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Summer semester 2017

Lecture: Riemann surfaces
Exercise sheet 5

Exercise 1.

(a) Show that any automorphism of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is given by a map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ of the form

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

Hint: Use the fact that any meromorphic function on $\mathbb{CP}^1$ is rational.

(b) On exercise sheet 3, you showed that any biholomorphic map defined on the complement of a finite subset P of a Riemann surface X extends to a automorphism of X. Use this to determine the automorphisms of $\mathbb{C}$, of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and of $\mathbb{C} \setminus \{0, 1\}$.

(c) Bonus: Show that $\mathrm{Aut}(\mathbb{C} \setminus \{0, 1, z\})$ (here, $z \notin \{0, 1\}$) contains at least one non-trivial element.

Exercise 2. Let X be a Riemann surface.

(a) For an open subset $U \subset X$ let $\mathcal{B}(U) \subset \mathcal{O}(U)$ be the set of bounded, holomorphic functions $f\colon U \to \mathbb{C}$. Prove that $\mathcal{B}$ defines a presheaf, but not a sheaf on X.

(b) Define $\mathcal{F}(U) = \mathcal{O}^*(U)/\exp \mathcal{O}(U)$. Show that this gives rise to another presheaf on X which is not a sheaf.

Exercise 3. Let $(\mathcal{F}, \rho)$ be a presheaf of Abelian groups on a topological space X and $\mathcal{F}_x$, $x \in X$ its stalks. We define the *sheafification* or *associated sheaf* of $\mathcal{F}$, denoted by $\mathcal{F}^\sharp$, as follows. For any open subset $U \subset X$, let $\mathcal{F}^\sharp(U)$ be the set of families $(\varphi_x)_{x \in U}$ of elements $\varphi_x \in \mathcal{F}_x$ with the following property: For every $x \in U$, there exists some open neighborhood $V \subset U$ and some $f \in \mathcal{F}(V)$ such that $\varphi_y = \rho_y^V(f)$ for every $y \in V$.

(a) Show that $\mathcal{F}^\sharp$, with the natural restriction homomorphisms $(\rho^\sharp)_V^U((\varphi_x)_{x\in U}) = (\varphi_x)_{x \in V}$, where $U, V \subset X$ are open sets, defines a sheaf.

(b) For any open $U \subset X$ open, define $\alpha_U\colon \mathcal{F}(U) \to \mathcal{F}^\sharp(U)$ by $\alpha_U(f) = (\rho_x^U(f))_{x \in U}$. Check that this map is well-defined and that it yields a *homomorphism* α *of presheaves*, i.e. it commutes with the restriction homomorphisms.

(c) Show that this map induces bijections $\alpha_x\colon \mathcal{F}_x \to \mathcal{F}_x^\sharp$ for every $x \in X$.

(d) Determine the sheafifications of the presheaves of the previous exercise.

Please hand in your solutions at the start of the exercise class on **Monday, June 12, 2017**.