A NOTE ON THE PURE MAPPING CLASS GROUP OF AN INFINITE-TYPE SURFACE OF FINITE GENUS

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Abstract. We show that the pure mapping class group of an infinite-type surface of finite genus is not generated by the collection of multitwists (i.e. products of powers of twists about disjoint non-accumulating curves).

1. Introduction

The mapping class group of a surface of finite type has been thoroughly studied since decades. In particular, multiple *simple* sets of generators are known. The Dehn–Lickorish theorem ([Deh38], [Lic64]), in combination with the Birman exact sequence ([Bir69]), shows that the pure mapping class group of a finite-type surface can be generated by finitely many Dehn twists about nonseparating curves, and we need to add finitely many half-twists to generate the full mapping class group. Humphries [Hum79] proved that, if the surface is closed and of genus $g \geq 2$, $2g+1$ Dehn twists about nonseparating curves suffice to generate the mapping class group, and moreover this number is optimal: fewer than $2g+1$ Dehn twists cannot generate. Other results show that mapping class groups can be generated by two elements (see e.g. [Waj96]), by finitely many involutions or by finitely many torsion elements (see e.g. [BF04]).

In the case of surfaces of infinite type, the (pure) mapping class group is uncountable, so in particular it is not finitely (nor countably) generated. For a special class of surfaces, Malestein and Tao [MT21] proved that mapping class groups are generated by involutions, and normally generated by a single involution, but to the best of our knowledge, no other generating set is known.

Note that the (pure) mapping class group of a surface of infinite type is endowed with an interesting topology, induced by the compact-open topology on the group of homeomorphisms of the surface. So *topological* generating sets (sets whose *closure* of the group they generate is the (pure) mapping class group) have been investigated as well. In particular, Patel and Vlamis [PV18] proved that the pure mapping class group of a surface is topologically generated by Dehn twists if the surface has at most one nonplanar end, and by Dehn twists and maps called *handle shifts* otherwise.

The goal of this note is to investigate a natural candidate for a set of generators of the pure mapping class group of a surface: the collection of *multitwists*. A multitwist is a (possibly infinite) product of powers of Dehn twists about a collection of simple closed curves which do not accumulate anywhere in the surface[1]. Our main result is a negative one:

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[1] The non-accumulation condition is necessary to have a well defined mapping class.

Theorem A. *Let S be an infinite-type surface of finite genus. Then the collection of multitwists does not generate the pure mapping class group.*

For proving this result, we explicitly construct a mapping class which is not a finite product of multitwists. We use work of Bestvina, Bromberg and Fujiwara [BBF16] to certify that the mapping class we construct is not in the subgroup generated by multitwists. These techniques do not allow us to extend the result to the case of surfaces of infinite genus, which raises the question of whether multitwists generate the pure mapping class group of some infinite-type surface.

2. Preliminaries

In this note, a surface is a connected, orientable, Hausdorff, second countable two-dimensional manifold, without boundary unless otherwise stated. One notable exception are subsurfaces, which will always have compact boundary. Given a surface S with boundary, $\tilde{S}$ will denote the surface obtained by gluing a once-punctured disk to each boundary component of S.

Surfaces are *of finite type* if their fundamental group is finitely generated and *of infinite type* otherwise. A surface S is *exceptional* if it has genus zero and at most four punctures or genus one and at most one puncture, otherwise it is *nonexceptional.*

The *mapping class group* of a surface S is the group $\mathrm{MCG}(S)$ of orientation preserving homeomorphisms of S up to homotopy. The *pure mapping class group* $\mathrm{PMCG}(S)$ is the subgroup of $\mathrm{MCG}(S)$ fixing all ends and boundary components.

A *curve* on a surface is the homotopy class of an essential (i.e. not homotopic to a point, an end or a boundary component) simple closed curve. Given a curve α, we denote by τ_α the Dehn twist about α.

An *integral weighted multicurve* μ is a formal sum $\sum_{i\in I} n_i\alpha_i$, where the α_i are pairwise disjoint curves not accumulating anywhere and the n_i are integers. Given an integral weighted multicurve μ, we define τ_μ to be the mapping class

$$\tau_\mu = \prod_{i\in I} \tau_{\alpha_i}^{n_i}.$$

Such a mapping class is called a *multitwist.*

We say that an integral weighted multicurve is *finite* if I is finite (i.e. it contains finitely many curves). An integral weighted multicurve ν is a *submulticurve* of an integral weighted multicurve $\mu = \sum_{i\in I} n_i\alpha_i$ if $\nu = \sum_{i\in J} n_i\alpha_i$, where $J \subset I$.

Given a group G, a *quasimorphism* $\varphi : G \to \mathbb{R}$ is a function such that

$$\Delta(\varphi) := \sup\{|\varphi(gh) - \varphi(g) - \varphi(h)| \mid g, h \in G\} < \infty.$$

$\Delta(\varphi)$ is called *defect* of the quasimorphisms. A quasimorphism φ is *homogeneous* if for every $g \in G$ and $n \in \mathbb{Z}$, $\varphi(g^n) = n\varphi(g)$.

We have (see e.g. [Cal09, Chapter 2]):

Proposition 1. *Let $\varphi : G \to \mathbb{R}$ be a quasimorphism. Then there is a unique homogeneous quasimorphism $\hat{\varphi}$ (the* homogenized quasimorphism*), given by*

$$\widehat{\varphi}(g) = \lim_{n\to\infty} \frac{\varphi(g^n)}{n} \ \ \forall g \in G,$$

such that φ differs from $\widehat{\varphi}$ by a bounded function.

A slight modification of the proof of [BBF16, Theorem 4.2] gives:

Proposition 2. *Let Σ be a finite-type surface, possibly with boundary, and F a non-exceptional subsurface of Σ. Let $f \in \mathrm{MCG}(\Sigma)$ with support on F and such that $g|_F$ is pseudo-Anosov. Then there is a homogeneous quasimorphism $\varphi : \mathrm{MCG}(\Sigma) \to \mathbb{R}$ which is unbounded on powers of f and zero on all multitwists.*

We add a sketch of proof for completeness. We refer to [BBF16] for the necessary definitions.

Proof. By [BBF16, Theorem 4.2], there is a homogeneous quasimorphism φ which is unbounded on powers of f, since f contains a single equivalence class, which is chiral and essential. Moreover, this quasimorphism is obtained as follows: we first look at the finite-index subgroup $\mathcal{S}$ of $\mathrm{MCG}(\Sigma)$ constructed in [BBF16, Proposition 2.5] and at its action on the projection complex $\mathcal{C}(\mathbb{Y})$, where $\mathbb{Y}$ is the $\mathcal{S}$-orbit of F (see [BBF16, Section 2.6]). This gives us a quasimorphism $\psi_1 : \mathcal{S} \to \mathbb{R}$. Then we choose coset representatives $1 = g_1, \dots, g_s$ of $\mathrm{MCG}(\Sigma)/\mathcal{S}$ and define $\psi_2 : \mathcal{S} \to \mathbb{R}$ by

$$\psi_2(h) = \sum_{i=1}^{n} \psi_1(g_i h g_i^{-1}).$$

The homogenized quasimorphism $\widehat{\psi_2}$ extends to a homogeneous quasimorphism $\varphi : \mathrm{MCG}(\Sigma) \to \mathbb{R}$ given by

$$\varphi(f) = \frac{1}{n}\widehat{\psi_2}(f^n),$$

where n is such that $f^n \in \mathcal{S}$.

Let τ_α be a twist. We claim that $\varphi(\tau_\alpha) = 0$. Let $n > 0$ be such that $\tau_\alpha^n \in \mathcal{S}$. For $i = 1, \dots, s$, $g_i \tau_\alpha^n g_i^{-1}$ is supported on the annulus with core curve $g_i(\alpha)$. So by [BBF16, Lemma 2.8], if it acts hyperbolically on $\mathcal{C}(\mathbb{Y})$, it has virtual quasiaxes intersecting $\mathcal{C}(F')$ in a uniformly bounded segment for every $F' \in \mathbb{Y}$. Thus the projection of its virtual quasiaxes onto the translations of the virtual quasiaxes of g are uniformly bounded. By [BBF16, Corollary 3.2(d)], this implies that ψ_1 is bounded on powers of $g_i \tau_\alpha^n g_i^{-1}$, so ψ_2 is bounded on powers of τ_α^n and thus $\widehat{\psi_2}(\tau_\alpha^n) = 0$. Hence $\varphi(\tau_\alpha) = 0$.

Now let τ_μ be a multitwist. Then it is the commuting product of powers of twists $\tau_{\alpha_k}^{n_k} \dots \tau_{\alpha_1}^{n_1}$; since φ is homogeneous, and thus in particular additive on a product of commuting elements,

$$\varphi(\tau_\mu) = \sum_{j=1}^{k} n_j \varphi(\tau_{\alpha_j}) = 0.$$

□

Remark 3. In Proposition 2, F is allowed to be equal to Σ.

3. Proof of Theorem A

In this section we will prove our main theorem. We will need an observation and two preliminary lemmas:

Remark 4. Let S be a surface and X a surface obtained from S by filling in some punctures. For any curve α on S, let $\pi(\alpha)$ be the homotopy class of α on X. Then for every α, β curves on S,

$$\tau_{\pi(\alpha)}(\pi(\beta)) = \pi(\tau_\alpha(\beta)).$$

Indeed, this can be seen by looking at a homeomorphism f of S realizing τ_α. Then f extended to the identity on $X \smallsetminus S$ realizes $\tau_{\pi(\alpha)}$. So for any b representative of β, $f(b)$ represents $\tau_\alpha(\beta)$ on S and $\tau_{\pi(\alpha)}(\pi(\beta))$ on X.

Lemma 5. *Let S be a surface, $f \in \mathrm{MCG}(S)$ a product of k multitwists with powers and $X \subset S$ an f-invariant subsurface of finite type. Then the map induced by f to $\tilde{X}$ is a product of at most k multitwists.*

Proof. We can think of $\tilde{X}$ as obtained from S by filling in some punctures. Suppose $f = \tau_{\mu_k} \circ \cdots \circ \tau_{\mu_1}$ is a product of k multitwists.

Note first that there are finite submulticurves $\nu_1, \ldots, \nu_k$ of $\mu_1, \ldots, \mu_k$ such that $f|_X = (\tau_{\nu_k} \circ \cdots \circ \tau_{\nu_1})|_X$. Indeed, if τ_μ is a multitwist, for any curve α, $\tau_\mu(\alpha) = \tau_\nu(\alpha)$, where ν is the submulticurve of μ given by curves intersecting α, and ν is finite since α is compact. Moreover, since X is of finite type, there are finitely many curves $\alpha_1, \ldots \alpha_N$ on X such that a mapping class of X is determined by the images of these curves. Applying these two observations allows us to find the multicurves ν_j as required.

By Remark 4

$$f|_{\tilde{X}} = \tau_{\pi(\nu_k)} \circ \cdots \circ \tau_{\pi(\nu_1)},$$

where for a curve α on S, $\pi(\alpha)$ denotes the homotopy class of α on $\tilde{X}$, and for an integral weighted multicurve $\mu = \sum_{i \in I} n_i \alpha_i$ on S, $\pi(\mu)$ denotes the multicurve on $\tilde{X}$ given by

$$\pi(\mu) = \sum_{\substack{i \,:\, \pi(\alpha_i) \\ \text{essential}}} n_i \pi(\alpha_i).$$

□

The second lemma we will need certifies the existence of a sequence of subsurfaces with specific topological properties.

Lemma 6. *Let S be an infinite-type surface of finite genus $g \geq 1$. Then S contains a sequence of subsurfaces X_n of genus g and 6 boundary components (some of which might be homotopic to a puncture), which can be decomposed as $X_n = X \cup P_n \cup Y_n$, where:*

- *X is a surface of genus g and one boundary component;*
- *each Y_n is a 6-holed sphere (where some boundary components might be homotopic to a puncture);*
- *P_n is a pair of pants with one boundary component in common with X and one boundary component in common with Y_n;*
- *the Y_n are pairwise disjoint and leave every compact;*
- *$Y_n \cap P_m$ is empty if $m \neq n$.*

Proof. Choose a surface X of genus g and one boundary component. Since S has finite genus and is of infinite type, there is an end e of S which is not isolated. Let

ℓ be a simple proper arc from a point $p \in \partial X$ to e, such that $\ell \cap X = \{p\}$. Then we can find (see for instance [FGM21]) a nested sequence of surfaces $U_n \subset S \setminus X$ such that:

- ∂U_n is a single separating boundary component,
- e is the only end contained in all U_n,
- $\partial U_n \cap \ell$ is a single point, denoted p_n, and
- $U_n \setminus U_{n+1}$ contains at least 5 ends.

So for every $n \geq 1$ we can find a 6-holed sphere $Y_n \subset U_n \setminus U_{n+1} \setminus \ell$. By construction, the Y_n are pairwise disjoint and leave every compact. Let γ_n be the boundary component of Y_n such that Y_n and X are contained in different components of $S \setminus \gamma_n$. In each $U_n \setminus U_{n+1}$, choose a simple compact arc ℓ_n from γ_n to ℓ (so that the interior of ℓ_n is in the interior of $U_n \setminus U_{n+1} \setminus Y_n \setminus \ell$). Let p_n be the intersection point of ℓ_n with ℓ and denote by $\ell|_{[p,p_n]}$ the subarc of ℓ between p and p_n. Define P_n to be the pair of pants with boundary components ∂X, γ_n and the boundary of a regular neighborhood of

$$\partial X \cup \ell|_{[p,p_n]} \cup \ell_n \cup \gamma_n.$$

Then by construction $X_n = X \cup P_n \cup Y_n$ satisfies all the required properties. $\square$

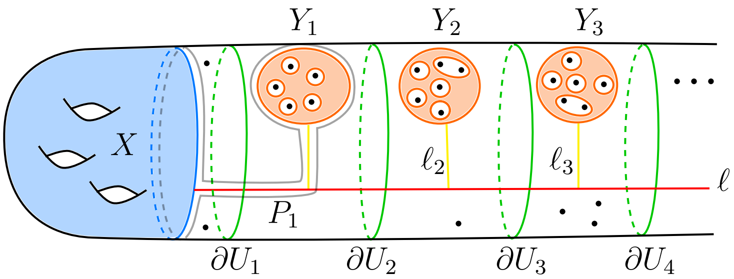

FIGURE 1. Finding subsurfaces asi in Lemma 6

We are now ready to prove our main theorem.

Proof of Theorem A. Let g be the genus of S. If $g = 0$, let $X_n = Y_n \subset S$ be a sequence of pairwise disjoint 6-holed spheres (where some boundary component might be homotopic to a puncture) leaving every compact. If $g \geq 1$, let X_n, X and Y_n be as in Lemma 6.

Fix a surface Σ homeomorphic to X_n, $F \subset \Sigma$ homeomorphic to Y_n, and homeomorphisms $\theta_n : \Sigma \to X_n$ restricting to homeomorphisms $F \to Y_n$. Let f be a pure mapping class on Σ, supported on F and such that $f|_F$ is a pseudo-Anosov. Let $\bar{f}$ be the mapping class on S given by

$$\bar{f} = \prod_{n=1}^{\infty} \theta_n \circ f^n \circ \theta_n^{-1}.$$

Informally, $\bar{f}$ is supported on $\bigcup Y_n$ and restricts to f^n on Y_n.

We claim that $\bar{f}$ is not in the group generated by multitwists. By contradiction, suppose that $\bar{f}$ is a product of k multitwists. Note that $\bar{f}$ leaves each X_n invariant and restricts to f^n on each Y_n, so by Lemma 5 the map induced by $\bar{f}$ on $\tilde{X}_n$ is a product of at most k multitwists. Tracing the definition of $\bar{f}$, this implies that for every $n \geq 1$, $f^n \in \mathrm{MCG}(\tilde{\Sigma})$ is a product of at most k multitwists. Let φ be a quasimorphism on $\mathrm{MCG}(\tilde{\Sigma})$ as in Proposition 2. Then

$$\lim_{n\to\infty} |\varphi(f^n)| = \infty,$$

but since f^n is a product of at most k multitwists,

$$|\varphi(f^n)| \leq k\Delta(\varphi),$$

a contradiction. □

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