LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Functional Analysis

Excercise Sheet 9

Problem 1 (Equality in Minkowski's inequality).

a) Let $1 < p < \infty$ and $0 \neq x, y \in \ell^p$. Prove that $\|x+y\|_p = \|x\|_p + \|y\|_p$ *iff* $y = \alpha x$ for some $\alpha \geq 0$.

b) Let $p \in \{1, \infty\}$. Prove that there exists $0 \neq x, y \in \ell^p$ s.t. $\|x+y\|_p = \|x\|_p + \|y\|_p$ and $y \neq \alpha x$ for all $\alpha \in \mathbb{R}$.

c) Let $1 \leq p \leq \infty$ and $q = \frac{p}{p-1}$. Prove that for every $x \in \ell^p$

$$\|x\|_p = \sup_{\substack{y \in \ell^q \\ \|y\|_q = 1}} \left| \sum_{n=1}^{\infty} x_n y_n \right| . \tag{1}$$

[5+2+3 Points]

Problem 2 (Inner product vs. parallelogram identity).

a) Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$. Prove, for $\mathbb{K} = \mathbb{R}$ and for $\mathbb{K} = \mathbb{C}$, that the norm in X is induced by an inner product *iff* it satisfies the parallelogram identity

$$\forall x, y \in X \;:\; \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 . \tag{2}$$

b) Let $p \in [1, \infty]$. Prove that ℓ^p is a Hilbert space *iff* $p = 2$.

[7+3 Points]

Problem 3 (Equality in inequalities).

a) Let $1 < p < \infty$. Let $x, y \in \ell^p$ with $\|x\|_p = \|y\|_p = 1$. Prove that $x = y$ if $\left\|\frac{x+y}{2}\right\|_p = 1$.

b) Let $1 < p \leq \infty$. Prove that $C([0,1])$ can be embedded isometrically in ℓ^p *iff* $p = \infty$.

[2+8 Points]

Problem 4 (Dual spaces). Consider the Banach spaces (on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$)

$$c = \{x = (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{K} \; \forall n \in \mathbb{N} \text{ and } \lim_{n \to \infty} x_n \text{ exists}\},$$

$$c_0 = \{x = (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{K} \; \forall n \in \mathbb{N} \text{ and } \lim_{n \to \infty} x_n = 0\} \subset c,$$

with the norm $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$.

a) Prove that c_0 and c are not isometrically isomorphic.

b) Prove that $c' \cong (c_0)' \cong \ell^1$.

[4+6 Points]

***Deadline: June 20, 2016 14:00**, for details see http://www.math.lmu.de/˜gottwald/16FA/.*