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Functional Analysis

Excercise Sheet 9

Remaining solution

Problem 1 (Equality in Minkowski's inequality).

a) Let $1 < p < \infty$ and $0 \neq x, y \in \ell^p$. Prove that $\|x+y\|_p = \|x\|_p + \|y\|_p$ *iff* $y = \alpha x$ for some $\alpha \geq 0$.

b) Let $p \in \{1, \infty\}$. Prove that there exists $0 \neq x, y \in \ell^p$ s.t. $\|x+y\|_p = \|x\|_p + \|y\|_p$ and $y \neq \alpha x$ for all $\alpha \in \mathbb{R}$.

c) Let $1 \leq p \leq \infty$ and $q = \frac{p}{p-1}$. Prove that for every $x \in \ell^p$

$$\|x\|_p = \sup_{\substack{y \in \ell^q \\ \|y\|_q = 1}} \left| \sum_{n=1}^{\infty} x_n y_n \right|. \tag{1}$$

[5+2+3 Points]

Proof. a) Recall that Minkowski's inequality follows from Hölder's inequality for all $x, y \in \ell^p$ s.t. $x + y \neq 0$ through

$$\begin{aligned} \|x+y\|_p^p &= \sum_{n=1}^{\infty} |x_n + y_n|^p \leq \sum_{n=1}^{\infty} |x_n||x_n+y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n||x_n+y_n|^{p-1} \\ &\leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n+y_n|^p\right)^{1-1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n+y_n|^p\right)^{1-1/p} \\ &= \left(\|x\|_p + \|y\|_p\right) \|x+y\|_p^{p-1}, \end{aligned} \tag{2}$$

where we used the triangle inequality for the first and Hölder's inequality for the second estimate. Since, by the convexity of $t \mapsto t^p$ for $t \geq 0$, we have that

$$\left|\frac{x_n + y_n}{2}\right|^p \leq \left(\frac{|x_n| + |y_n|}{2}\right)^p \leq \frac{|x_n|^p + |y_n|^p}{2},$$

and hence that

$$\|x+y\|_p^p = \sum_{n=1}^{\infty} |x_n + y_n|^p \leq 2^{p-1} \left(\sum_{n=1}^{\infty} |x_n|^p + \sum_{n=1}^{\infty} |y_n|^p\right) = 2^{p-1}(\|x\|_p^p + \|y\|_p^p) < \infty,$$

we can devide in (2) by $\|x+y\|_p^{p-1} < \infty$ to get Minkowski's inequality.

"$\Longrightarrow$": Assume that $\|x+y\|_p = \|x\|_p + \|y\|_p$ for some $0 \neq x, y \in \ell^p$. Then the inequalities in (2) (divided by $\|x+y\|_p^{p-1}$) must be equalities. In particular we must have equality in both Hölder inequalities, i.e.

$$\sum_{n=1}^{\infty} |x_n||x_n+y_n|^{p-1} = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n+y_n|^p\right)^{1-1/p}$$
$$\sum_{n=1}^{\infty} |y_n||x_n+y_n|^{p-1} = \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n+y_n|^p\right)^{1-1/p}.$$

Equality in Hölder holds true *iff* there exists $\lambda_1^p, \lambda_2^p \geq 0$ s.t. for all $n \in \mathbb{N}$:

$$|x_n+y_n|^p = \lambda_1^p |x_n|^p \quad \text{and} \quad |x_n+y_n|^p = \lambda_2^p |y_n|^p,$$

which is equivalent to

$$|x_n+y_n| = \lambda_1 |x_n| \quad \text{and} \quad |x_n+y_n| = \lambda_2 |y_n| \tag{3}$$

for all $n \in \mathbb{N}$. If $\lambda_1 = 0$ or $\lambda_2 = 0$ then $x = -y$ and from the assumption $\|x+y\|_p = \|x\|_p + \|y\|_p$ follows the contradiction $x = y = 0$. Thus $\lambda_1, \lambda_2 > 0$ and hence

$$|y_n| = \frac{\lambda_1}{\lambda_2} |x_n| \tag{4}$$

for all $n \in \mathbb{N}$. This proves in particular that $x_n = 0$ iff $y_n = 0$. Hence we can assume that $|x_n|, |y_n| > 0$.

It remains to prove that x_n and y_n have the same phase (or sign if $\mathbb{K} = \mathbb{R}$) for all $n \in \mathbb{N}$. Again from (3) and the assumption follows

$$\|x\|_p + \|y\|_p = \|x+y\|_p = \left(\sum_{n=1}^{\infty} |x_n+y_n|^p\right)^{1/p} = \begin{cases} \lambda_1 \|x\|_p \\ \lambda_2 \|y\|_p \end{cases},$$

and hence $\|y\|_p = (\lambda_1 - 1)\|x\|_p$ and $\|x\|_p = (\lambda_2 - 1)\|y\|_p = (\lambda_1 - 1)(\lambda_2 - 1)\|x\|_p$, which implies that

$$\lambda_1 \lambda_2 = \lambda_1 + \lambda_2. \tag{5}$$

Denote $x_n = |x_n| e^{i\alpha_n}$ and $y_n = |y_n| e^{i\beta_n}$ for some $\alpha_n, \beta_n \in [0, 2\pi)$. From (3) and (4) follows that

$$\lambda_1 |x_n| = |x_n+y_n| = |x_n| \left| e^{i\alpha_n} + \frac{\lambda_1}{\lambda_2} e^{i\beta_n} \right|,$$

and hence from (5) that $\lambda_1 + \lambda_2 = \lambda_1 \lambda_2 = \left|\lambda_1 + \lambda_2 e^{i(\alpha_n - \beta_n)}\right|$, which implies that $\alpha_n = \beta_n$ for all $n \in \mathbb{N}$.

With $\alpha \coloneqq \frac{\lambda_1}{\lambda_2}$ we finally get from (3) for all $n \in \mathbb{N}$ that

$$y_n = |y_n| e^{i\beta_n} = \alpha |x_n| e^{i\beta_n} = \alpha x_n e^{i(\beta_n - \alpha_n)} = \alpha x_n, \tag{6}$$

and hence that $y = \alpha x$.

"$\Longleftarrow$": Assume that $x = \alpha y$ for some $\alpha \geq 0$. Then $\|x+y\|_p = (1+\alpha)\|x\|_p = \|x\|_p + \|y\|_p$.

b) • $p = 1$: Consider $x := (1, 0, 0, \dots)$ and $y := (0, 1, 0, \dots)$ then $\|x\|_1 = \|y\|_p = 1$, $\|x + y\|_1 = 2$ and hence $\|x + y\|_1 = \|x\|_1 + \|y\|_1$, but $x \neq \alpha y$ for all $\alpha \geq 0$.

• $p = \infty$: Consider $x := (1, 0, 0, \dots)$ and $y := (1, 1, 0, 0, \dots)$ then $\|x\|_\infty = \|y\|_\infty = 1$, $\|x + y\|_\infty = 2$ and hence $\|x + y\|_1 = \|x\|_1 + \|y\|_1$, but $x \neq \alpha y$ for all $\alpha \geq 0$.

c) • $1 < p \leq \infty$: From the lecture we know that $\ell^p \cong (\ell^q)'$ i.e. that $T : \ell^p \to (\ell^q)'$ with $x \mapsto \phi_x$, where $\phi_x(y) = \sum_{n=1}^\infty x_n y_n$ for all $y \in \ell^q$, is an isometric isomorphism. Let $x \in \ell^p$. Then we have

$$\|x\|_p = \|\phi_x\|_{(\ell^q)'} = \sup_{\substack{y \in \ell^q \\ \|y\|_q = 1}} |\phi_x(y)| = \sup_{\substack{y \in \ell^q \\ \|y\|_q = 1}} \left| \sum_{n=1}^\infty x_n y_n \right|$$

• $p = 1$: Let $0 \neq x \in \ell^1$. From Hölder's inequality we get, for $y \in \ell^\infty$ s.t. $\|y\|_\infty = 1$, that

$$\left| \sum_{n=1}^\infty x_n y_n \right| \leq \|x\|_1 \|y\|_\infty = \|x\|_1 ,$$

and hence that $\sup_{\substack{y \in \ell^q \\ \|y\|_q = 1}} \left|\sum_{n=1}^\infty x_n y_n\right| \leq \|x\|_1$.

For the reverse inequality consider the sequence $\tilde{y} := (\tilde{y}_n)_{n \in \mathbb{N}}$ with $\tilde{y}_n := \frac{|x_n|}{x_n}$ if $x_n \neq 0$ and $\tilde{y}_n := 0$ if $x_n = 0$. Then $\|\tilde{y}\|_\infty = \sup_{n \in \mathbb{N}} |\tilde{y}_n| = 1$, since $x \neq 0$, and we have

$$\sup_{\substack{y \in \ell^q \\ \|y\|_q = 1}} \left| \sum_{n=1}^\infty x_n y_n \right| \geq \left| \sum_{n=1}^\infty x_n \tilde{y}_n \right| = \sum_{n=1}^\infty |x_n| = \|x\|_1$$

Therefore $\|x\|_1 = \sup_{\substack{y \in \ell^q \\ \|y\|_q = 1}} \left|\sum_{n=1}^\infty x_n y_n\right|$.

□

Problem 2 (Inner product vs. parallelogram identity).

a) Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$. Prove, for $\mathbb{K} = \mathbb{R}$ and for $\mathbb{K} = \mathbb{C}$, that the norm in X is induced by an inner product *iff* it satisfies the parallelogram identity

$$\forall x, y \in X \ : \ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 . \tag{7}$$

b) Let $p \in [1, \infty]$. Prove that ℓ^p is a Hilbert space *iff* $p = 2$.

[7+3 Points]

Proof.

a) "$\Longrightarrow$": Assume that X is an inner product space with inner product $\langle \cdot, \cdot \rangle$. Let $\|\cdot\|$ denote the induced norm, i.e. $\|x\| := \sqrt{\langle x, x \rangle}$ for all $x \in X$. Then for $x, y \in X$:

$$\|x + y\|^2 = \|x\|^2 + 2\Re \langle x, y \rangle + \|y\|^2 ,$$
$$\|x - y\|^2 = \|x\|^2 - 2\Re \langle x, y \rangle + \|y\|^2 .$$

Summing both equations gives (7). This was also proven in tutorial 9.

"$\Longleftarrow$": Assume that the norm $\|\cdot\|$ of X satisfies (7).

- For the case $\mathbb{K} = \mathbb{R}$ define $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$, by

$$\langle x, y \rangle := \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right), \quad \forall x, y \in X. \tag{8}$$

Hint: This is called a real *polarisation identity.*

We have to prove, that $\langle \cdot, \cdot \rangle$ defines a real inner product and that $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$. The latter, symmetry and definiteness are obvious.

To prove linearity of $\langle \cdot, \cdot \rangle$ observe first, that we have for all $x, y, z \in X$, by (7), that

$$\begin{aligned} 4 \langle x + y, z \rangle &\overset{(8)}{=} \|x + y + z\|^2 - \|x + y - z\|^2 \\ &= \|(x + z) + y)\|^2 + \|(x + z) - y\|^2 - \|x - (y - z)\|^2 - \|x + (y - z)\|^2 \\ &\overset{(7)}{=} 2 \left(\|x + z\|^2 + \|y\|^2 \right) - 2 \left(\|x\|^2 + \|y - z\|^2 \right), \end{aligned}$$

and similarly, by exchanging x with y, that

$$4 \langle x + y, z \rangle = 2 \left(\|y + z\|^2 + \|x\|^2 \right) - 2 \left(\|y\|^2 + \|x - z\|^2 \right).$$

Summing both equations leads to

$$\begin{aligned} 8 \langle x + y, z \rangle &= 2 \left(\|x + z\|^2 - \|x - z\|^2 \right) + 2 \left(\|y + z\|^2 - \|y - z\|^2 \right) \\ &= 8 \langle x, z \rangle + 8 \langle y, z \rangle. \end{aligned}$$

Hence $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y \in X$.

By induction follows $\langle nx, z \rangle = n \langle x, z \rangle$ for all $n \in \mathbb{N}$. Directly from the definition we get $\langle 0x, y \rangle = 0 = 0 \langle x, y \rangle$ and $\langle -x, y \rangle = - \langle x, y \rangle$ for all $x, y \in X$. Hence $\langle nx, y \rangle = n \langle x, y \rangle$ for all $n \in \mathbb{Z}$ and all $x, y \in X$.

For $\alpha = \frac{m}{n} \in \mathbb{Q}$ and $x, y \in X$ we get $n \langle \alpha x, y \rangle = \langle mx, y \rangle = m \langle x, y \rangle$ and hence $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

For the last step observe, that $x \mapsto \langle x, y \rangle$ is continuous for all (fixed) $y \in X$, since the norm ist continuous. Let $\alpha \in \mathbb{R}$ and $\alpha_k \in \mathbb{Q}$ s.t. $\alpha_k \to \alpha$ for $k \to \infty$. Then $\langle \alpha x, y \rangle = \langle \lim_{k \to \infty} \alpha_k x, y \rangle = \lim_{k \to \infty} \langle \alpha_k x, y \rangle = \lim_{k \to \infty} \alpha_k \langle x, y \rangle = \alpha \langle x, y \rangle$.

In total we have for $\alpha \in \mathbb{R}$ and $x, y, z \in X$ that $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$. This proves the linearity of $\langle \cdot, \cdot \rangle$.

- For the case $\mathbb{K} = \mathbb{C}$ define $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$, by

$$\langle x, y \rangle := \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right) - \frac{i}{4} \left(\|x + iy\|^2 - \|x - iy\|^2 \right), \quad \forall x, y \in X. \tag{9}$$

Hint: This is called a complex *polarisation identity.*

We have to prove, that $\langle \cdot, \cdot \rangle$ defines a complex inner product and that $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$. The proof is analog to the real case.

b) "$\Longleftarrow$": Assume $p = 2$. Then $\ell^p = \ell^2$ is a well-known Hilbert space and its norm is induced by the inner product.

"$\Longrightarrow$": Assume that ℓ^p is Hilbert. Then it satisfies the parallelogram identity and we have, for $e_1 := (1, 0, \ldots)$ and $e_2 := (0, 1, 0, \ldots)$, that $\|e_1 + e_2\|_p^2 + \|e_1 - e_2\|_p^2 = 2 \|e_1\|_p^2 + 2 \|e_2\|_p^2$. For $p < \infty$ this leads to $2^{2/p} + 2^{2/p} = 2 + 2$, which is true only for $p = 2$. For $p = \infty$ this leads to the contradiction $1 + 1 = 2 + 2$. Thus $p = 2$.

$\square$