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Summer semester 2017

Lecture: Riemann surfaces
Exercise sheet 8

Exercise 1.

(a) Consider $\mathbb{C}^n$ with its standard Hermitian inner product $(v, w) \mapsto \langle v, w\rangle$. Show that

$$\langle v, w\rangle = (v, w) - i(iv, w)$$

where (v, w) denotes the standard inner product on $\mathbb{R}^{2n}$.

Now consider an almost complex manifold X, i.e. a manifold equipped with an endomorphism J of TX such that $J^2 = -\operatorname{id}_{TX}$. Further assume that X comes equipped with a Riemannian metric g which satisfies $g(Jv, Jw) = g(v, w)$, i.e. J is g-orthogonal (or *compatible*). Then use the above to obtain a *Hermitian metric* (i.e. a fiberwise Hermitian inner product) h on TX.

(b) Show that the imaginary part of h defines a two-form $\omega \in \Lambda^2 T^*X$. ω is called the *fundamental form*. Why is the fundamental form on a Riemann surface always closed?

Exercise 2. Let $X = \mathbb{C}/\Gamma$ be a torus and consider an arbitrary homomorphism $\pi_1(X) \to \mathbb{C}$. Show that this homomorphism can be realized as the period homomorphism of a smooth one-form $\alpha \in \mathcal{E}^1(X)$.

Exercise 3. Show that there is no meromorphic function on $X = \mathbb{C}/\Gamma$ with a single pole of order one.

Exercise 4. Show that for a torus $X = \mathbb{C}/\Gamma$, $H^0(X, \Omega^1) \cong \mathbb{C}$.

Please hand in your solutions at the start of the exercise class on **Monday, July 3, 2017**.