LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis

Excercise Sheet 11

Problem 1 (Polynomials and completeness). Prove:

a) Any closed proper subspace of a normed space X is nowhere dense.

b) Any Hamel basis in an infinite dimensional Banach space X is uncountable.

c) The space $\mathcal{P}$ of polynomials cannot be equipped with a norm $\|\cdot\|$ such that $(\mathcal{P}, \|\cdot\|)$ is complete.

[4+4+2 Points]

Problem 2 (Uniform boundedness principle).

a) Let X and Y be Banach spaces, and let $\tau : X \times Y \to \mathbb{C}$ be such that for each $x \in X$ and $y \in Y$ we have $\tau(x, \cdot) \in Y'$ and $\tau(\cdot, y) \in X'$. Prove that if $x_n \to x$ and $y_n \to y$ then $\tau(x_n, y_n) \to \tau(x, y)$ as $n \to \infty$.

b) Let $a := (a_n)_{n=1}^{\infty}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} a_n b_n$ exists for all $(b_n)_{n=1}^{\infty} \in c_0$. Prove that $a \in \ell^1$.

[5+5 Points]

Problem 3 (Orthogonal projection). Consider

$$\mathcal{M} := \left\{ f \in L^2([\tfrac{1}{2}, 2]) \mid f(x) = f(\tfrac{1}{x}) \text{ for a.e. } x \in [\tfrac{1}{2}, 2] \right\} \subseteq L^2([\tfrac{1}{2}, 2]).$$

a) Prove that $\mathcal{M}^{\perp} = \left\{ g \in L^2([\tfrac{1}{2}, 2]) \,|\, g(\tfrac{1}{x}) = -x^2 g(x) \text{ for a.e. } x \in [\tfrac{1}{2}, 2] \right\}$.
(Hint: Observe that $[\tfrac{1}{2}, 2] = [\tfrac{1}{2}, 1] \cup [1, 2]$.)

b) Find the orthogonal projection of the function $f_0(x) = x$ onto $\mathcal{M}$.
(Hint: Prove first that $L^2([\tfrac{1}{2}, 2]) = \mathcal{M} \oplus \mathcal{M}^{\perp}$.)

[5+5 Points]

Problem 4 (Inverse mapping and closed graph theorem). (Updated) Let X, Y, Z be Banach spaces and let $J : Y \to Z$ be an injective bounded linear map with closed range. Let $T : X \to Y$ be a linear map such that JT is bounded.

a) Prove that T is bounded.

b) Is it necessary that J has closed range for T to be bounded?

[10+0 Points]

***Deadline: July 4, 2016 14:00**, for details see http://www.math.lmu.de/~gottwald/16FA/.*