Dr. Mark Hamilton
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Summer semester 2017

Lecture: Riemann surfaces
Exercise sheet 1

Exercise 1. Let $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Gamma' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ be two lattices in the complex plane $\mathbb{C}$. Show that $\Gamma = \Gamma'$ if and only if there exists a matrix

$$A \in \mathrm{GL}(2,\mathbb{Z})$$

(with $\det A = \pm 1$) such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Exercise 2.

(a) Let $\Gamma, \Gamma' \subset \mathbb{C}$ be two lattices. Suppose that $\alpha \in \mathbb{C}^*$ satisfies $\alpha\Gamma \subset \Gamma'$. Show that the map $\mathbb{C} \to \mathbb{C}, z \mapsto \alpha z$ induces a holomorphic map $\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma$ and that this map is biholomorphic if and only if $\alpha\Gamma = \Gamma'$.

(b) Prove that every complex torus $X = \mathbb{C}/\Gamma$ is isomorphic to a complex torus of the form

$$X(\tau) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau),$$

where $\tau \in \mathbb{C}$ satisfies $\mathrm{Im}(\tau) > 0$.

(c) Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \quad \text{and} \quad \mathrm{Im}(\tau) > 0.$$

Let

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Show that $\mathrm{Im}(\tau') > 0$ and that the complex tori $X(\tau)$ and $X(\tau')$ are isomorphic.

Exercise 3. Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$$

and define the fractional linear transformation

$$f(z) = \frac{az + b}{cz + d}.$$

Prove that f defines a meromorphic function on $\mathbb{CP}^1$ and that the induced map $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is biholomorphic, i.e. an automorphism of $\mathbb{CP}^1$.

Please hand in your solutions at the start of the exercise class on **Monday, May 8, 2017**.