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Summer semester 2017

Lecture: Riemann surfaces
Exercise sheet 2

Exercise 1. Consider a function f, holomorphic on a punctured disk $U = D_r(z_0)\setminus\{z_0\}$, where $D_r(z_0) = \{|z - z_0| < r\} \subset \mathbb{C}$. Then f has a Laurent series expansion on U of the form

$$f = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k \qquad \text{with} \qquad a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{k+1}}\mathrm{d}z$$

where C is a circle of radius less than r around z_0. We call z_0 a **removable singularity** if $a_k = 0$ for every $k < 0$. Prove that if f is bounded in a neighbourhood of z_0, then z_0 is a removable singularity.

Exercise 2. A smooth real-valued function f on an open set $U \subset \mathbb{R}^2 \cong \mathbb{C}$ is called **harmonic** if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv 0.$$

(a) Let $U, V \subset \mathbb{C}$ be open subsets and $\phi\colon U \to V$ a holomorphic map. Prove that both the real and imaginary part of ϕ are harmonic.

(b) Let f be a real-valued harmonic function on an open set $U \subset \mathbb{R}^2 \cong \mathbb{C}$. Let $V \subset \mathbb{C}$ be open as well and $\varphi\colon V \to U$ a holomorphic map. Prove that $f \circ \varphi$ is harmonic. Note that this implies that the notion of a harmonic function has a well-defined meaning on any Riemann surface.

(c) Let X be a Riemann surface and $f\colon X \to \mathbb{R}$ a non-constant harmonic function. Prove that f does not attain its maximum and deduce that if X is compact, f must be constant.

Exercise 3. Let f be a holomorphic function defined on a punctured disk around $z_0 \in \mathbb{C}$. Then z_0 is called an **essential singularity** of f if the Laurent series of f around z_0 has infinitely many terms with a negative exponent.

(a) Suppose that f is a holomorphic function on a punctured disk U around z_0 in $\mathbb{C}$ and $|f(z) - \alpha| > s$ for some $\alpha \in \mathbb{C}$, $s > 0$ for all $z \in U$. Show that f does not have an essential singularity in z_0.

(b) Suppose that X is a compact Riemann surface and $P = \{p_1, \dots, p_n\}$ a finite set of points in X. Let $X' = X \setminus P$ and suppose that

$$f\colon X' \longrightarrow \mathbb{C}$$

is a non-constant holomorphic function. Show that the image of f comes arbitrarily close to every $c \in \mathbb{C}$.

Please hand in your solutions at the start of the exercise class on **Monday, May 15, 2017**.