}\left(\frac{\partial}{\partial\lambda}v(s,y)-\frac{\partial}{\partial\lambda}v(s,x)\right)+o(1)+\xi(t-s,x)\,\frac{\partial}{\partial\lambda}v(s,x)+o(1),$$

as $\lambda\to 0$. We set $\lambda=0$ and use equation (2.5) to obtain a differential equation for w:

$$\begin{cases}\frac{\partial w}{\partial s}(s,x)=\kappa\Delta w(s,x)+\xi(t-s,x)\,w(s,x), & (s,x)\in[0,t]\times\mathbb{Z}^d;\\ w(0,x)=\delta_z(x), \quad x\in\mathbb{Z}^d.\end{cases} \tag{2.12}$$

We use the Feynman-Kac formula (1.3) to obtain a stochastic representation for $w(t,x)$. Let $(X_s,\mathbb{P}_x)$ denote a random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$, then

$$\begin{aligned} w(t,x) &= \mathbb{E}_x\exp\left\{\int_0^t\xi(s,X_s)\,ds\right\}w(0,X_t)\\ &= \mathbb{E}_x\exp\left\{\int_0^t\xi(s,X_s)\,ds\right\}\delta_z(X_t). \end{aligned} \tag{2.13}$$

A time reversion of X_s on the time interval $[0,t]$ leads to

$$w(t,x)=\mathbb{E}_x\exp\left\{\int_0^t\xi(s,X_s)\,ds\right\}\delta_z(X_t)=\mathbb{E}_z\exp\left\{\int_0^t\xi(t-s,X_s)\,ds\right\}\delta_x(X_t). \tag{2.14}$$

We combine (2.8), (2.13) and (2.14) to obtain an equation for u,

$$\begin{aligned} u(t,z) &= \sum_{x\in\mathbb{Z}^d}w(t,x)=\sum_{x\in\mathbb{Z}^d}\mathbb{E}_z\exp\left\{\int_0^t\xi(t-s,X_s)\,ds\right\}\delta_x(X_t)\\ &= \mathbb{E}_z\exp\left\{\int_0^t\xi(t-s,X_s)\,ds\right\}. \end{aligned} \tag{2.15}$$

We use again the Feynman-Kac formula (1.3) to see that u solves the parabolic Anderson problem (1.1).

Chapter 3

Preliminaries on Random Walks

In this chapter we state some auxiliary results needed for the proofs in Chapter 4 below. Lemma 3.1 and 3.2 deal with features of simple symmetric random walks. The statements are well known, we give the proofs for completeness.

Denote by

$$Q_L := [-L, L]^d \cap \mathbb{Z}^d$$

a centered box with side length 2L and $Q_L^p := Q_L \times \cdots \times Q_L$ the p-th cartesian product of Q_L in $\mathbb{Z}^{pd}$.

Let $(Y_t, \mathbb{P}_y)$ denote a simple symmetric random walk on $\mathbb{Z}^d$ with generator $\rho\Delta$.

Lemma 3.1 (Exit probability from a finite box)
For sufficiently large values of t (depending on ρ and the dimension d),

$$\mathbb{P}_0\left(Y_t \notin Q_{t\log^2 t}\right) \le 2^{d+1} \mathrm{e}^{t\log^2 t}. \tag{3.1}$$

Proof. For $L > 0$, let $\tau(L)$ denote the first exit time of the random walk Y_t from the box Q_L. From Lemma 4.3 in Gärtner and Molchanov [6] we know that for arbitrary $L > 0$ and $t > 0$,

$$\mathbb{P}_0(\tau(L) > t) \le 2^{d+1} \exp\left\{-L \log \frac{L}{d\rho t} + L\right\}, \tag{3.2}$$

which implies

$$\begin{aligned}\mathbb{P}_0\left(Y_t \notin Q_L\right) &\le \mathbb{P}_0\left(\tau\left(L\right) > t\right)\\ &\le 2^{d+1} \exp\left\{-L \log \frac{L}{d\rho t} + L\right\}.\end{aligned}$$

Substituting $L = t\log^2 t$, we get

$$\mathbb{P}_0\left(Y_t \notin Q_{t\log^2 t}\right) \le 2^{d+1} \exp\left\{-t\log^2 t\left[\log\left(\log^2 t\right) - \log\left(\rho d\right)\right] + t\log^2 t\right\}, \tag{3.3}$$

but $\left[\log\left(\log^2 t\right) - \log\left(\rho d\right)\right] \ge 2$ for sufficiently large t, which proves the lemma. ■

Denote by

$$p(t,x) := \mathbb{P}_0(Y_t = x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d,$$

the transition function of the random walk $(Y_t, \mathbb{P}_y)$. Then $p(t,0)$ is the probability that the process is again in its starting point at time t. Due to the spatial shift invariance of simple symmetric random walks,

$$p(t,0) = \mathbb{P}_y\left(Y_t = y\right)$$

for all $y \in \mathbb{Z}^d$.

Lemma 3.2
The transition function $p(t,x)$ of the random walk $(Y_t, \mathbb{P}_y)$ satisfies

(i) *$p(t,0)$ is monotonically decreasing in ρ;*

(ii) *the decay of $p(t,0)$ as $t \to \infty$ is at most polynomial, i.e.,*

$$\liminf_{t\to\infty} \frac{p(t,0)}{(\rho t)^{-d/2}} \geq 1. \tag{3.4}$$

Proof. The proof uses Fourier analysis. The transition function p satisfies the initial value problem

$$\frac{\partial p}{\partial t} = \rho \Delta p, \qquad p|_{t=0} = \delta_0.$$

The Fourier transformation in the spatial variables of $p(t,\cdot)$ leads to the formula

$$p(t,x) = \frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi)^d} \exp\left\{-\rho t \hat{\varphi}(k) - ikx\right\} dk,$$

where

$$\hat{\varphi}(k) := \sum_{\substack{x\in\mathbb{Z}^d\\ |x|=1}} (1 - \cos(k\cdot x)), \qquad k \in [-\pi,\pi)^d.$$

Substituting $x = 0$, we get

$$p(t,0) = \frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi)^d} \exp\left\{-\rho t \hat{\varphi}(k)\right\} dk$$

and this is clearly monotonically decreasing in ρ, because $\hat{\varphi} \geq 0$. This proves assertion (i). From a Taylor expansion for $\hat{\varphi}$ we obtain

$$\hat{\varphi}(k) \leq \sum_{i=1}^{d} \frac{k_i^2}{2}, \qquad k = (k_1,\ldots,k_d) \in [-\pi,\pi)^d.$$

We proceed by substituting $y := \sqrt{\rho t}\, k_1$ and obtain

$$\begin{aligned} p(t,0) &\geq \frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi)^d} \exp\left\{-\rho t \sum_{i=1}^d \frac{k_i^2}{2}\right\} dk \\ &= \left(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \exp\left\{-\rho t\, \frac{k_1^2}{2}\right\} dk_1\right)^d \\ &= \left(\frac{1}{\sqrt{\rho t}}\right)^d \underbrace{\left(\frac{1}{2\pi} \int\limits_{-\pi\sqrt{\rho t}}^{\pi\sqrt{\rho t}} \mathrm{e}^{-y^2/2}\, dy\right)^d}_{\to 1 \text{ as } t \to \infty}. \end{aligned}$$

In other words,

$$\liminf_{t\to\infty} p(t,0) \cdot (\rho t)^{d/2} \geq 1.$$

This proves assertion (ii). ■

The following proposition is a statement on the transformation of random walks. We will see that it is a main ingredient in the proof of Theorem 4.1 below. The proof is based on a more general theorem on the transformation of the state space of Markov processes, which we cite in Appendix B.

Proposition 3.3 (Transformation of Random Walks)
Let $(X_t^1, \mathbb{P}_x^X), \dots, (X_t^p, \mathbb{P}_x^X)$ and $(Y_t, \mathbb{P}_y^Y)$ denote independent random walks on $\mathbb{Z}^d$ with generators $\kappa\Delta, \dots, \kappa\Delta$ and $\rho\Delta$, respectively. Then,

$$(Z_t^1, \dots, Z_t^p) := (X_t^1 - Y_t, \dots, X_t^p - Y_t)$$

defines a Markov process on $\mathbb{Z}^{pd}$, and its generator A^p is of the form

$$A^p = \kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho B^p,$$

with Δ_i acting on the i-th component of the argument ($i = 1, \dots, p$) and

$$\begin{aligned} B^p f(x_1, \dots, x_p) = \sum_{\substack{v\in\mathbb{Z}^d \\ |v|=1}} [f(x_1+v, \dots, x_p+v) - f(x_1, \dots, x_p)], \\ f \in \ell^2(\mathbb{Z}^{pd}),\ x_1, \dots, x_p \in \mathbb{Z}^d. \end{aligned} \tag{3.5}$$

Proof. The proof uses Theorem B.1 of the Appendix. We consider the process $(X_t^1, \dots, X_t^p, Y_t)$ on $\mathbb{Z}^{(p+1)d}$ with the joint probability measure

$$\mathbb{P}_{x_1,\dots,x_p,y}^{X^1,\dots,X^p,Y} = \mathbb{P}_{x_1}^X \otimes \cdots \otimes \mathbb{P}_{x_p}^X \otimes \mathbb{P}_y^Y.$$

Due to the independence of the processes $X_t^1, \dots, X_t^p$ and Y_t,

$$\begin{aligned} & \mathbb{P}_{x_1,\dots,x_p,y}^{X^1,\dots,X^p,Y}\left(X_t^1 \in \Gamma_1, \dots, X_t^p \in \Gamma_p, Y_t \in \Gamma_{p+1}\right) \\ = \; & \mathbb{P}_{x_1}^X\left(X_t^1 \in \Gamma_1\right) \cdots \mathbb{P}_{x_p}^X\left(X_t^p \in \Gamma_p\right) \cdot \mathbb{P}_y^Y\left(Y_t \in \Gamma_{p+1}\right), \end{aligned} \tag{3.6}$$

for all sets $\Gamma_1, \dots, \Gamma_p, \Gamma_{p+1} \subset \mathbb{Z}^d$, all $x_1, \dots, x_p, y \in \mathbb{Z}^d$, and for all $t \in \mathbb{R}^+$. The joint process $(X_t^1, \dots, X_t^p, Y_t)$ is again a Markov process and its generator $\bar{A}^p$ can be written as the sum of the generators of the single processes, i.e.,

$$\bar{A}^p = \kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho\Delta_{p+1}. \tag{3.7}$$

Additionally, we consider the transformation

$$\gamma\colon \mathbb{Z}^{(p+1)d} \to \mathbb{Z}^{pd}, \quad \gamma(x_1, \dots, x_p, y) = (x_1 - y, \dots, x_p - y).$$

Then we have

$$(Z_t^1, \dots, Z_t^p)_{t\ge 0} = \gamma\left(X_t^1, \dots, X_t^p, Y_t\right)_{t\ge 0}.$$

Let $\mathcal{F}$ denote the σ-Algebra generated by the sets $\{(Z_t^1, \dots, Z_t^p) \in \tilde{\Gamma}\}$, $t \ge 0$, $\tilde{\Gamma} \in \mathbb{Z}^{pd}$, and define the family of probability measures

$$\mathbb{P}_{\gamma(x_1,\dots,x_p,y)}^{Z^1,\dots,Z^p}(A) := \mathbb{P}_{x_1,\dots,x_p,y}^{X^1,\dots,X^p,Y}(\gamma^{-1}A), \qquad (x_1, \dots, x_p, y) \in \mathbb{Z}^{pd},\ A \in \mathcal{F}.$$

1° We show that

$$\left((Z_t^1, \dots, Z_t^p), \mathbb{P}_{z_1,\dots,z_p}^{Z^1,\dots,Z^p}\right)$$

is again a Markov process. We introduce the transition function

$$p(t, z, \tilde{z}) := \mathbb{P}_z^{Z^1,\dots,Z^p}\left((Z_t^1, \dots, Z_t^p) = \tilde{z}\right), \qquad t \ge 0,\ z, \tilde{z} \in \mathbb{Z}^{pd}. \tag{3.8}$$

Suppose we are given a couple $(x_1, \dots, x_p, y)$, $(x_1', \dots, x_p', y') \in \mathbb{Z}^{(p+1)d}$ satisfying

$$\gamma(x_1, \dots, x_p, y) = \gamma(x_1', \dots, x_p', y') \tag{3.9}$$

and arbitrary $(z_1, \dots, z_p) \in \mathbb{Z}^{pd}$, $t \in \mathbb{R}^+$. Equation (3.9) is equivalent to

$$(x_1 - y, x_2 - y, \dots, x_p - y) = (x_1' - y', x_2' - y', \dots, x_p' - y'). \tag{3.10}$$

By Theorem B.1, we have to show that

$$p\left(t, (x_1, \dots, x_p, y), \gamma^{-1}(z_1, \dots, z_p)\right) = p\left(t, (x_1', \dots, x_p', y'), \gamma^{-1}(z_1, \dots, z_p)\right). \tag{3.11}$$

Clearly,

$$\begin{aligned} \gamma^{-1}(z_1, \dots, z_p) &= \{(x_1, \dots, x_p, y) \mid (x_1 - y, \dots, x_p - y) = (z_1, \dots, z_p)\} \\ &= \{(z_1 + k, \dots, z_p + k, k) \mid k \in \mathbb{Z}^d\}. \end{aligned} \tag{3.12}$$

Using equation (3.10), we can extend (3.12) to

$$\begin{aligned}\gamma^{-1}(z_1,\ldots,z_p) &= \{(z_1+k+(y'-y),\ldots,z_p+k+(y'-y),k+(y'-y))\,|\, k\in\mathbb{Z}^d\}\\ &= \{(z_1+k+(x_1'-x_1),\ldots,z_p+k+(x_p'-x_p),k+(y'-y))\,|\, k\in\mathbb{Z}^d\}.\end{aligned} \tag{3.13}$$

The transition function of the joint process p can be written as the tensor product of the transition functions of the single processes $p^{X^1},\ldots,p^{X^p},p^Y$. The transition functions are spatially shift invariant, hence for all $k\in\mathbb{Z}^d$,

$$\begin{aligned}&p\left(t,(x_1,\ldots,x_p,y),(z_1+k,\ldots,z_p+k,k)\right)\\ =\;& p^{X^1}\left(t,x_1,k+z_1\right)\cdots p^{X^p}\left(t,x_p,k+z_p\right)\cdot p^Y\left(t,y,k\right)\\ =\;& p^{X^1}\left(t,x_1',k+z_1+(x_1'-x_1)\right)\cdots p^{X^p}\left(t,x_p',k+z_p+(x_p'-x_p)\right)\cdot p^Y\left(t,y',k+(y'-y)\right)\\ =\;& p\left(t,(x_1',\ldots,x_p',y'),(k+z_1+(x_1'-x_1),\ldots,k+z_p+(x_p'-x_p),k+(y'-y))\right).\end{aligned}$$

Summation over k and equations (3.12), (3.13) yield the desired equation

$$\begin{aligned}&p\left(t,(x_1,\ldots,x_p,y),\gamma^{-1}(z_1,\ldots,z_p)\right)\\ &=\sum_{k\in\mathbb{Z}^d} p\left(t,(x_1,\ldots,x_p,y),(k+z_1,\ldots,k+z_p,k)\right)\\ &=\sum_{k\in\mathbb{Z}^d} p\left(t,(x_1',\ldots,x_p',y'),(k+z_1+(x_1'-x_1),\ldots,k+z_p+(x_p'-x_p),k+(y'-y))\right)\\ &=p\left(t,\left(x_1',\ldots,x_p',y'\right),\gamma^{-1}(z_1,\ldots,z_p)\right).\end{aligned}$$

2° We next show that the Markov process $(Z_t^1,\ldots,Z_t^p)$ has generator

$$A^p := \kappa\Delta_1+\cdots+\kappa\Delta_p+\rho B^p,$$

where

$$B^p f(x_1,\ldots,x_p)=\sum_{\substack{v\in\mathbb{Z}^d\\ |v|=1}}[f(x_1+v,\ldots,x_p+v)-f(x_1,\ldots,x_p)],\quad f\in\ell^2(\mathbb{Z}^{pd}),\ x_1,\ldots,x_p\in\mathbb{Z}^d.$$

Denote by $B(\mathbb{Z}^{pd})$ and $B(\mathbb{Z}^{(p+1)d})$ the Banach space of bounded functionals on $\mathbb{Z}^{pd}$ and $\mathbb{Z}^{(p+1)d}$, respectively. We introduce the mapping

$$\begin{aligned}\gamma^*\colon B(\mathbb{Z}^{pd}) &\to B(\mathbb{Z}^{(p+1)d}),\\ (\gamma^* f)(x_1,\ldots,x_p,y) &= f(\gamma(x_1,\ldots,x_p,y))\\ &= f(x_1-y,\ldots,x_p-y),\\ &\qquad f\in B(\mathbb{Z}^{pd}),\ (x_1,\ldots,x_p,y)\in\mathbb{Z}^{(p+1)d}.\end{aligned}$$

Let $\bar{A}^p$ and A^p be the infinitesimal operators of the semigroups associated with the processes $(X_t^1, \ldots, X_t^p, Y_t)$ and $(Z_t^1, \ldots, Z_t^p)$, respectively. We recall from equation (3.7) that

$$\bar{A}^p = \kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho\Delta_{p+1}.$$

By Theorem B.1,

$$\gamma^* A^p = \bar{A}^p \gamma^*. \tag{3.14}$$

To finish the proof, it is sufficient to show that $\gamma^* A^p = \gamma^* \left(\kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho B^p\right)$ on $B(\mathbb{Z}^{(p+1)d})$, because γ is surjective.

Let $f \in B(\mathbb{Z}^{pd})$, $g := \gamma^* f \in B(\mathbb{Z}^{(p+1)d})$, $(\bar{x}_1, \ldots, \bar{x}_p, \bar{y}) \in \mathbb{Z}^{(p+1)d}$. Then

$$\begin{aligned}
& (\gamma^* A^p f)(\bar{x}_1, \ldots, \bar{x}_p, \bar{y}) \\
= \ & (\bar{A}^p \gamma^* f)(\bar{x}_1, \ldots, \bar{x}_p, \bar{y}) \\
= \ & (\bar{A}^p g)(\bar{x}_1, \ldots, \bar{x}_p, \bar{y}) \\
= \ & \left((\kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho\Delta_{p+1}) g\right)(\bar{x}_1, \ldots, \bar{x}_p, \bar{y}) \\
= \ & \kappa \sum_{x_1 \sim \bar{x}_1} \left[g(x_1, \bar{x}_2, \ldots, \bar{x}_p, \bar{y}) - g(\bar{x}_1, \ldots, \bar{x}_p, \bar{y})\right] \\
& + \cdots \\
& + \kappa \sum_{x_p \sim \bar{x}_p} \left[g(\bar{x}_1, \ldots, \bar{x}_{p-1}, x_p, \bar{y}) - g(\bar{x}_1, \ldots, \bar{x}_p, \bar{y})\right] \\
& + \rho \sum_{y \sim \bar{y}} \left[g(\bar{x}_1, \ldots, \bar{x}_p, y) - g(\bar{x}_1, \ldots, \bar{x}_p, \bar{y})\right] \\
= \ & \kappa \sum_{z_1 \sim \bar{x}_1 - \bar{y}} \left[f(z_1, \bar{x}_2 - \bar{y}, \ldots, \bar{x}_p - \bar{y}) - f(\bar{x}_1 - \bar{y}, \ldots, \bar{x}_p - \bar{y})\right] \\
& + \cdots \\
& + \kappa \sum_{z_p \sim \bar{x}_p - \bar{y}} \left[f(\bar{x}_1 - \bar{y}, \ldots, \bar{x}_{p-1} - \bar{y}, z_p) - f(\bar{x}_1 - \bar{y}, \ldots, \bar{x}_p - \bar{y})\right] \\
& + \rho \sum_{v \sim 0} \left[f(\bar{x}_1 - \bar{y} + v, \ldots, \bar{x}_p - \bar{y} + v) - f(\bar{x}_1 - \bar{y}, \ldots, \bar{x}_p - \bar{y})\right] \\
= \ & \left(\kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho B^p\right) f(\bar{x}_1 - \bar{y}, \ldots, \bar{x}_p - \bar{y}) \\
= \ & \left(\gamma^* (\kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho B^p) f\right)(\bar{x}_1, \ldots, \bar{x}_p, \bar{y}).
\end{aligned}$$

This completes the proof. ■

Corollary 3.4
Let $\left(X_t, \mathbb{P}_x^X\right)$ and $\left(Y_t, \mathbb{P}_y^Y\right)$ denote two independent random walks on $\mathbb{Z}^d$ with generators $\kappa\Delta$ and $\rho\Delta$, respectively. Then $Z_t := X_t - Y_t$ is a simple symmetric random walk on $\mathbb{Z}^d$ with generator $(\kappa + \rho)\Delta$.

This corollary follows immediately from Proposition 3.3 when $p = 1$. In addition, the statement is heuristically quite clear. The process Z_t jumps whenever X_t or Y_t jump, and

Z_t jumps to one of its $2d$ neighbours with equal probability. The jump times of X_t and Y_t are independent and exponentially distributed with parameters $2d\kappa$ and $2d\rho$. Hence, the minimum of both is also exponentially distributed, but with parameter $2d(\kappa + \rho)$.

Chapter 4

Existence and Spectral Characterization of the Lyapunov Exponents

We recall from Proposition 3.3 that the generator of the Markov process $(X_t^1 - Y_t, \ldots, X_t^p - Y_t)$ is $A^p = \kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho B^p$, where

$$B^p f(x_1, \ldots, x_p) = \sum_{\substack{v \in \mathbb{Z}^d \\ |v|=1}} [f(x_1+v, \ldots, x_p+v) - f(x_1, \ldots, x_p)], \quad f \in \ell^2(\mathbb{Z}^{pd}),\, x_1, \ldots, x_p \in \mathbb{Z}^d.$$

We define the Hamilton operator

$$\begin{aligned} \mathcal{H}^p &:= A^p + \gamma\delta_0^{(1)} + \cdots + \gamma\delta_0^{(p)} && (4.1) \\ &= \kappa\Delta_1 + \cdots + \kappa\Delta_p + \rho B^p + \gamma\delta_0^{(1)} + \cdots + \gamma\delta_0^{(p)} && (4.2) \end{aligned}$$

on $\ell^2(\mathbb{Z}^{pd})$. Here $\delta_0^{(i)}(x_1, \ldots, x_p) = 1$ if $x_i = 0$, and 0 else. In our model, the term $\gamma(\delta_0^{(1)} + \cdots + \delta_0^{(p)})$ can be regarded as a *potential*.

Remember that $u(t,x)$ is the solution of the parabolic Anderson problem (1.1) with the time-dependent potential $\xi(t,x) = \gamma\delta_{Y_t}(x)$. The following theorem links the asymptotic behaviour of $\langle u(t,x)^p \rangle$ as $t \to \infty$ to the ℓ^2-spectrum $\mathrm{Sp}(\mathcal{H}^p)$ of the operator $\mathcal{H}^p$.

Theorem 4.1 (Existence and spectral characterization)
For each $p \in \mathbb{N}$, the Lyapunov exponent

$$\lambda_p = \lim_{t\to\infty} \frac{1}{t} \log \langle u(t,x)^p \rangle$$

exists, is finite, and

$$\lambda_p = \sup \mathrm{Sp}(\mathcal{H}^p). \qquad (4.3)$$

A natural start for the analysis of $\langle u(t,x)^p\rangle$ is the Feynman-Kac formula (1.3). Together with Fubini's theorem we obtain

$$\begin{aligned}\langle u(t,x)^p\rangle &= \mathbb{E}^{X^1,\dots,X^p;Y}_{x,\dots,x;0}\exp\left\{\gamma\int_0^t\sum_{i=1}^p\delta_{Y_{t-s}}(X_s^i)ds\right\}\\ &= \sum_{z\in\mathbb{Z}^d}\mathbb{E}^{X^1,\dots,X^p;Y}_{x,\dots,x;0}\exp\left\{\gamma\int_0^t\sum_{i=1}^p\delta_0(X_s^i-Y_{t-s})ds\right\}\delta_z(Y_t),\end{aligned}$$

where $(X_t^1)_{t\geq 0},\dots,(X_t^p)_{t\geq 0}$ are p independent random walks on $\mathbb{Z}^d$ with generator $\kappa\Delta$. A time reversion of Y yields

$$\langle u(t,x)^p\rangle = \sum_{z\in\mathbb{Z}^d}\mathbb{E}^{X^1,\dots,X^p;Y}_{x,\dots,x;z}\exp\left\{\gamma\int_0^t\sum_{i=1}^p\delta_0(X_s^i-Y_s)ds\right\}\delta_0(Y_t). \tag{4.4}$$

Proceeding from equation (4.4), we prepare the proof of Theorem 4.1. The first lemma allows us to reduce the analysis to $\lim_{t\to\infty}\frac{1}{t}\log\langle u(t,0)^p\rangle$. The following two lemmas are asymptotic statements needed for the proof. Starting from equation (4.4) we show in the first part of the proof that $\sup\mathrm{Sp}(\mathcal{H}^p)$ is an upper bound for $\limsup_{t\to\infty}\frac{1}{t}\log\langle u(t,0)^p\rangle$, afterwards we show that it is also a lower bound for $\liminf_{t\to\infty}\frac{1}{t}\log\langle u(t,0)^p\rangle$.

First we show that, although the random field $\xi(t,x)$ is not spatially shift-invariant, the exponential growth of $\langle u(t,x)^p\rangle$ is independent of x.

Lemma 4.2
If $\kappa,\rho>0$ and the limit

$$\lim_{t\to\infty}\frac{1}{t}\log\langle u(t,0)^p\rangle$$

exists, then, for all $x\in\mathbb{Z}^d$,

$$\lim_{t\to\infty}\frac{1}{t}\log\langle u(t,x)^p\rangle = \lim_{t\to\infty}\frac{1}{t}\log\langle u(t,0)^p\rangle. \tag{4.5}$$

Proof. We start with equation (4.4) and only consider paths of $X^1,\dots,X^p$ that are in y_2 at time $t=1$. Then we use the Markov-property (MP) for $t=1$, which yields for all $y_1,y_2\in\mathbb{Z}^d$,

$$\begin{aligned}\langle u(t,y_1)^p\rangle &= \sum_{z\in\mathbb{Z}^d}\mathbb{E}^{X^1,\dots,X^p;Y}_{y_1,\dots,y_1;z}\exp\left\{\gamma\int_0^t\sum_{i=1}^p\delta_0(X_s^i-Y_s)ds\right\}\delta_0(Y_t)\\ &\geq \sum_{z\in\mathbb{Z}^d}\mathbb{E}^{X^1,\dots,X^p;Y}_{y_1,\dots,y_1;z}\exp\left\{\gamma\int_1^t\sum_{i=1}^p\delta_0(X_s^i-Y_s)ds\right\}\delta_0(Y_t)\\ &\qquad\times\delta_{y_2}(X_1^1)\cdots\delta_{y_2}(X_1^p)\,\delta_z(Y_1)\end{aligned}$$

$$\overset{(MP)}{=} \sum_{z\in\mathbb{Z}^d} \mathbb{P}_{y_1}(X_1^1 = y_2)\cdots \mathbb{P}_{y_1}(X_1^p = y_2)\,\mathbb{P}_z(Y_1 = z)$$
$$\times\, \mathbb{E}_{y_2,\dots,y_2;z}^{X^1,\dots,X^p;Y} \exp\left\{\gamma \int_0^{t-1} \sum_{i=1}^p \delta_0(X_s^i - Y_s)ds\right\} \delta_0\left(Y_{t-1}\right).$$

In the last transformation, we took into account that $X_t^1,\dots,X_t^p,Y_t$ are independent. As $X_t^1,\dots,X_t^p$ are identically distributed and $\mathbb{P}_0(Y_1^1=0)\ge \mathrm{e}^{-\rho}$,

$$\langle u(t,y_1)^p\rangle \ge \left[\mathbb{P}_{y_1}(X_1^1 = y_2)\right]^p \mathrm{e}^{-\rho}\langle u(t-1,y_2)^p\rangle.$$

Thus, for $y_1 = x, y_2 = 0$,

$$\liminf_{t\to\infty} \frac{1}{t}\log\langle u(t,x)^p\rangle \ge \liminf_{t\to\infty}\frac{1}{t}\log\langle u(t,0)^p\rangle,$$

whereas, for $y_1 = 0, y_2 = x$,

$$\limsup_{t\to\infty} \frac{1}{t}\log\langle u(t,0)^p\rangle \ge \limsup_{t\to\infty}\frac{1}{t}\log\langle u(t,x)^p\rangle,$$

which completes the proof. ■

We need the following lemma to derive the upper bound in the proof of Theorem 4.1. It states that we can restrict equation (4.4) to paths that start and finish in the finite box $Q_{t\log^2 t}$.

Lemma 4.3
As $t\to\infty$,

$$\langle u(t,0)^p\rangle = (1+o(1)) \sum_{z\in Q_{t\log^2 t}} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \exp\left\{\gamma\int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s)ds\right\}\delta_0\left(Y_t\right)$$
$$\times\, 1\!\!1_{(X_t^1,\dots,X_t^p)\in Q_{t\log^2 t}^p}. \tag{4.6}$$

Proof. We abbreviate $A_t := \gamma\int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s)ds$. Then $1\le \mathrm{e}^{A_t}\le \mathrm{e}^{\gamma t p}$. It remains to check that

$$r(t) := \frac{\sum_{z\in\mathbb{Z}^d}\mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y}\mathrm{e}^{A_t}\,\delta_0(Y_t) - \sum_{z\in Q_{t\log^2 t}}\mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y}\mathrm{e}^{A_t}\,\delta_0(Y_t)\,1\!\!1_{(X^1,\dots,X^p)\in Q_{t\log^2 t}^p}}{\sum_{z\in Q_{t\log^2 t}}\mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y}\mathrm{e}^{A_t}\,\delta_0(Y_t)\,1\!\!1_{(X^1,\dots,X^p)\in Q_{t\log^2 t}^p}}$$

tends to 0 as $t \to \infty$. Obviously, $r(t) \geq 0$. Furthermore,

$$\begin{aligned}
r(t) &\leq \mathrm{e}^{\gamma t p} \frac{\sum_{z \notin Q_{t \log^2 t}} \mathbb{E}_{0,\ldots,0;z}^{X^1,\ldots,X^p;Y} \delta_0(Y_t) + \sum_{z \in \mathbb{Z}^d} \mathbb{E}_{0,\ldots,0;z}^{X^1,\ldots,X^p;Y} \delta_0(Y_t) \, \mathbb{1}_{(X^1,\ldots,X^p) \in Q^p_{t \log^2 t}}}{\mathbb{E}_{0,\ldots,0;0}^{X^1,\ldots,X^p;Y} \delta_0(Y_t)} \\
&= \mathrm{e}^{\gamma t p} \frac{\sum_{z \notin Q_{t \log^2 t}} \mathbb{P}_z^Y(Y_t = 0) + \mathbb{E}_{0,\ldots,0}^{X^1,\ldots,X^p} \mathbb{1}_{(X_t^1,\ldots,X_t^p) \notin Q^p_{t \log^2 t}}}{\mathbb{P}_0^Y(Y_t = 0)} \\
&= \mathrm{e}^{\gamma t p} \frac{\mathbb{P}_0^Y \left(Y_t \notin Q_{t \log^2 t}\right) + \mathbb{P}_{0,\ldots,0}^{X^1,\ldots,X^p} \left((X_t^1,\ldots,X_t^p) \notin Q^p_{t \log^2 t}\right)}{\mathbb{P}_0^Y(Y_t = 0)} \qquad (4.7)
\end{aligned}$$

In the last transformation we used again a time reversal for Y. Applying Lemma 3.1 and Lemma 3.2 to the right hand side of (4.7), we get

$$\begin{aligned}
r(t) &\leq \mathrm{e}^{\gamma t p} \frac{2^{d+1} \, \mathrm{e}^{-t \log^2 t} + \left(2^{d+1} \, \mathrm{e}^{-t \log^2 t}\right)^p}{(\rho t)^{-d/2}} \\
&\leq 2 \, \mathrm{e}^{\gamma t p} \frac{\left(2^{d+1} \, \mathrm{e}^{-t \log^2 t}\right)^p}{(\rho t)^{-d/2}} \\
&= 2^{1+p(d+1)} \exp\left\{\gamma t p - p t \log^2 t + \frac{d}{2} \log(\rho t)\right\}
\end{aligned}$$

for large t. Since the expression in the last line tends to 0 as $t \to \infty$, we get

$$\lim_{t \to \infty} r(t) = 0.$$

■

The next lemma is again an asymptotic statement. We need it for the lower bound in the proof of Theorem 4.1. It states that paths starting outside the finite box $Q_{t \log^2 t}$ are asymptotically negligible.

Lemma 4.4
As $t \to \infty$,

$$\begin{aligned}
&\sum_{y \in Q_{t \log^2 t}} \mathbb{E}_{0,\ldots,0;0}^{X^1,\ldots,X^p;Y} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s) ds\right\} \delta_y(Y_t) \, \delta_y\left(X_t^1\right) \ldots \delta_y\left(X_t^p\right) \qquad (4.8) \\
= \; &(1 + o(1)) \sum_{y \in \mathbb{Z}^d} \mathbb{E}_{0,\ldots,0;0}^{X^1,\ldots,X^p;Y} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s) ds\right\} \delta_y(Y_t) \, \delta_y\left(X_t^1\right) \ldots \delta_y\left(X_t^p\right)
\end{aligned}$$

Proof. The technique of the proof is very similar to the previous lemma. We have to show that

$$r(t) := \frac{\sum_{y \notin Q_{t \log^2 t}} \mathbb{E}_{0,\ldots,0;0}^{X^1,\ldots,X^p;Y} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s) ds\right\} \delta_y(Y_t) \, \delta_y(X_t^1) \ldots \delta_y(X_t^p)}{\sum_{y \in \mathbb{Z}^d} \mathbb{E}_{0,\ldots,0;0}^{X^1,\ldots,X^p;Y} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s) ds\right\} \delta_y(Y_t) \, \delta_y(X_t^1) \ldots \delta_y(X_t^p)}$$

tends to 0 as $t \to \infty$. We use Lemma 3.1 and Lemma 3.2 and the independence of $X^1, \dots, X^p$ to obtain

$$
\begin{aligned}
r(t) &\le \frac{e^{\gamma tp}\, \mathbb{P}_0^Y \left(Y_t \notin Q_{t\log^2 t}\right)\ \mathbb{P}_{0,\dots,0}^{X^1,\dots,X^p}\left((X_t^1,\dots,X_t^p) \notin Q^p_{t\log^2 t}\right)}{\mathbb{P}_0^Y \left(Y_t = 0\right)\ \mathbb{P}_{0,\dots,0}^{X^1,\dots,X^p}\left(X_t^1 = 0,\dots,X_t^p = 0\right)} \\
&\le e^{\gamma pt}\, 2^{(p+1)(d+1)}\, e^{-t\log^2 t}\, (\rho t)^{d/2}\, (\kappa t)^{pd/2}.
\end{aligned}
$$

But this vanishes as $t \to \infty$. ∎

Now we have made enough preparations to present the proof of Theorem 4.1.

Proof of Theorem 4.1. The proof will be split into two parts:
1° $\limsup\limits_{t\to\infty} \frac{1}{t}\log\langle u(t,0)^p\rangle \le \sup \mathrm{Sp}(\mathcal{H}^p)$,
2° $\liminf\limits_{t\to\infty} \frac{1}{t}\log\langle u(t,0)^p\rangle \ge \sup \mathrm{Sp}(\mathcal{H}^p)$.
Once we have proved this, the existence of the limit is established.

1° *Upper bound.* Using Lemma 4.3, we have

$$
\begin{aligned}
\langle u(t,0)^p\rangle &= (1+o(1)) \quad \textstyle\sum_{z\in Q_{t\log^2 t}} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s)ds\right\} \\
&\qquad \times \delta_0(Y_t)\ \mathbb{1}_{(X_t^1,\dots,X_t^p)\in Q^p_{t\log^2 t}}.
\end{aligned} \tag{4.9}
$$

Since $\delta_0(Y_t)\cdot \mathbb{1}_{(X_t^1,\dots,X_t^p)\in Q^p_{t\log^2 t}} \le \mathbb{1}_{(X_t^1-Y_t,\dots,X_t^p-Y_t)\in Q^p_{t\log^2 t}}$, we find that

$$
\begin{aligned}
\langle u(t,0)^p\rangle &\le (1+o(1)) \sum_{z\in Q_{t\log^2 t}} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s)ds\right\} \\
&\qquad \times \mathbb{1}_{(X_t^1-Y_t,\dots,X_t^p-Y_t)\in Q^p_{t\log^2 t}}.
\end{aligned}
$$

Now we apply the transformation

$$
(Z_t^1,\dots,Z_t^p) := (X_t^1 - Y_t,\dots,X_t^p - Y_t)
$$

which yields

$$
\begin{aligned}
\langle u(t,0)^p\rangle &\le (1+o(1)) \sum_{z\in Q_{t\log^2 t}} \mathbb{E}_{z,\dots,z}^{Z_t^1,\dots,Z_t^p} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(Z_s^i)ds\right\} \\
&\qquad \times \mathbb{1}_{(Z_t^1,\dots,Z_t^p)\in Q^p_{t\log^2 t}} \\
&\le (1+o(1)) \sum_{z_1,\dots,z_p\in Q_{t\log^2 t}} \mathbb{E}_{z_1,\dots,z_p}^{Z^1,\dots,Z^p} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(Z_s^i)ds\right\} \\
&\qquad \times \mathbb{1}_{(Z_t^1,\dots,Z_t^p)\in Q^p_{t\log^2 t}}.
\end{aligned} \tag{4.10}
$$

We can rewrite the sum in (4.10) with the help of the semigroup $\{e^{t\mathcal{H}^p} \mid t \geq 0\}$, acting on $f \in \ell^2(\mathbb{Z}^{pd})$ as

$$\left(e^{t\mathcal{H}^p} f\right)(z_1, \dots, z_p) = \mathbb{E}_{z_1,\dots,z_p}^{Z^1,\dots,Z^p} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(Z_s^i) ds\right\} f(Z_t^1, \dots, Z_t^p), \tag{4.11}$$

and obtain

$$\begin{aligned} &\sum_{z_1,\dots,z_p \in Q_{t\log^2 t}} \mathbb{E}_{z_1,\dots,z_p}^{Z^1,\dots,Z^p} \exp\left\{\gamma \int_0^t \sum_{i=1}^p \delta_0(Z_s^i) ds\right\} \mathbb{1}_{(Z_t^1,\dots,Z_t^p) \in Q^p_{t\log^2 t}} \\ &= \left(e^{t\mathcal{H}^p} \mathbb{1}_{Q^p_{t\log^2 t}}, \mathbb{1}_{Q^p_{t\log^2 t}}\right)_{\ell^2(\mathbb{Z}^{pd})}. \end{aligned} \tag{4.12}$$

Finally, we use the spectral theorem for bounded, self-adjoint operators. The operator $\mathcal{H}^p$ is bounded and symmetric on the Hilbert space $\ell^2(\mathbb{Z}^{pd})$, thus it is self-adjoint. Define $\mu := \sup \mathrm{Sp}(\mathcal{H}^p)$ and denote $\{E_\lambda; \lambda \leq \mu\}$ the family of spectral projectors associated with the operator $\mathcal{H}^p$. During the following, we write $(\cdot,\cdot)$ for the inner product in $\ell^2(\mathbb{Z}^{pd})$ and $\|\cdot\|$ for the corresponding norm. Then our semigroup admits the spectral representation

$$e^{t\mathcal{H}^p} = \int_{(-\infty,\mu]} e^{t\lambda}\, dE_\lambda,$$

thus

$$\begin{aligned} \left(e^{t\mathcal{H}^p} \mathbb{1}_{Q_{t\log^2 t}}, \mathbb{1}_{Q_{t\log^2 t}}\right) &= \int_{(-\infty,\mu]} e^{t\lambda}\, d\left(E_\lambda \mathbb{1}_{Q^p_{t\log^2 t}}, \mathbb{1}_{Q^p_{t\log^2 t}}\right) \\ &\leq e^{t\mu} \int_{(-\infty,\mu]} d\left\|E_\lambda \mathbb{1}_{Q^p_{t\log^2 t}}\right\|^2 \\ &\leq e^{t\mu}\, |Q^p_{t\log^2 t}|^2. \end{aligned} \tag{4.13}$$

Combining equations (4.10), (4.12) and (4.13) we get

$$\langle u(t,0)^p \rangle \;\leq (1+o(1))\, e^{t\mu}\, |Q^p_{t\log^2 t}|^2,$$

hence

$$\limsup_{t\to\infty} \frac{1}{t} \log\langle u(t,0)^p\rangle \leq \mu + \limsup_{t\to\infty} \frac{1}{t} \log |Q^p_{t\log^2 t}|^2 = \mu,$$

and this proves the upper bound.

2° *Lower bound.* For convenience, we abbreviate

$$A_t := \gamma \int_0^t \sum_{i=1}^p \delta_0(X_s^i - Y_s)\, ds.$$

Restricting the expectation on the right hand side of equation (4.4) to paths of X^i ($i = 1,\dots,p$) and Y that start and end at 0, we get

$$\begin{aligned}
&\langle u(t,0)^p\rangle \\
&\quad= \sum_{z\in\mathbb{Z}^d} \mathbb{E}_{0,\dots,0;z}^{X^1,\dots,X^p;Y}\left[\mathrm{e}^{A_t}\,\delta_0(Y_t)\right] \\
&\quad\geq \qquad \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y}\left[\mathrm{e}^{A_t}\,\delta_0(Y_t)\,\delta_0(X_t^1)\cdots\delta_0(X_t^p)\right] \\
&\quad= \sum_{x_1,\dots,x_p,y\in\mathbb{Z}^d} \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y}\Big[\mathrm{e}^{A_{t/2}}\,\delta_y(Y_{\frac{t}{2}})\,\delta_{x_1}(X_{\frac{t}{2}}^1)\cdots\delta_{x_p}(X_{\frac{t}{2}}^p) \\
&\qquad\qquad\qquad\qquad \mathrm{e}^{A_t-A_{t/2}}\,\delta_0(Y_t)\,\delta_0(X_t^1)\cdots\delta_0(X_t^p)\Big].
\end{aligned}\tag{4.14}$$

We use the Markov property, which transforms (4.14) into

$$\begin{aligned}
&\sum_{x_1,\dots,x_p,y\in\mathbb{Z}^d} \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y}\left[\mathrm{e}^{A_{t/2}}\,\delta_y(Y_{\frac{t}{2}})\,\delta_{x_1}(X_{\frac{t}{2}}^1)\cdots\delta_{x_p}(X_{\frac{t}{2}}^p)\right] \\
&\qquad\times\ \mathbb{E}_{x_1,\dots,x_p;y}^{X^1,\dots,X^p;Y}\left[\mathrm{e}^{A_{t/2}}\,\delta_0(Y_{\frac{t}{2}})\,\delta_0(X_{\frac{t}{2}}^1)\cdots\delta_0(X_{\frac{t}{2}}^p)\right].
\end{aligned}$$

After a time reversion in the second line, we get

$$\begin{aligned}
\langle u(t,0)^p\rangle &\geq \sum_{\substack{x_1,\dots,x_p,\\ y\in\mathbb{Z}^d}} \left(\mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y}\left[\mathrm{e}^{A_{t/2}}\,\delta_y(Y_{\frac{t}{2}})\,\delta_{x_1}(X_{\frac{t}{2}}^1)\cdots\delta_{x_p}(X_{\frac{t}{2}}^p)\right]\right)^2 \\
&\geq \sum_{y\in Q_{t\log^2 t}} \left(\mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y}\left[\mathrm{e}^{A_{t/2}}\,\delta_y(Y_{\frac{t}{2}})\,\delta_y(X_{\frac{t}{2}}^1)\cdots\delta_y(X_{\frac{t}{2}}^p)\right]\right)^2.
\end{aligned}$$

The convexity of the parabola implies that

$$\left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2 \leq \frac{1}{n}\sum_{i=1}^n x_i^2, \qquad x_1,\dots,x_n\in\mathbb{R}.$$

The above inequality and Lemma 4.4 yield

$$\begin{aligned}
\langle u(t,0)^p\rangle &\geq \frac{1}{|Q_{t\log^2 t}|}\left(\sum_{y\in Q_{t\log^2 t}} \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y}\left[\mathrm{e}^{A_{t/2}}\,\delta_y(Y_{\frac{t}{2}})\,\delta_y(X_{\frac{t}{2}}^1)\cdots\delta_y(X_{\frac{t}{2}}^p)\right]\right)^2 \\
&= \frac{1+o(1)}{|Q_{t\log^2 t}|}\left(\sum_{y\in\mathbb{Z}^d} \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y}\left[\mathrm{e}^{A_{t/2}}\,\delta_y(Y_{\frac{t}{2}})\,\delta_y(X_{\frac{t}{2}}^1)\cdots\delta_y(X_{\frac{t}{2}}^p)\right]\right)^2,
\end{aligned}$$

which, finally, is

$$\geq \frac{1+o(1)}{|Q_{t\log^2 t}|}\left(\mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y}\left[\mathrm{e}^{A_{t/2}}\,\delta_0(X_{\frac{t}{2}}^1-Y_{\frac{t}{2}})\cdots\delta_0(X_{\frac{t}{2}}^p-Y_{\frac{t}{2}})\right]\right)^2.$$

As before, we apply the transformation

$$(Z_t^1, \dots, Z_t^p) = (X_t^1 - Y_t, \dots, X_t^p - Y_t).$$

Hence,

$$\langle u(t,0)^p \rangle \geq \frac{1+o(1)}{|Q_{t\log^2 t}|} \left(\mathbb{E}_{0,\dots,0}^{Z^1,\dots,Z^p} \left[\exp\left\{ \gamma \int_0^{\frac{t}{2}} \sum_{i=1}^{p} \delta_0(Z_s^i) ds \right\} \delta_0(Z_{\frac{t}{2}}^1) \cdots \delta_0(Z_{\frac{t}{2}}^p) \right] \right)^2 . \quad (4.15)$$

Again, we express (4.15) with the help of the semigroup $\left\{ \mathrm{e}^{t\mathcal{H}^p}; t \geq 0 \right\}$. We apply equation (4.11) and obtain

$$\langle u(t,0)^p \rangle \geq \frac{1+o(1)}{|Q_{t\log^2 t}|} \left[\left(\mathrm{e}^{\frac{t}{2}\mathcal{H}^p} \delta_0, \delta_0 \right)_{\ell^2(\mathbb{Z}^{pd})} \right]^2 . \quad (4.16)$$

If there is a positive eigenfunction v corresponding to the principal eigenvalue $\mu := \sup \mathrm{Sp}(\mathcal{H}^p)$ and μ has algebraic multiplicity 1, then

$$\begin{aligned} \left(\mathrm{e}^{\frac{t}{2}\mathcal{H}^p} \delta_0, \delta_0 \right) &= \mathrm{e}^{\frac{t}{2}\mu} \left(v, \delta_0 \right)^2 + \int_{(-\infty,\mu)} \mathrm{e}^{\frac{t}{2}\lambda} d\left(E_\lambda \delta_0, \delta_0 \right) \\ &\geq \mathrm{e}^{\frac{t}{2}\mu} v(0)^2, \end{aligned} \quad (4.17)$$

which is positive, because the eigenfunction v is positive.

Otherwise (if there is no such eigenfunction), we restrict the ℓ^2-operator to a finite box with Dirichlet boundary conditions and apply the Perron-Frobenius theory for nonnegative, irreducible matrices. This is done as follows.

For each $n \in \mathbb{N}$, we define $\mathcal{H}_n^p$ as the restriction of the operator $\mathcal{H}^p$ to the subspace $\ell^2(Q_n^p)$. We embed $\mathcal{H}_n^p$ into $\ell^2(\mathbb{Z}^{pd})$ with Dirichlet boundary conditions. Then, the semigroup generated by $\mathcal{H}_n^p$ acts on $f \in \ell^2(\mathbb{Z}^{pd})$ as

$$\mathrm{e}^{t\mathcal{H}_n^p} f(z_1, \dots, z_p) = \mathbb{E}_{z_1,\dots,z_p}^{Z^1,\dots,Z^p} \left[\exp\left\{ \gamma \int_0^t \sum_{i=1}^{p} \delta_0(Z_s^i) ds \right\} f(Z_t^1, \dots, Z_t^p) \mathbb{1}_{\tau_n > t} \right], \quad (4.18)$$

where $\tau_n := \inf\{t | (Z_t^1, \dots, Z_t^p) \notin Q_n^p\}$ denotes the stopping time of the first exit from the box Q_n^p. For $f_n \in \ell^2(Q_n^p)$, $f \in \ell^2(\mathbb{Z}^{pd})$ with $f_n(x) = f(x)$ for every $x \in Q_n^p$, we have

$$\left(\mathrm{e}^{t\mathcal{H}_n^p} f, f \right)_{\ell^2(\mathbb{Z}^{pd})} = \left(\mathrm{e}^{t\mathcal{H}_n^p} f_n, f_n \right)_{\ell^2(Q_n^p)} . \quad (4.19)$$

It follows immediately that

$$\left(\mathrm{e}^{t\mathcal{H}_n^p} f, f \right)_{\ell^2(\mathbb{Z}^{pd})} \leq \left(\mathrm{e}^{t\mathcal{H}^p} f, f \right)_{\ell^2(\mathbb{Z}^{pd})} \quad (4.20)$$

for all $f \in \ell^2(\mathbb{Z}^{pd})$. We identify $\ell^2(Q_n^p) = \mathbb{R}^{|Q_n^p|}$ and consider $\mathcal{H}_n^p$ as a matrix. Then, $\mathcal{H}_n^p + 2d\kappa\, I$ is a nonnegative matrix. Furthermore, it is irreducible in the sense of Theorem C.1, because we can put $|Q_n^p|$ into an order such that the minor diagonals of $\mathcal{H}_n^p + 2d\kappa\, I$ are nonzero:

$$\mathcal{H}_n^p + 2d\kappa\, I = \begin{pmatrix} 0 & \star & & & \\ \star & 0 & \star & & \\ & \star & 0 & \ddots & \\ & & \ddots & \ddots & \star \\ & & & \star & 0 \end{pmatrix}, \qquad \star \neq 0.$$

Hence, by Theorem C.1, there exists a positive eigenfunction (eigenvector) $v_n > 0$, corresponding to the largest eigenvalue of $\mathcal{H}_n^p + 2d\kappa\, I$. So v_n is also an eigenfunction to the largest eigenvalue of $\mathcal{H}_n^p$ (but corresponds to a different eigenvalue μ_n), and also of $e^{\frac{t}{2}\mathcal{H}_n^p}$. Denote by $\{E_\lambda^n; \lambda < \mu_n\}$ the family of spectral projectors associated with the operator $\mathcal{H}_n^p$. Analogous to (4.17),

$$\begin{aligned} \left(e^{\frac{t}{2}\mathcal{H}_n^p}\delta_0, \delta_0\right) &= e^{\frac{t}{2}\mu_n}\,(v_n, \delta_0)^2 + \int\limits_{(-\infty,\mu_n)} e^{\frac{t}{2}\lambda} d\,(E_\lambda^n \delta_0, \delta_0) \\ &\geq e^{\frac{t}{2}\mu_n} v_n(0)^2. \end{aligned} \tag{4.21}$$

Since v_n is positive, the above inequality implies

$$\liminf_{t\to\infty} \frac{1}{t} \log\left(e^{\frac{t}{2}\mathcal{H}_n^p}\delta_0, \delta_0\right) \geq \frac{\mu_n}{2}. \tag{4.22}$$

We combine the inequalities (4.16), (4.20) and (4.22) to obtain for all $n \in \mathbb{N}$ that

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t} \log \langle u(t,0)^p \rangle &\geq \liminf_{t\to\infty} \frac{1}{t} \log \left\{ \frac{1+o(1)}{|Q_{t\log^2 t}|} \left(e^{\frac{t}{2}\mathcal{H}^p}\delta_0, \delta_0\right)^2 \right\} \\ &= 2 \liminf_{t\to\infty} \frac{1}{t} \log\left(e^{\frac{t}{2}\mathcal{H}^p}\delta_0, \delta_0\right) \\ &\geq 2 \liminf_{t\to\infty} \frac{1}{t} \log\left(e^{\frac{t}{2}\mathcal{H}_n^p}\delta_0, \delta_0\right) \\ &\geq \mu_n. \end{aligned}$$

It remains to show that

$$\lim_{n\to\infty} \mu_n \to \mu.$$

If A is a bounded and self-adjoint operator on $\ell^2(\mathbb{Z}^{pd})$, then the *Rayleigh-Ritz formula* for bounded and self-adjoint operators states that

$$\sup \mathrm{Sp}(A) = \sup_{\substack{f \in \ell^2(\mathbb{Z}^{pd}) \\ \|f\|=1}} (Af, f)\,. \tag{4.23}$$

For $f \in \ell^2(\mathbb{Z}^{pd})$ we write $f\big|_n := f \cdot \mathbb{1}_{Q_n^p}$. Then, for all $f \in \ell^2(\mathbb{Z}^{pd})$,

$$\left(\mathcal{H}^p f\big|_n, f\big|_n\right) = \left(\mathcal{H}_n^p f\big|_n, f\big|_n\right), \tag{4.24}$$

hence we can write the Rayleigh-Ritz formula (4.23) as

$$\mu_n = \sup_{\substack{\|f\|=1 \\ \mathrm{supp}(f)\subset Q_n^p}} (\mathcal{H}^p f, f). \tag{4.25}$$

Here $\mathrm{supp}(f) = \{x \in \mathbb{Z}^d \,|\, f(x) \neq 0\}$ denotes the support of f. Additionally, we see from (4.25) that μ_n is nondecreasing. It is sufficient that, for any $f \in \ell^2(\mathbb{Z}^{pd})$,

$$\lim_{n\to\infty} \left(\mathcal{H}^p f\big|_n, f\big|_n\right) = (\mathcal{H}^p f, f), \tag{4.26}$$

because this validates

$$\sup_{\|f\|=1} (\mathcal{H}^p f, f) = \sup_{\substack{\|f\|=1 \\ |\mathrm{supp}(f)|<\infty}} (\mathcal{H}^p f, f) \tag{4.27}$$

and

$$\begin{aligned}
\mu &= \sup_{\|f\|=1} (\mathcal{H}^p f, f) \\
&= \sup_{n\in\mathbb{N}} \sup_{\substack{\|f\|=1 \\ \mathrm{supp}(f)\subset Q_n^p}} (\mathcal{H}^p f, f) \\
&= \sup_{n\in\mathbb{N}} \mu_n \\
&= \lim_{n\to\infty} \mu_n.
\end{aligned}$$

In order to show (4.26), we calculate

$$\begin{aligned}
(\mathcal{H}^p f, f) - \left(\mathcal{H}^p f\big|_n, f\big|_n\right) &= \left(\mathcal{H}^p f - \mathcal{H}^p f \mathbb{1}_{Q_n^p}, f\right) + \left(\mathcal{H}^p f, f \mathbb{1}_{\mathbb{Z}^{pd}\setminus Q_n^p}\right) \\
&= \left(\mathcal{H}^p f \mathbb{1}_{\mathbb{Z}^{pd}\setminus Q_n^p}, f\right) + \left(\mathcal{H}^p f, f \mathbb{1}_{\mathbb{Z}^{pd}\setminus Q_n^p}\right) \\
&= 2\left(\mathcal{H}^p f, f \mathbb{1}_{\mathbb{Z}^{pd}\setminus Q_n^p}\right).
\end{aligned}$$

In the last transformation, we used the self-adjointness of the operator $\mathcal{H}^p$. The Cauchy-Schwarz inequality yields

$$\left|(\mathcal{H}^p f, f) - \left(\mathcal{H}^p f\big|_n, f\big|_n\right)\right| \leq 2\,\|\mathcal{H}^p\|\,\|f\|\,\left\|f \mathbb{1}_{\mathbb{Z}^{pd}\setminus Q_n^p}\right\|,$$

but $\left\|f \mathbb{1}_{\mathbb{Z}^{pd}\setminus Q_n^p}\right\| \to 0$ as $n \to \infty$ as a remainder of a convergent series. This completes the proof. ■

Chapter 5

Analysis of the First Order Lyapunov Exponent

In this chapter we investigate the behaviour of the first moment of the solution u. More precisely, we want to compute the *first moment Lyapunov exponent*

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log \langle u(t,0)\rangle.$$

In this case, we can calculate the whole ℓ^2-spectrum of the Hamilton operator $\mathcal{H}^1 = (\kappa+\rho)\Delta + \gamma\delta_0$ (cf. Corollary 3.4). With Theorem 4.1, we obtain λ_1 by the identity

$$\lambda_1 = \sup \mathrm{Sp}(\mathcal{H}^1).$$

Throughout this chapter we will assume that $\kappa + \rho > 0$.

Denote by $(P_t)_{t\geq 0}$ the semigroup of operators associated with the discrete Laplacian Δ and observe that the transition function $p(t,x)$ of the Markov process with generator Δ may be written as

$$p(t,x) = P_t\,\delta_0(x). \tag{5.1}$$

We further denote by G_d the Green's function associated with Δ in zero as a function of the exponential stopping parameter

$$G_d(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda t} p(t,0)\,dt \tag{5.2}$$

and introduce the constant

$$r_d := \left(G_d(0)\right)^{-1}. \tag{5.3}$$

Then it is clear that

Claim 5.1

$$r_d \begin{cases} = 0, & \text{if } d = 1,2; \\ > 0, & \text{if } d \geq 3. \end{cases} \tag{5.4}$$

This is an elementary result. For the sake of completeness, we give the proof at the end of the chapter.

Theorem 5.2 (Asymptotics of the first moment)
If $\gamma \leq r_d(\kappa+\rho)$, then

$$\lambda_1 = 0.$$

Otherwise, λ_1 equals the unique positive solution of the equation

$$\gamma\, G_d(\cdot) = \kappa + \rho.$$

Combining Theorem 5.2 with Claim 5.1 we obtain the following conclusion. In dimension $d = 1, 2$, the first moment $\langle u(t,x)\rangle$ always grows exponentially fast, whereas in dimension $d \geq 3$ we have exponential growth if $\gamma/(\kappa+\rho)$ exceeds the critical value r_d. Otherwise $\langle u(t,x)\rangle$ grows subexponentially.

We introduce the operator

$$\mathcal{H}_\gamma := \Delta + \gamma\delta_0 \tag{5.5}$$

on $\ell^2(\mathbb{Z}^d)$. The proof of Theorem 5.2 is based on a characterization of the spectrum of $\mathcal{H}_\gamma$. We will compute the spectrum of $\mathcal{H}_\gamma$ and use this result to derive the spectrum of $\mathcal{H}^1$ by a scaling argument. The following lemma is well-known and can be found e.g. in [4], Lemma 1.3.1.

Lemma 5.3 (The spectrum of $\mathcal{H}_\gamma$)

(i) The ℓ^2-spectrum of the operator $\mathcal{H}_\gamma$ has the form

$$\mathrm{Sp}(\mathcal{H}_\gamma) = [-4d, 0] \cup \{\mu\}, \tag{5.6}$$

with

$$\mu \begin{cases} = 0, & \text{if } \gamma \leq r_d, \\ > 0, & \text{if } \gamma > r_d, \end{cases}$$

and, in the latter case, μ is the unique positive solution of the equation

$$\gamma G_d(\cdot) = 1; \tag{5.7}$$

(ii) if $\mu > 0$, the principal Eigenvalue is simple and corresponds to a positive eigenfunction of $\mathcal{H}_\gamma$.

Proof of Lemma 5.3. (i) A sketch of this proof is given in Gärtner and den Hollander [4].

Let $\hat{F}\colon \ell^2(\mathbb{Z}^d) \to L^2([-\pi,\pi)^d)$ denote the Fourier transformation defined by

$$\left(\hat{F}v\right)(k) = \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-ikx} v(x), \qquad k \in [-\pi,\pi)^d,$$

and let $\hat{\mathcal{H}}_\gamma$ denote the Fourier transform of the operator $\mathcal{H}_\gamma$ given by

$$\hat{F}^{-1}\hat{\mathcal{H}}_\gamma\hat{F} = \mathcal{H}_\gamma.$$

Then $\hat{\mathcal{H}}_\gamma$ acts on $\hat{v} \in L^2([-\pi,\pi)^d)$ as

$$\left(\hat{\mathcal{H}}_\gamma\hat{v}\right)(k) = -\hat{\varphi}(k)\,\hat{v}(k) + \gamma\frac{1}{(2\pi)^d}\int\limits_{[-\pi,\pi)^d}\hat{v}(l)\,dl$$

with

$$\hat{\varphi}(k) := \sum_{\substack{x\in\mathbb{Z}^d\\|x|=1}}(1-\cos(k\cdot x)), \qquad k\in[-\pi,\pi)^d.$$

Since $\hat{\mathcal{H}}_\gamma$ is isometric equivalent to $\mathcal{H}_\gamma$, it follows that $\mathrm{Sp}(\mathcal{H}_\gamma) = \mathrm{Sp}(\hat{\mathcal{H}}_\gamma)$. Additionally, $\hat{\mathcal{H}}_\gamma$ and $\mathcal{H}_\gamma$ are self-adjoint, hence the spectrum of $\hat{\mathcal{H}}_\gamma$ (and of $\mathcal{H}_\gamma$) consists of those $\lambda\in\mathbb{R}$, where $\lambda-\hat{\mathcal{H}}_\gamma$ is not invertible. Thus we have to find all λ, such that for a given $g\in L^2([-\pi,\pi)^d)$ there exists no $f\in L^2([-\pi,\pi)^d)$ which satisfies the equation $(\lambda-\hat{\mathcal{H}}_\gamma)f = g$, i.e.,

$$(\lambda+\hat{\varphi}(k))\,f(k) - \gamma\frac{1}{(2\pi)^d}\int\limits_{[-\pi,\pi)^d}f(l)\,dl = g(k) \qquad \text{Lebesgue–a.s. in } [-\pi,\pi)^d. \tag{5.8}$$

Observe that the range of the continuous mapping $\hat{\varphi}$ is $[0,4d]$. Hence, for $\lambda\in[-4d,0]$, there exists $k^*\in[-\pi,\pi)^d$ such that $\lambda+\hat{\varphi}(k^*)=0$ and

$$\begin{aligned}|\lambda+\hat{\varphi}(k)| &= |(\lambda+\hat{\varphi}(k))-(\lambda+\hat{\varphi}(k^*))|\\ &= |\hat{\varphi}(k)-\hat{\varphi}(k^*)|\\ &\le 2d\,\|k-k^*\|\,.\end{aligned}$$

We know, that a function of the type $f(x) = |x|^{-\alpha}$, $x\in[-\pi,\pi)^d$ is in $L^2([-\pi,\pi)^d)$ if $0\le\alpha<d/2$. Consider the function

$$g(k) = \|\,k-k^*\|^{-(2d-1)/4} \qquad \in L^2\left([-\pi,\pi)^d\right).$$

For

$$r := \begin{cases}\pi, & \text{if } \int_{[-\pi,\pi)^d}f(l)\,dl\ge 0;\\ \left(-\frac{2\gamma}{(2\pi)^d}\int_{[-\pi,\pi)^d}f(l)\,dl\right)^{-4/(2d-1)}, & \text{if } \int_{[-\pi,\pi)^d}f(l)\,dl<0;\end{cases} \tag{5.9}$$

is always $r>0$ and, for every $k\in\{k\,|\,k\in[-\pi,\pi)^d \text{ and } \|k-k^*\|\le r\}$ with $\lambda+\hat{\varphi}(k)\ne 0$,

$$-\frac{\gamma}{(2\pi)^d}\int\limits_{[-\pi,\pi)^d}f(l)\,dl \le \frac{1}{2}\,\|k-k^*\|^{-(2d-1)/4}. \tag{5.10}$$

Then equation (5.8) transforms into

$$\begin{aligned}(\lambda+\hat{\varphi}(k))\,f(k) &= \|k-k^*\|^{-(2d-1)/4}+\frac{\gamma}{(2\pi)^d}\int\limits_{[-\pi,\pi)^d} f(l)\,dl \\ &\geq \frac{1}{2}\,\|k-k^*\|^{-(2d-1)/4} \qquad (5.11)\end{aligned}$$

which leads to the inequality

$$|f(k)|=\left|\frac{g(k)+\gamma/(2\pi)^d\int_{[-\pi,\pi)^d}f(l)\,dl}{\lambda+\hat{\varphi}(k)}\right|\geq\frac{1/2\;\|k-k^*\|^{-(2d-1)/4}}{2d\;\|k-k^*\|}=\frac{1}{4d}\;\|k-k^*\|^{-(2d+3)/4}.$$

The last inequality shows that f^2 is not (locally) integrable, hence $f\notin L^2([-\pi,\pi)^d)$. In other words, the operator $(\lambda-\hat{\mathcal{H}}_\gamma)$ is not surjective for $\lambda\in[-4d,0]$, i.e., $[-4d,0]\subset\mathrm{Sp}(\hat{\mathcal{H}}_\gamma)$.

Consider the case $\lambda>0$. Then also $\lambda+\hat{\varphi}(k)>0$ for all $k\in[-\pi,\pi)^d$ and equation (5.8) transforms into

$$f(k)-\frac{\gamma}{\lambda+\hat{\varphi}(k)}\int\limits_{[-\pi,\pi)^d}\frac{1}{(2\pi)^d}f(l)\,dl=\frac{g(k)}{\lambda+\hat{\varphi}(k)}. \tag{5.12}$$

Multiplying by $(2\pi)^{-d}$ and integrating over k gives

$$\begin{aligned}&\int\limits_{[-\pi,\pi)^d}\frac{1}{(2\pi)^d}f(k)\,dk-\int\limits_{[-\pi,\pi)^d}\frac{1}{(2\pi)^d}\frac{\gamma}{\lambda+\hat{\varphi}(k)}\,dk\int\limits_{[-\pi,\pi)^d}\frac{1}{(2\pi)^d}f(l)\,dl\\ &=\int\limits_{[-\pi,\pi)^d}\frac{1}{(2\pi)^d}\frac{g(k)}{\lambda+\hat{\varphi}(k)}\,dk. \qquad (5.13)\end{aligned}$$

The Fourier representation for equation (5.2) reads

$$G_d(\lambda)=\int_{[-\pi,\pi)^d}\frac{1}{(2\pi)^d}\frac{1}{(\lambda+\hat{\varphi}(k))}\,dk. \tag{5.14}$$

Combining (5.13) and (5.14), we obtain

$$(1-\gamma G_d(\lambda))\int\limits_{[-\pi,\pi)^d}\frac{1}{(2\pi)^d}f(l)\,dl=\int\limits_{[-\pi,\pi)^d}\frac{1}{(2\pi)^d}\frac{g(k)}{\lambda+\hat{\varphi}(k)}\,dk. \tag{5.15}$$

If $1-\gamma G_d(\lambda)=0$, then the operator $(\lambda-\hat{\mathcal{H}}_\gamma)$ is not injective, hence λ is an eigenvalue of $\hat{\mathcal{H}}_\gamma$. If, on the other hand, $1-\gamma G_d(\lambda)\neq 0$, then we get a unique solution by plugging (5.12) into (5.15),

$$f(k)=\frac{1}{\lambda+\hat{\varphi}(k)}\left(g(k)+\frac{\gamma}{1-\gamma G_d(\lambda)}\int\limits_{[-\pi,\pi)^d}\frac{1}{(2\pi)^d}\frac{g(l)}{\lambda+\hat{\varphi}(l)}\,dl\right). \tag{5.16}$$

Hence $(\lambda - \hat{\mathcal{H}}_\gamma)$ is invertible and $\lambda \notin \mathrm{Sp}(\hat{\mathcal{H}}_\gamma)$.

Finally, consider $\lambda < -4d$. Then $\lambda + \varphi(k) < 0$ for all $k \in [-\pi, \pi)^d$, hence $G_d(\lambda) < 0$ and $\gamma G_d(\lambda) \neq 1$. Therefore, f can be written in the form of equation (5.16) and $\lambda \notin \mathrm{Sp}(\hat{\mathcal{H}}_\gamma)$.

We summarize that the spectrum $\mathrm{Sp}(\hat{\mathcal{H}}_\gamma)$ consists of the interval $[-4d, 0]$ and the (unique) positive solution of the equation $1 - \gamma G_d(\lambda) = 0$, if it exists. This solution exists if and only if $\gamma > r_d$, because $r_d = 1/G_d(0)$ and $G_d(\lambda)$ is monotonically decreasing.

(ii) The proof of the second assertion uses the resolvent method. For $\lambda > 0$, denote by R_λ the resolvent of the discrete Laplacian Δ,

$$R_\lambda = (\lambda - \Delta)^{-1} . \tag{5.17}$$

Then, for all $f \in \ell^2(\mathbb{Z}^d)$, $x \in \mathbb{Z}^d$,

$$\begin{aligned}
(R_\lambda f)(x) &= \int_0^\infty \mathrm{e}^{-\lambda t} P_t f(x)\, dt \\
&= \int_0^\infty \mathrm{e}^{-\lambda t} \sum_{y \in \mathbb{Z}^d} p(t, x-y) f(y)\, dt \\
&= \sum_{y \in \mathbb{Z}^d} \underbrace{\left(\int_0^\infty \mathrm{e}^{-\lambda t} p(t, x-y)\, dt \right)}_{=: r_\lambda(x-y)} f(y).
\end{aligned} \tag{5.18}$$

Consider the case that the principal eigenvalue $\mu := \sup \mathrm{Sp}(\mathcal{H}_\gamma)$ is positive. The eigenequation for μ reads

$$\Delta v + \gamma \delta_0 v = \mu v. \tag{5.19}$$

This is equivalent to

$$(\mu - \Delta)\, v = \gamma\, v(0)\, \delta_0,$$

hence

$$v(x) = \gamma\, v(0)\, (R_\mu \delta_0)(x) = \gamma\, v(0)\, r_\mu(x).$$

Since $r_\mu > 0$, the eigenfunction v corresponding to μ can be chosen positive. The last equation holds for all eigenfunctions of $\mathcal{H}_\gamma$ corresponding to the eigenvalue μ, hence the dimension of the corresponding eigenspace is 1. ■

Proof of Theorem 5.2. With the Rayleigh-Ritz formula (4.23) and Corollary 3.4 we get

$$\begin{aligned}
\lambda_1 &= \sup \mathrm{Sp}(\mathcal{H}^1) \\
&= \sup_{\|f\|=1} \left[(\kappa + \rho) \Delta f + \gamma \delta_0 f \right] \\
&= \sup_{\|f\|=1} (\kappa + \rho) \left[\Delta f + \frac{\gamma}{\kappa + \rho} \delta_0 f \right] \\
&= (\kappa + \rho) \sup \mathrm{Sp}(\mathcal{H}_{\gamma/(\kappa+\rho)}).
\end{aligned}$$

Now the statement of the theorem is a consequence of Lemma 5.3. ■

Proof of Claim 5.1. By equation (5.14) is r_d positive if and only if the integral $\int_{[-\pi,\pi)^d} \frac{dk}{\hat{\varphi}(k)}$ converges. The Taylor expansion for $\hat{\varphi}(k)$ at 0 yields for $k \to 0$,

$$\hat{\varphi}(k) = \sum_{j=1}^{d} \left(k_j^2 + O(k_j^4)\right).$$

Hence, it remains to show, whether the integral

$$\int_{[-\pi,\pi)^d} \frac{1}{\sum_{j=1}^d k_j^2}\, dk$$

is finite or infinite. Denote by $B_R^d := \left\{x \in \mathbb{R}^d \middle|\, \|x\| \le R\right\}$ the d-dimensional ball of radius $R > 0$ and set

$$f(k_1, \dots, k_d) := \frac{1}{\sum_{j=1}^d k_j^2}, \qquad (k_1, \dots, k_d) \in \mathbb{R}^d.$$

Let

$$\pi\colon (r, \varphi) \mapsto (r\varphi_1, \dots, r\varphi_d), \qquad (r, \varphi) \in (0, \infty) \times S^{d-1},$$

denote the generalized polar coordinate transformation. Here, S^{d-1} represents the unit sphere in $\mathbb{R}^d$. For all $R > 0$,

$$\begin{aligned}
\int\limits_{B_R^d} f(k)\, dk &= \int\limits_{[0,R]} \int\limits_{S^{d-1}} f(\pi(r,\varphi))\, r^{d-1}\, dr\, d\varphi \\
&= \int\limits_{S^{d-1}} d\varphi \int\limits_{[0,R]} r^{-2}\, r^{d-1}\, dr \\
&= \omega_n \int\limits_{[0,R]} r^{d-3}\, dr,
\end{aligned}$$

where ω_n represents the area of the surface of the unit sphere S^{d-1}. Hence for $d \le 2$,

$$\int\limits_{[-\pi,\pi)^d} f(k)\, dk \ge \int\limits_{B_\pi^d} f(k)\, dk = \infty,$$

whereas for $d \ge 3$,

$$\int\limits_{[-\pi,\pi)^d} f(k)\, dk \le \int\limits_{B_{\pi\sqrt{d}}^d} f(k)\, dk < \infty.$$

■

Chapter 6

Analysis of the Higher Lyapunov Exponents and Intermittency

In this chapter we study the behaviour of λ_p for varying $p \in \mathbb{N}$ and analyse the intermittency of the system.

6.1 General p-intermittency

As before, let u denote the solution of the parabolic Anderson problem (1.1). The statements of this section are valid for any nonnegative potential ξ, given that the Lyapunov exponents (1.4) exist.

Lemma 6.1 (General properties of Lyapunov exponents)
(i) *For all* $p \in \mathbb{N}$,

$$\frac{\lambda_p}{p} \leq \frac{\lambda_{p+1}}{p+1};$$

(ii) *the mapping* $p \mapsto \lambda_p$ *is convex, i.e., for all* $p, q \in \mathbb{N}$, $\alpha \in (0,1)$ *with* $\alpha p + (1-\alpha) q \in \mathbb{N}$,

$$\lambda_{\alpha p + (1-\alpha) q} \leq \alpha \lambda_p + (1-\alpha) \lambda_q.$$

Proof.

(i) We use Hölder's inequality, which states that, for any $r, s > 1$ with $1/r + 1/s = 1$ and any nonnegative random variables f, g,

$$\mathbb{E}[fg] \leq \mathbb{E}[f^r]^{1/r} \mathbb{E}[g^s]^{1/s}.$$

Applying this inequality to $f = u(t,x)^{p+1}, g = u(t,x)^{p/(p+1)}, r = (p+1)/p, s = p+1$ yields

$$\langle u(t,x)^p \rangle^{\frac{1}{p}} \leq \langle u(t,x)^{p+1} \rangle^{\frac{1}{p+1}}.$$

Hence $\lambda_p / p \leq \lambda_{p+1}/(p+1)$.

(ii) Let $\alpha \in (0,1)$ and $p, q, \alpha p + (1-\alpha)q \in \mathbb{N}$. Applying Hölder's inequality, we find that for $f = u(t,x)^{p\alpha}$, $g = u(t,x)^{q(1-\alpha)}$, $r = 1/\alpha$, $s = 1/(1-\alpha)$,

$$\langle u(t,x)^{\alpha p + (1-\alpha)q} \rangle \leq \langle u(t,x)^p \rangle^\alpha \langle u(t,x)^q \rangle^{1-\alpha}.$$

This implies the desired inequality $\lambda_{\alpha p+(1-\alpha)q} \leq \alpha \lambda_p + (1-\alpha)\lambda_q$. ■

Remark. We had to restrict the convexity to those $\alpha \in (0,1)$ with $\alpha p + (1-\alpha)q \in \mathbb{N}$, because otherwise limit (1.4) may not exist.

Corollary 6.2
If $\lambda_p/p < \lambda_{p+1}/(p+1)$ for some $p \in \mathbb{N}$, then $\lambda_q/q < \lambda_{q+1}/(q+1)$ for all $q \in \mathbb{N}$ with $q > p$.

Proof. It is sufficient to show the corollary for $q = p+1$. We proceed indirectly by assuming that $\lambda_p/p < \lambda_{p+1}/(p+1)$ but $\lambda_{p+1}/(p+1) = \lambda_{p+2}/(p+2)$. Then, by Lemma 6.1,

$$\lambda_{p+1} \leq \frac{1}{2}\lambda_p + \frac{1}{2}\lambda_{p+2} < \frac{1}{2}\left(\frac{p}{p+1}\lambda_{p+1} + \frac{p+2}{p+1}\lambda_{p+1}\right) = \lambda_{p+1},$$

which is a contradiction. Therefore, $\lambda_{p+1}/(p+1) < \lambda_{p+2}/(p+2)$. ■

6.2 Large p behaviour

We consider again the case that the random potential ξ has the form (1.2). In this section we will prove a result for the asymptotic behaviour of λ_p as p tends to infinity. Denote by $\bar{\lambda}$ the upper boundary of the spectrum $\mathrm{Sp}(\kappa\Delta + \gamma\delta_0)$.

Lemma 6.3
If $\rho = 0$, then

$$\frac{\lambda_p}{p} = \bar{\lambda}, \qquad \text{for all } p \in \mathbb{N}. \tag{6.1}$$

The statement of the lemma implies that in the setting of a fixed catalyst ($\rho = 0$) the system is not intermittent.

Proof. Let $\rho = 0$. We have $\mathcal{H}^1 = \kappa\Delta + \gamma\delta_0$ by Corollary 3.4. For $p > 1$,

$$\begin{aligned} \mathcal{H}^p &= \kappa\Delta_1 + \cdots + \kappa\Delta_p + \gamma\delta_0^{(1)} + \cdots + \gamma\delta_0^{(p)} \\ &= \sum_{i=1}^{p} \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes \mathcal{H}^1 \otimes \underbrace{I \otimes \cdots \otimes I}_{p-i \text{ times}}, \end{aligned}$$

and by Theorem VIII.33 in Reed-Simon [10],

$$\mathrm{Sp}(\mathcal{H}^p) = \sum_{i=1}^{p} \mathrm{Sp}(\mathcal{H}^1),$$

where $\sum$ refers to the addition of sets. Therefore, by Theorem 4.1, $\lambda_p = p\lambda_1 = p\bar{\lambda}$. ■

Theorem 6.4
As $p \nearrow \infty$,

$$\frac{\lambda_p}{p} \nearrow \bar{\lambda}. \tag{6.2}$$

Proof. By Lemma 6.1, λ_p/p is monotonically increasing. The Rayleigh-Ritz formula (4.23) yields

$$\begin{aligned}
\lambda_p &= \sup_{\substack{f\in\ell^2(\mathbb{Z}^{pd})\\ \|f\|=1}} (\mathcal{H}^p f, f) \\
&= \sup_{\substack{f\in\ell^2(\mathbb{Z}^{pd})\\ \|f\|=1}} \left[\Big((\kappa\Delta_1 + \cdots + \kappa\Delta_p + \gamma\delta_0^{(1)} + \cdots + \gamma\delta_0^{(p)})f, f\Big) + (\rho B^p f, f)\right].
\end{aligned}$$

We recall from 3.5 that for $f \in \ell^2(\mathbb{Z}^{pd})$,

$$\begin{aligned}
B^p f(x_1,\ldots,x_p) &= \sum_{\substack{v\in\mathbb{Z}^d\\ v\sim 0}} [f(x_1+v,\ldots,x_p+v) - f(x_1,\ldots,x_p)] \\
&= \sum_{i=1}^{d} [f(x_1+e_i,\ldots,x_p+e_i) - f(x_1,\ldots,x_p)] \\
&\quad + \sum_{i=1}^{d} [f(x_1-e_i,\ldots,x_p-e_i) - f(x_1,\ldots,x_p)],
\end{aligned}$$

where e_i stands for the i-th unit vector in $\mathbb{Z}^d$. Let $f \in \ell^2(\mathbb{Z}^{pd})$. We expand the Dirichlet form for the operator B^p,

$$\begin{aligned}
(B^p f, f) &= \sum_{i=1}^{d} \sum_{x_1,\ldots,x_p\in\mathbb{Z}^d} [f(x_1+e_i,\ldots,x_p+e_i) - f(x_1,\ldots,x_p)]\, f(x_1,\ldots,x_p) \\
&\quad + \sum_{i=1}^{d} \underbrace{\sum_{x_1,\ldots,x_p\in\mathbb{Z}^d} [f(x_1-e_i,\ldots,x_p-e_i) - f(x_1,\ldots,x_p)]\, f(x_1,\ldots,x_p)}_{=\sum_{x_1,\ldots,x_p\in\mathbb{Z}^d}[f(x_1,\ldots,x_p)-f(x_1+e_i,\ldots,x_p+e_i)]\,f(x_1+e_i,\ldots,x_p+e_i)} \\
&= \sum_{i=1}^{d} \sum_{x_1,\ldots,x_p\in\mathbb{Z}^d} -[f(x_1+e_i,\ldots,x_p+e_i) - f(x_1,\ldots,x_p)]^2 \\
&\le 0.
\end{aligned}$$

It is enough to show

$$\lim_{p\to\infty} \frac{1}{p}\,(B^p f, f) = 0.$$

Since

$$2\,|f(x_1+e_i,\dots,x_p+e_i)f(x_1,\dots,x_p)| \le f(x_1+e_i,\dots,x_p+e_i)^2 + f(x_1,\dots,x_p)^2,$$

and $\|f\|=1$, we obtain

$$\begin{aligned}
|(B^p f,f)| &\le \sum_{i=1}^{d}\Bigg[\overbrace{\sum_{x_1,\dots,x_p\in\mathbb{Z}^d} f(x_1+e_i,\dots,x_p+e_i)^2}^{=1} + \overbrace{\sum_{x_1,\dots,x_p\in\mathbb{Z}^d} f(x_1,\dots,x_p)^2}^{=1} \\
&\qquad + \sum_{x_1,\dots,x_p\in\mathbb{Z}^d} 2\,|f(x_1+e_i,\dots,x_p+e_i)f(x_1,\dots,x_p)|\Bigg] \\
&\le 4d
\end{aligned}$$

and

$$\left|\frac{1}{p}\,(B^p f,f)\right| \le \frac{4d}{p} \overset{p\to\infty}{\longrightarrow} 0$$

completes the proof. ■

6.3 Properties of the Lyapunov exponents $\lambda_p(\kappa,\rho)$

In this section we regard the p-th moment Lyapunov exponent $\lambda_p = \lambda_p(\kappa,\rho)$ as a function of κ and ρ and study its dependence on κ. If we impose γ to the model, a scaling argument gives

$$\lambda_p(\kappa,\rho,\gamma) = \gamma\cdot\lambda_p\left(\frac{\kappa}{\gamma},\frac{\rho}{\gamma},1\right), \tag{6.3}$$

hence $\lambda_p(\kappa,\rho)$ contains the full information for all three parameters κ, ρ and γ.

Theorem 6.5
For each $p\in\mathbb{N}$, the function $\lambda_p(\kappa,\rho)$, $(\kappa,\rho)\in[0,\infty)^2$, is continuous, convex, non-increasing in κ and ρ and

$$\lambda_p(\kappa,\rho)=0 \qquad \text{for all } \kappa \ge \frac{\gamma}{r_d}. \tag{6.4}$$

Proof. The proof uses ideas from Carmona and Molchanov [1]. Fix $p\in\mathbb{N}$.

1° *Convexity.* With the help of the Rayleigh-Ritz-Formula (4.23)we can write

$$\begin{aligned}
\lambda_p &= \sup \mathrm{Sp}\,(\mathcal{H}^p) \\
&= \sup_{\substack{f\in\ell^2(\mathbb{Z}^{pd})\\ \|f\|=1}} \left[\kappa\,((\Delta_1+\dots+\Delta_p)f,f) + \rho\,(B^p f,f) + \gamma\left((\delta_0^{(1)}+\dots+\delta_0^{(p)})f,f\right)\right].
\end{aligned}$$

For a fixed function $f \in \ell^2(\mathbb{Z}^{pd})$, the expression in square brackets $[\cdots]$ is linear in κ and ρ and describes a hyperplane. But the supremum of linear functions is convex.

2° *Large* κ*.* By Theorem 6.4, $\lambda_p(\kappa,\rho) \le \bar{\lambda}$. If we consider the model with $\rho = 0$, then, by Lemma 6.3, $\lambda_1 = \bar{\lambda}$ and, by Theorem 5.2, $\lambda_1(\kappa, 0) = 0$ for $\kappa \ge \gamma/r_d$. Therefore $\lambda_p(\kappa,\rho) = 0$ for $\kappa \ge \gamma/r_d$ and all $\rho \in \mathbb{R}^+$.

3° *Monotonicity in* κ*.* From the convexity and the fact that $\lambda_p(\kappa,\rho) = 0$ for $\kappa \ge \gamma/r_d$, it follows immediately that $\lambda_p(\cdot,\rho)$ is monotonically decreasing.

4° *Monotonicity in* ρ*.* Fix $\kappa \in [0,\infty)$. We want to show that $\lambda_p(\kappa,\cdot)$ is monotonically decreasing. We assume the contrary. That means, there exist values $0 \le \rho_1 < \rho_2$ with $\lambda(\kappa,\rho_1) < \lambda(\kappa,\rho_2)$. Due to the convexity this implies that $\lambda(\kappa,\rho) \to \infty$ as $\rho \to \infty$. This contradicts the fact that λ_p has an upper boundary:

$$\begin{aligned}
\langle u(t,0)^p \rangle &= \mathbb{E}_{0,\dots,0;0}^{X^1,\dots,X^p;Y} \exp\left\{ \gamma \int_0^t \sum_{i=1}^p \delta_0(X_s - Y_{t-s})\, ds \right\} \\
&\le \mathrm{e}^{\gamma t p},
\end{aligned}$$

hence

$$\lambda_p(\kappa,\rho) = \lim_{t\to\infty} \frac{1}{t} \log\langle u(t,0)^p \rangle \le \gamma p.$$

5° *Continuity.* Since $(\kappa,\rho) \mapsto \lambda_p(\kappa,\rho)$ is convex, it is continuous on $(0,\infty)^2$. Hence, it remains to check the continuity at $\kappa = 0$ and $\rho = 0$.

We start with $\kappa = 0$. With the Feynman-Kac formula (1.3) we can write

$$\begin{aligned}
\langle u(t,0)^p \rangle &= \left\langle \left(\mathbb{E}_0^X \exp\left\{ \gamma \int_0^t \delta_0(X_s - Y_{t-s})\, ds \right\} \right)^p \right\rangle \\
&\ge \left\langle \left(\mathbb{E}_0^X \exp\left\{ \gamma \int_0^t \delta_0(0 - Y_{t-s})\, ds \right\} 1\!\!1_{\{X_s = 0; 0 \le s \le t\}} \right)^p \right\rangle \\
&= \left\langle \left(\exp\left\{ \gamma \int_0^t \delta_0(Y_{t-s})\, ds \right\} \mathbb{P}_0^X \left(X_s = 0; 0 \le s \le t \right) \right)^p \right\rangle \\
&= \mathrm{e}^{-2d\kappa p t} \left\langle \left(\exp\left\{ \gamma \int_0^t \delta_0(Y_{t-s})\, ds \right\} \right)^p \right\rangle,
\end{aligned}$$

where we used the fact that $(X_t)_{t\ge 0}$ has generator $\kappa\Delta$ and hence the stopping time of the first jump of $(X_t)_{t\ge 0}$ is exponentially distributed with parameter $2d\kappa$. This leads to the inequality

$$\lim_{t\to\infty} \frac{1}{t} \log \langle (u(t,0))^p \rangle \ge -2d\kappa p + \lambda_p(0,\rho).$$

Taking into account that $\lambda_p(\kappa)$ is monotonically decreasing, one has

$$\lambda_p(0,\rho) - 2pd\kappa \le \lambda_p(\kappa,\rho) \le \lambda_p(0,\rho), \tag{6.5}$$

which implies $\lim_{\kappa\to 0}\lambda_p(\kappa,\rho)=\lambda_p(0,\rho)$.

The case $\rho=0$ is very similar. Let again $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ denote random walks on $\mathbb{Z}^d$ with generator $\kappa\Delta$ and $\rho\Delta$, respectively. Then,

$$\begin{aligned}
\langle u(t,0)^p\rangle &= \left\langle\left(\mathbb{E}_0^X \exp\left\{\gamma\int_0^t \delta_0(X_s-Y_{t-s})\,ds\right\}\right)^p\right\rangle\\
&\geq \left\langle\left(\mathbb{E}_0^X \exp\left\{\gamma\int_0^t \delta_0(X_s-0)\,ds\right\}\right)^p 1\!\!1_{\{Y_s=0;0\leq s\leq t\}}\right\rangle\\
&= \mathbb{P}_0^Y\left(Y_s=0;0\leq s\leq t\right)\left\langle\left(\exp\left\{\gamma\int_0^t \delta_0(X_s)\,ds\right\}\right)^p\right\rangle\\
&= \mathrm{e}^{-2d\rho t}\left\langle\left(\exp\left\{\gamma\int_0^t \delta_0(X_s)\,ds\right\}\right)^p\right\rangle,
\end{aligned}$$

hence,

$$\lambda_p(\kappa,\rho)\geq\lambda_p(\kappa,0)-2d\rho.$$

Together with the monotonicity of $\lambda(\kappa,\cdot)$ we get the desired limit

$$\lim_{\rho\to 0}\lambda_p(\kappa,\rho)=\lambda_p(\kappa,0).$$

■

We now consider $\kappa=0$. Later, we will use the following lemma to derive a statement on intermittency in this case.

Lemma 6.6
In the case $\kappa=0$,

$$\lambda_p(0,\rho,\gamma)=\lambda_1(0,\rho,p\gamma). \tag{6.6}$$

Proof. Let $\kappa=0$. Then $X_s^i=0$ for $s\in[0,t]$ almost sure and

$$\begin{aligned}
\langle u(t,0)^p\rangle &= \left\langle\mathbb{E}_{0,\ldots,0}^{X^1,\ldots,X^p}\exp\left\{\gamma\int_0^t\sum_{i=1}^p\delta_0(X_s^{(i)}-Y_{t-s})\,ds\right\}\right\rangle\\
&= \left\langle\exp\left\{p\gamma\int_0^t\delta_0(Y_{t-s})\,ds\right\}\right\rangle\\
&= \left\langle\mathbb{E}_0^{X^1}\exp\left\{p\gamma\int_0^t\delta_0(Y_{t-s})\,ds\right\}\right\rangle
\end{aligned}$$

which, by (1.4), leads to the desired equation

$$\lambda_p(0,\rho,\gamma)=\lambda_1(0,\rho,p\gamma).$$

■

Alternatively, one can prove the previous lemma analytically. With the Rayleigh-Ritz formula it is sufficient to show that

$$\sup \mathrm{Sp}\left(\rho B^p + \gamma(\delta_0^{(1)} + \cdots + \delta_0^{(p)})\right)_{\ell^2(\mathbb{Z}^{pd})} = \sup \mathrm{Sp}\left(\rho\Delta + (p\gamma)\delta_0\right)_{\ell^2(\mathbb{Z}^d)}. \tag{6.7}$$

This seems to be quite natural, because a jump process on $\mathbb{Z}^{pd}$ with generator B^p makes only jumps in diagonal directions.

6.4 Intermittency of the parabolic Anderson model

Finally, we want to analyse the intermittency behaviour of the system.

Theorem 6.7 (Intermittency)
Let $\rho > 0$. If $\kappa\, r_d < \gamma$, then there exists a $p \in \mathbb{N}\backslash\{1\}$ such that the system is p-intermittent, whereas for $\kappa\, r_d \geq \gamma$ the system is not intermittent. Furthermore, for $(\kappa + \rho)\, r_d < \gamma$, the system shows full intermittency.

For completeness, we recall from Lemma 6.3 that, for $\rho = 0$, the system is not intermittent.

We have shown in Claim 5.1 that $r_d = 0$ in dimension $d = 1, 2$. This implies that, for $d = 1, 2$, the system shows full intermittency for all $\kappa \in [0, \infty)$, $\rho, \gamma \in (0, \infty)$.

In the case $\kappa = 0$, there exists some $p \in \mathbb{N} \setminus \{1\}$ such that the system is p-intermittent. One may think that the system shows always full intermittency for $\kappa = 0$, but this is not generally true in dimension $d \geq 3$. By Theorem 5.2, $\lambda_1(0, \rho, \gamma) > 0$ if and only if

$$\gamma > r_d\, \rho.$$

Hence, with the help of Lemma 6.6, we conclude that $\lambda_p(0, \rho, \gamma) > 0$ if and only if

$$p\,\gamma > r_d\, \rho,$$

and this is obviously satisfied for sufficiently large $p \in \mathbb{N}$.

We start proving Theorem 6.7. To this end, we need the following lemma.

Lemma 6.8
If λ_1 is an eigenvalue of $\mathcal{H}^1$ corresponding to a positive eigenfunction and $\rho > 0$, then $\lambda_2/2 > \lambda_1$, i.e., the system shows full intermittency.

Proof. We assume that λ_1 corresponds to the positive eigenfunction v and v is normalized as $\|v\| = 1$. We introduce the operator

$$\tilde{\mathcal{H}^2} := \mathcal{H}^1 \otimes \mathcal{H}^1$$

acting on $\mathbb{Z}^d \times \mathbb{Z}^d$ as

$$\tilde{\mathcal{H}}^2 = (\kappa+\rho)(\Delta_1+\Delta_2) + \gamma(\delta_0^{(1)} + \delta_0^{(2)}).$$

Then $v \otimes v(x,y) = v(x)\, v(y)$ is an eigenfunction of $\tilde{\mathcal{H}}^2$ corresponding to the eigenvalue $2\lambda_1$ and, with the Rayleigh-Ritz formula (4.23),

$$\begin{aligned} \lambda_2 - 2\lambda_1 &= \sup \mathrm{Sp}(\mathcal{H}^2) - \sup \mathrm{Sp}(\tilde{\mathcal{H}}^2) \\ &= \sup_{\|f\|=1} \left(\mathcal{H}^2 f, f\right) - \left(\tilde{\mathcal{H}}^2\, v \otimes v, v \otimes v\right) \\ &\geq \left(\left(\mathcal{H}^2 - \tilde{\mathcal{H}}^2\right) v \otimes v, v \otimes v\right). \end{aligned}$$

We compute

$$\begin{aligned} & \left(\left(\mathcal{H}^2 - \tilde{\mathcal{H}}^2\right) v \otimes v, v \otimes v\right) \\ = \;& \rho\left(\left(B^2 - \Delta_1 - \Delta_2\right) v \otimes v, v \otimes v\right) \\ = \;& 2\rho \sum_{x,y\in\mathbb{Z}^d} \sum_{i=1}^d \left[v(x)v(y+e_i) - v(x)v(y)\right]\, \left[v(x-e_i)v(y) - v(x)v(y)\right] \\ = \;& 2\rho \sum_{x,y\in\mathbb{Z}^d} \sum_{i=1}^d v(x)\left[v(x-e_i) - v(x)\right] v(y) \left[v(y+e_i) - v(y)\right] \\ = \;& \rho \sum_{x,y\in\mathbb{Z}^d} \sum_{i=1}^d \left[v(x-e_i) - v(x)\right]^2 \left[v(y+e_i) - v(y)\right]^2, \end{aligned}$$

but this expression vanishes if and only if $v \equiv 0$, because $v \in \ell^2(\mathbb{Z}^d)$ implies $\lim_{|z|\to\infty} v(z) = 0$. Therefore, $\lambda_2 - 2\lambda_1 > 0$. ■

Proof of Theorem 6.7. With Lemma 6.8, it remains to show that for $\gamma \leq (\kappa+\rho) r_d$ and $\kappa\, r_d < \gamma$ there exists $p \in \mathbb{N} \setminus \{1\}$ such that

$$\frac{\lambda_{p-1}}{p-1} < \frac{\lambda_p}{p}.$$

For $\kappa\, r_d < \gamma \leq (\kappa+\rho) r_d$ we have $\lambda_1 = 0$ and $\bar{\lambda} > 0$. By Theorem 6.4, $\lambda_p/p \nearrow \bar{\lambda}$ as $p \to \infty$, hence there exists a $p \in \mathbb{N}$ such that $\lambda_p > 0$. Set $p^* := \min\{p \in \mathbb{N} | \lambda_p > 0\}$. Then the system is p^*-intermittent. ■

Appendix A

Feynman-Kac Formula

We consider the (deterministic) Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \kappa\Delta u(t,x) + \xi(t,x)\,u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d; \\ u(0,x) = u_0(x), & x \in \mathbb{Z}^d; \end{cases} \tag{A.1}$$

with $\xi(t,x) = \gamma\delta_{Y_t}(x)$ where $(Y_s)_{0\le s\le t}$ is a fixed, piecewise constant path in $\mathbb{Z}^d$. Then, $\xi(t,\cdot)$ is a field $\mathbb{Z}^d \to \mathbb{R}^+$ which is piecewise constant in the time variable with (deterministic) jump times $0 = \tau_0 < \tau_1 < \cdots < \tau_{r-1} < \tau_r = t$.

We say that the bounded nonnegative function $u\colon \mathbb{R}^+ \times \mathbb{Z}^d \to \mathbb{R}$ solves Problem (A.1) if $u(t,x)$ is continuous and, moreover, on the open time interval (τ_{i-1},τ_i), $i = 1,\dots,r$, it is differentiable with respect to t and solves the heat equation

$$\frac{\partial u}{\partial t}(t,x) = \kappa\Delta u(t,x) + \xi(t,x)\,u(t,x).$$

In this Appendix, we will show that there exists a unique nonnegative solution to problem (A.1), which has the form (1.3).

We therefore split the function u into $u_1,\dots,u_r$ (see Figure A.1) by

$$u_l(s,x) = u(\tau_r - \tau_l + s, x), \qquad 1 \le l \le r,\ 0 \le s \le \tau_{l-1} - \tau_{l-2},\ x \in \mathbb{Z}^d,$$

with the initial condition

$$u(0,x) = u_0(x) = u_1(0,x), \qquad x \in \mathbb{Z}^d,$$

$$u(t,x) = u_r(\tau_r - \tau_{r-1}, x), \qquad x \in \mathbb{Z}^d,$$

and the continuity property

$$u_{l-1}(\tau_{l-1} - \tau_{l-2}, x) = u_l(0,x), \qquad 2 \le l \le r,\ x \in \mathbb{Z}^d. \tag{A.2}$$

Let $(X_t, \mathcal{F}_t, \mathbb{P}_x)$ denote the simple symmetric random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$ and corresponding family of time shift operators $\{\theta_s; 0 \le s \le t\}$.

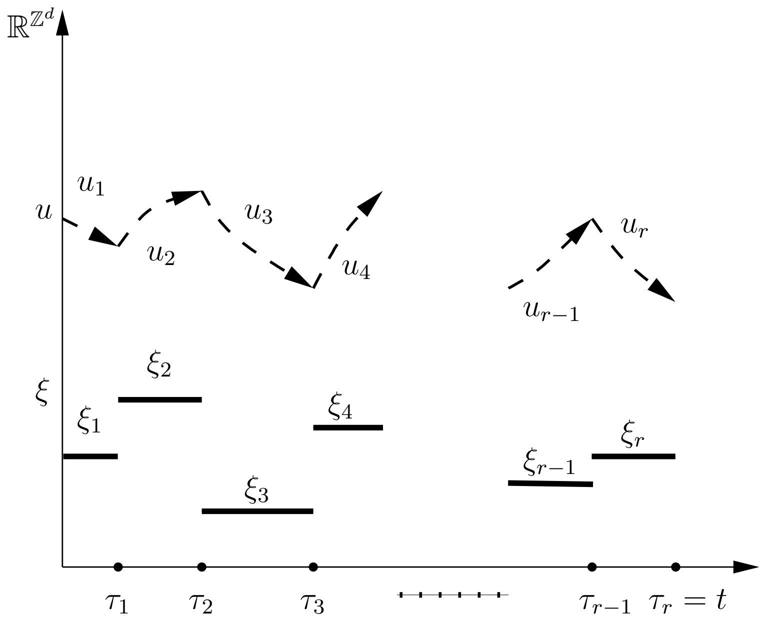

Figure A.1: The time evolution of ξ and u

We use the Feynman-Kac formula for the time-independent parabolic Anderson model to obtain

$$u_l(s,x) = \mathbb{E}_x\, u_l(0, X_s) \exp\left\{ \int_0^s \xi_l(X_w)\, dw \right\}, \qquad 1 \le l \le r,\ s \in [0, \tau_l - \tau_{l-1}],\ x \in \mathbb{Z}^d. \tag{A.3}$$

It has been shown by Gärtner and Molchanov (cf. [6], Section 2.2) that u_l is the unique nonnegative solution of the corresponding Cauchy problem, if the expression on the right hand side in (A.3) is finite and ξ is non-percolating from below. Both is fulfilled if ξ is bounded.

We use the Markov property (MP) for the process X to obtain

$$\begin{aligned} u(t,x) &= u_r(\tau_r - \tau_{r-1}, x) \\ &\overset{(A.3)}{=} \mathbb{E}_x u_r(0, X_{\tau_r - \tau_{r-1}}) \exp\left\{ \int_0^{\tau_r - \tau_{r-1}} \xi_r(X_s)\, ds \right\} \\ &\overset{(A.2)}{=} \mathbb{E}_x u_{r-1}(\tau_{r-1} - \tau_{r-2}, X_{\tau_r - \tau_{r-1}}) \exp\left\{ \int_0^{\tau_r - \tau_{r-1}} \xi_r(X_s)\, ds \right\}, \end{aligned}$$

but

$$
\begin{aligned}
& u_{r-1}(\tau_{r-1}-\tau_{r-2}, X_{\tau_r-\tau_{r-1}}) \\
\overset{(A.3)}{=} \quad & \mathbb{E}_{X_{\tau_r-\tau_{r-1}}} u_{r-1}(0, X_{\tau_{r-1}-\tau_{r-2}}) \exp\left\{ \int_0^{\tau_{r-1}-\tau_{r-2}} \xi_{r-1}(X_s)\, ds \right\} \\
\overset{(MP)}{=} \quad & \mathbb{E}_x \left[u_{r-1}(0, X_{\tau_{r-1}-\tau_{r-2}} \circ \theta_{\tau_r-\tau_{r-1}}) \exp\left\{ \int_0^{\tau_{r-1}-\tau_{r-2}} \xi_{r-1}(X_s \circ \theta_{\tau_r-\tau_{r-1}})\, ds \right\} \middle| \mathcal{F}_{\tau_r-\tau_{r-1}} \right] \\
= \quad & \mathbb{E}_x \left[u_{r-1}(0, X_{\tau_r-\tau_{r-2}}) \exp\left\{ \int_{\tau_r-\tau_{r-1}}^{\tau_r-\tau_{r-2}} \xi_{r-1}(X_s)\, ds \right\} \middle| \mathcal{F}_{\tau_r-\tau_{r-1}} \right],
\end{aligned}
$$

such that

$$
\begin{aligned}
u(t,x) &= \mathbb{E}_x \underbrace{\left[\exp\left\{ \int_0^{\tau_r-\tau_{r-1}} \xi_r(X_s)\, ds \right\} \right.}_{\mathcal{F}_{\tau_r-\tau_{r-1}}\text{-measurable}} \\
&\qquad \left. \times\, \mathbb{E}_x \left[u_{r-1}(0, X_{\tau_r-\tau_{r-2}}) \exp\left\{ \int_{\tau_r-\tau_{r-1}}^{\tau_r-\tau_{r-2}} \xi_{r-1}(X_s)\, ds \right\} \middle| \mathcal{F}_{\tau_r-\tau_{r-1}} \right] \right] \\
&= \mathbb{E}_x\, u_{r-1}(0, X_{\tau_r-\tau_{r-2}}) \exp\left\{ \int_{\tau_{r-2}}^{\tau_r} \xi(t-s, X_s)\, ds \right\} \\
&= \cdots \\
&= \mathbb{E}_x\, u_1(0, X_{\tau_r-\tau_0}) \exp\left\{ \int_0^{\tau_r} \xi(t-s, X_s)\, ds \right\} \\
&= \mathbb{E}_x\, u_0(X_t) \exp\left\{ \int_0^{t} \xi(t-s, X_s)\, ds \right\}.
\end{aligned}
$$

We have derived the Feynman-Kac formula for the *time-dependent* parabolic Anderson problem (A.1). It has the unique nonnegative solution

$$
u(t,x) = \mathbb{E}_x \exp\left\{ \int_0^t \xi(t-s, X_s) \right\} u_0(X_t). \tag{A.4}
$$

Appendix B

Transformation of Markov Processes

Theorem B.1 (Transformation of the state space)
Let $\mathbb{X} = (X_t, \mathbb{P}_x)$ be a Markov process on the countable state space E with transition function $p(t, x, \Gamma)$. Let γ be a measurable transformation of E into the (countable) state space $\tilde{E}$ such that $\gamma(E) = \tilde{E}$ and such that the following condition is satisfied:

For all $\Gamma \subset E$ and any $x, x' \in E$ such that $\gamma(x) = \gamma(x')$,

$$p(t, x, \gamma^{-1}\Gamma) = p(t, x', \gamma^{-1}\Gamma).$$

Set $\tilde{X}_t = \gamma X_t$ and denote by $\mathcal{F}$ the σ-Algebra generated by the sets $\{\tilde{X}_t \in \tilde{\Gamma}\}$, $t \geq 0$, $\tilde{\Gamma} \subset \tilde{E}$. Further, let $\tilde{\mathbb{P}}_{\gamma x}(A) = \mathbb{P}_x(A)$, $A \in \mathcal{F}$. The system $\tilde{\mathbb{X}} = (\tilde{X}_t, \tilde{\mathbb{P}_x})$ defines a Markov process on the state space $\tilde{E}$ with transition function

$$\tilde{p}(t, x, \Gamma) = p(t, x, \gamma^{-1}\Gamma).$$

If $\mathbb{X}$ is a strong Markov process, so is the process $\tilde{\mathbb{X}}$.

Denote by $B(E)$ and $B(\tilde{E})$ the Banach space of bounded functionals on E and $\tilde{E}$, respectively. We define a transformation γ^* of the space $B(\tilde{E})$ into the space $B(E)$ by the formula

$$\gamma^* f(x) = f(\gamma x) \qquad (f \in B(\tilde{E}), x \in E).$$

Let $T_t, \tilde{T}_t$ be the semigroups of operators corresponding to the processes $\mathbb{X}$ and $\tilde{\mathbb{X}}$, respectively, and let $A, \tilde{A}$ be the infinitesimal operators (generators) of these semigroups. Then,

$$\gamma^* \tilde{T}_t = T_t \gamma^*,$$

$$\gamma^* \tilde{A}_t = A_t \gamma^*,$$

moreover, $f \in D_{\tilde{A}}$ if and only if $\gamma^* f \in D_A$.

We will say that the process $\tilde{\mathbb{X}}$ is obtained from $\mathbb{X}$ by the transformation γ of the state space and write $\tilde{\mathbb{X}} = \gamma(\mathbb{X})$.

Reference: Dynkin [3], Chapter X, Theorem 10.13. In fact, Dynkin formulates the theorem for general state spaces. We limit ourselves to countable state spaces and omit filtrations.

Appendix C

Perron-Frobenius Theory

Let $A \in \mathbb{R}^{n\times n}$ be a matrix with nonnegative entries. We define A to be reducible, if there is a permutation matrix P such that

$$P^T A P = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

We call A to be irreducible if no such P exists. This is equivalent to the fact, that there exists $p \in \mathbb{N}$ such that the matrix A^p has strictly positive entries.

Theorem C.1 (Perron-Frobenius)
If the matrix $A \in \mathbb{R}^{n\times n}$ is (pointwise) nonnegative and irreducible, then,

(i) the matrix A has a positive eigenvalue r, equal to the spectral radius of A;

(ii) there is a positive (right) eigenvector associated with the eigenvalue r;

(iii) the eigenvalue r has algebraic multiplicity 1.

Reference: Lancaster and Tismenetsky [9], Paragraph 15.3, Theorem 1.

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