LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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## Lecture: Riemann surfaces
## Exercise sheet 10

**Exercise 1.** Let $D$ be a divisor on $\mathbb{C}\mathrm{P}^1$. Use the Riemann-Roch theorem to prove:

(a) $\dim H^0(\mathbb{C}\mathrm{P}^1, \mathcal{O}_D) = \max(0, 1 + \deg D)$.

(b) $\dim H^1(\mathbb{C}\mathrm{P}^1, \mathcal{O}_D) = \max(0, -1 - \deg D)$.

**Exercise 2.** Let $X = \mathbb{C}/\Gamma$ be a torus and consider the divisor $P$ which vanishes everywhere except at the point $P \in X$, where it takes the value $1$.

(a) Show that

$$\dim H^0(X, \mathcal{O}_{nP}) = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ n & n \geq 1 \end{cases}$$

*Hint:* For the last case, start with the case $n = 1$ and proceed by induction, making use of the Weierstrass $\wp$-function.

(b) Calculate $\dim H^1(X, \mathcal{O}_{nP})$ by means of the Riemann-Roch theorem.

**Exercise 3.** Let $\mathfrak{D}$ denote the sheaf of divisors on a Riemann surface $X$, defined as follows: For every open $U \subset X$, $\mathfrak{D}(U)$ consists of all maps $D\colon U \to \mathbb{Z}$ such that, for every compact subset $K \subset U$, there are only finitely many points in $K$ where $D$ does not vanish.

(a) Show that $\mathfrak{D}$ is indeed a sheaf.

(b) Show that $H^1(X, \mathfrak{D}) = 0$, imitating the proof that $H^1(X, \mathcal{E}) = 0$, using an integer-valued (discontinuous) partition of unity.

(c) Let $\mathcal{M}^*$ be the sheaf that assigns to any open set $U \subset X$ the set of meromorphic functions which do not vanish identically on any connected component of $U$. Let $\beta : \mathcal{M}^* \to \mathfrak{D}$ be the map that assigns to every meromorphic function its divisor. Show that there is a short exact sequence

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \longrightarrow \mathfrak{D} \longrightarrow 0$$

and, in particular, there exists a *long* exact sequence

$$0 \to H^0(X, \mathcal{O}^*) \to H^0(X, \mathcal{M}^*) \to \mathrm{Div}(X) \to H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{M}^*) \to 0$$

Please hand in your solutions at the start of the exercise class on **Monday, July 17, 2017**.