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Summer semester 2017

Lecture: Riemann surfaces

Exercise sheet 7

Exercise 1. Recall the (meromorphic) Weierstrass $\wp$-function on $\mathbb{C}$, associated to a lattice $\Gamma \subset \mathbb{C}$. In this exercise, we will use it to determine the structure of the field of doubly periodic, meromorphic functions (also called *elliptic functions*) with respect to Γ, $K(\Gamma) = \mathcal{M}(\mathbb{C}/\Gamma)$.

(a) Start by showing by induction that any even, elliptic function $f \in K(\Gamma)$ whose only poles occur at the lattice points can be expressed as a polynomial in $\wp$.

(b) Show that any even, elliptic function $g \in K(\Gamma)$ can be written as a rational function of $\wp$. *Hint:* Use $\wp$ to eliminate any (always finitely many) singularities that lie outside Γ, and apply the previous result.

(c) Show that an arbitrary elliptic function is the sum of an odd and an even elliptic function, and use the fact that $\wp'$ is odd to express any $f \in K(\Gamma)$ as $f = R(\wp) + \wp' S(\wp)$, where R and S are rational functions. This shows that $K(\Gamma)$ is a two-dimensional vector space over $\mathbb{C}(\wp)$, the (fraction) field of rational functions of $\wp$.

Exercise 2. (Bonus):

(a) Let $U \subset \mathbb{C}$ be a sufficiently small open disk around $0 \in \mathbb{C}$ (i.e. the radius should be smaller than $\min_{\omega\in\Gamma} |\omega|$). Show by induction that

$$\wp(z) - \frac{1}{z^2} = \sum_{n=1}^{\infty} a_{2n} z^{2n} \qquad\qquad a_{2n} = (2n+1) \sum_{\omega\in\Gamma\setminus\{0\}} \frac{1}{\omega^{2(n+1)}}$$

Now, we define the *Eisenstein series of weight* $2n$ with respect to the lattice Γ by $G_{2n} = \sum_{\omega\in\Gamma\setminus\{0\}} \omega^{-2n}$ $(n \geq 2)$. Then we see that

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2(n+1)} z^{2n}$$

The Eisenstein series turn out to be important in (analytic) number theory and the theory of modular forms.

(b) As an application of the first exercise, express $(\wp')^2$ as a polynomial of $\wp$ and $\wp'$. Observe that differentiating this equation yields expressions for all derivatives of $\wp$ (and corresponding identities between Eisenstein series, after equating Laurent coefficients). The coefficients of the resulting polynomial are known as the *modular invariants* of the lattice.

Exercise 3.

(a) Show that the $U_0 = \mathbb{C}\mathrm{P}^1 \backslash \{0\}$ and $U_1 = \mathbb{C}\mathrm{P}^1 \backslash \{\infty\}$ define a Leray covering $\mathcal{U} = \{U_0, U_1\}$ for the sheaf Ω of holomorphic one-forms on $\mathbb{C}\mathrm{P}^1$.

(b) Prove that $H^1(\mathbb{C}\mathrm{P}^1, \Omega) \cong \mathbb{C}$, and that it is generated by the holomorphic one-form $z^{-1}\mathrm{d}z \in \Omega(U_0 \cap U_1)$.

Please hand in your solutions at the start of the exercise class on **Monday, June 26, 2017**.