LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Lecture: Riemann surfaces

Exercise sheet 3

Exercise 1. Let X be a compact Riemann surface and X' the complement of a finite set of points $\{x_1, \ldots, x_n\}$ in X. Prove that every automorphism $f : X' \to X'$ extends to an automorphism of X.

Exercise 2. Let $\Gamma \subset \mathbb{C}$ be a lattice and consider the Weierstrass $\wp_\Gamma$-function on $\mathbb{C}$, which is defined by:

$$\wp_\Gamma(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

(a) Prove that for every compact disk $K_r := \{z \in \mathbb{C} \mid |z| \leq r\}$ there exists a finite subset $\Gamma_0 \subset \Gamma$ such that $\omega \notin K_r$ for every $\omega \in \Gamma \setminus \Gamma_0$ and the series

$$\sum_{\omega \in \Gamma \setminus \Gamma_0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

converges uniformly on K_r. This implies that $\wp_\Gamma$ is a well-defined meromorphic function on $\mathbb{C}$ with poles of order two at the lattice points.

(b) Show that $\wp_\Gamma$ induces a meromorphic function $\mathbb{C}/\Gamma \to \mathbb{CP}^1$. To do this, first show that its derivative is doubly periodic.

Exercise 3. Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be the holomorphic map defined by $f(z) = \frac{1}{z} + z$. Find its branch points and show that the preimage of any other point consists of two points.

Please hand in your solutions at the start of the exercise class on **Monday, May 22, 2017**.