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Functional Analysis

Excercise Sheet 10

Problem 1 (Finite rank operators are compact)**.** Let X, Y be normed spaces. A linear operator $T : X \to Y$ is called *finite-rank* if $\operatorname{rank}(T) \coloneqq \dim R(T) < \infty$. Prove that a bounded linear operator is compact if it is finite-rank. [5 Points]

Proof. Let $A \subseteq X$ be bounded. Then there exists $R > 0$ s.t. $\|x\| < R$ for all $x \in A$. Since T is bounded we have for $x \in A$ that $\|Tx\| \leq \|T\| \, \|x\| < \|T\| \, R$. Thus

$$TA = \{Tx \mid x \in A\} \subseteq B_{\|T\|R}(0) \subset \overline{B_{\|T\|R}(0)}$$

is bounded and hence $\overline{TA}$ as well, as $\overline{TA}$ is the smallest closed set containing TA. Clearly $\overline{TA}$ is closed (observe that A and TA are sets and not spaces in general).

Claim: A subset $S \subseteq V$ of a finite dimensional normed $\mathbb{K}$-vectorspace V is compact *iff* it is closed and bounded.
Proof: From linear algebra we know that V is isometrically isomorphic to $\mathbb{K}^d$, where $d = \dim V < \infty$. Let $\varphi : V \to \mathbb{K}^d$ be such an isometric isomorphism. Since φ is in particular an homeomorphism, $\varphi(S) \subseteq \mathbb{K}^d$ is { closed, bounded, compact } *iff* S is. In $\mathbb{K}^d$ we know by Heine-Borel, that $\varphi(S)$ is compact *iff* it is closed and bounded. □

With $S \coloneqq \overline{TA} \subseteq V \coloneqq R(T)$ follows by the claim that T is compact. □

Alternative proof: Recall that an operator $T : X \to Y$ is compact *iff* for every bounded sequence $\{x_n\}_n \subseteq X$ the sequence $\{Tx_n\}_n \subseteq Y$ has a convergent subsequence.

Let $N \coloneqq \operatorname{rank}(T) < \infty$ and $\{e_j\}_{j=1}^N$ be a basis of $V \coloneqq R(T)$. Consider the 1-norm on V w.r.t. $\{e_j\}_{j=1}^N$, i.e. $\|y\|_1 \coloneqq \sum_{j=1}^N |\alpha_j|$ for all $y = \sum_{j=1}^N \alpha_j e_j$, where $\{\alpha_j\}_{j=1}^N = \{\alpha_j(y)\}_{j=1}^N \subseteq \mathbb{C}^N$. Since all norms in finite-dimensional vector spaces are equivalent, there exist $c, C > 0$ such that $c \|y\|_Y \leq \|y\|_1 \leq C \|y\|_Y$ for all $y \in V \subseteq Y$.
Let $(x_n)_n \subseteq X$ be a bounded sequence in X, wlog $\|x_n\|_X \leq 1$ for all $n \in \mathbb{N}$. Let $\{\alpha_j^{(n)}\}_{j=1}^N$ denote the coordinates of $y_n \coloneqq Tx_n \in V$ w.r.t. $\{e_j\}_{j=1}^N$. Then we have for all $j = 1, \ldots, N$ that

$$|\alpha_j^{(n)}| \leq \sum_{j=1}^N |\alpha_j^{(n)}| = \|Tx_n\|_1 \leq C \|Tx_n\|_Y \leq \|T\|_{B(X,Y)} \|x_n\|_X \leq \|T\|_{B(X,Y)} .$$

Hence, for each j, $(\alpha_j^{(n)})_n$ is a bounded sequence in $\mathbb{C}$ and therefore admits a convergent subsequence. Thus we can iteratively extract convergent subsequences

$$\alpha_1^{(\tilde{n}_k)} \to \alpha_1, \quad \alpha_2^{(\tilde{n}_{k_l})} \to \alpha_2, \quad \ldots .$$

such that after N steps, each component converges to some $\alpha_j \in \mathbb{C}$. Denoting this subsequence by $(\alpha^{(n_k)})_k$ and $y := \sum_j \alpha_j e_j$, we find that

$$c\,\|Tx_{n_k} - y\|_Y \leq \|Tx_{n_k} - y\|_1 = \sum_{n=1}^{N} |\alpha_j^{(n_k)} - \alpha_j| \to 0.$$

Thus for every bounded sequence $\{x_n\}_n \subseteq X$ the sequence $\{Tx_n\}_n \subseteq Y$ has a convergent subsequence. Hence T is compact. □

Problem 2 (Relative compactness)**.** Consider the following subsets of $C([0,1])$:

a) $E := \{f \in C^1([0,1]) \mid f(0) = 0 \wedge \forall x \in [0,1] : |f'(x)| \leq 1\}$.

b) $E := \{x \mapsto x^n \mid n \in \mathbb{N}\}$.

c) $E := \{x \mapsto \sin(nx) \mid n \in \mathbb{N}\}$.

d) $E := \{x \mapsto \sin(\alpha + x) \mid \alpha \in \mathbb{R}\}$.

e) $E := \{f \in C^2([0,1]) \mid f(0) = 0 \wedge \forall x \in [0,1] : |f''(x)| \leq 1\}$.

Prove whether or not these sets are relatively compact[1] in $(C([0,1]), \|\cdot\|_\infty)$.

[3 + 3 + 3 + 3 + 3 Points]

Problem 3 (Compact operator)**.** Let $k \in C([0,1]^2)$ and for $f \in C([0,1])$ and $x \in [0,1]$ define

$$(Tf)(x) := \int_0^x k(x,y)\, f(y)\, dy. \tag{1}$$

Prove the following:

a) This defines a bounded linear map $T : (C([0,1]), \|\cdot\|_\infty) \to (C([0,1]), \|\cdot\|_\infty)$.

b) T is compact.

[5+5 Points]

Problem 4 (Extension)**.** Let $\mathcal{H}$ be a separable Hilbert space and let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$.

a) Consider $A, B > 0$, let $a = (a_n)_{n\in\mathbb{N}}, b = (b_n)_{n\in\mathbb{N}}$ s.t. $a_n, b_n > 0$ for all $n \in \mathbb{N}$, and let $T : \operatorname{span}\{e_n\}_{n\in\mathbb{N}} \to \mathcal{H}$ be a linear operator such that

$$\sum_{n=1}^{\infty} b_n |\langle e_m, Te_n\rangle| \leq A a_m \;\forall m \in \mathbb{N} \quad \text{and} \quad \sum_{m=1}^{\infty} a_m |\langle e_m, Te_n\rangle| \leq B b_n \;\forall n \in \mathbb{N}. \tag{2}$$

Prove that T extends uniquely to a bounded linear operator on $\mathcal{H}$ with $\|T\| \leq \sqrt{AB}$.

b) Let $T : \operatorname{span}\{e_n\}_{n\in\mathbb{N}} \to \mathcal{H}$ be a linear operator such that $\langle e_m, Te_n\rangle = \frac{1}{n+m-1}$ for all $n, m \in \mathbb{N}$.

Prove that T extends uniquely to a bounded linear operator on $\mathcal{H}$ with $\|T\| \leq \pi$.

[1] $K \subseteq X$ is called relatively compact iff $\overline{K}$ is compact.

c) Let $T : \operatorname{span}\{e_n\}_{n\in\mathbb{N}} \to \mathcal{H}$ be a linear operator such that $\langle e_m, Te_n\rangle = \frac{1}{2^{n+m-1}}$ for all $n, m \in \mathbb{N}$.

Prove that T extends uniquely to a bounded linear operator on $\mathcal{H}$ and compute $\|T\|$.

[5+2+3 Points]

***Deadline: June 27, 2016 14:00**, for details see http://www.math.lmu.de/˜gottwald/16FA/.*