LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Prof. T. Ø. Sørensen PhD
A. Groh, S. Gottwald

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Functional Analysis
Excercise Sheet 7

Problem 1 (Operator norms). Consider the Banach space $(C([0,1]), \|\cdot\|_\infty)$. Let the maps $T_j : C([0,1]) \to C([0,1])$, $j = 1, \dots, 4$, for $f \in C([0,1])$ and $x \in [0,1]$ be defined as follows:

a) $(T_1 f)(x) \coloneqq \int_0^x f(y)\mathrm{d}y$.

b) $(T_2 f)(x) \coloneqq x^2 f(0)$.

c) $(T_3 f)(x) \coloneqq f(x^2)$.

d) $(T_4 f)(x) \coloneqq (T_1^n f)(x)$, where T_1^n, $n \in \mathbb{N}$, is the n-fold composition of T_1.

Prove in each case that this defines a bounded linear operator, and compute its norm.

[2+2+2+4 Points]

Problem 2 (Separability).

a) Let $1 \leq p < \infty$. Prove that ℓ^p is separable.

b) Prove that ℓ^∞ is *not* separable.

[6+4 Points]

Problem 3 (Bounded maps). Let X be any set and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Define the set of ($\mathbb{K}$-valued) bounded maps on X by

$$\mathfrak{B}(X, \mathbb{K}) \coloneqq \{f : X \to \mathbb{K} \mid \exists C_f \geq 0 \, \forall x \in X : |f(x)| \leq C_f\}.$$

For $f, g : X \to \mathbb{K}$, $\alpha \in \mathbb{K}$ define $\alpha f + g : X \to \mathbb{K}$ by $(\alpha f + g)(x) \coloneqq \alpha f(x) + g(x)$ for all $x \in X$, $fg : X \to \mathbb{K}$ by $(fg)(x) \coloneqq f(x)g(x)$ for all $x \in X$, and for $f \in \mathfrak{B}(X, \mathbb{K})$ let $\|f\|_\infty \coloneqq \sup\{|f(x)| \mid x \in X\}$.

a) Prove that $(\mathfrak{B}(X, \mathbb{K}), (f, g) \mapsto fg)$ is a $\mathbb{K}$-algebra[1].

b) Find an example of a finite, and a countable X such that each $\mathfrak{B}(X, \mathbb{K})$ is isometrically isomorphic to a well-known space introduced in class.

c) Prove that $\|\cdot\|_\infty$ is a norm on $\mathfrak{B}(X, \mathbb{K})$ satisfying the inequality $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$ for all $f, g \in \mathfrak{B}(X, \mathbb{K})$.

d) Prove that $(\mathfrak{B}(X, \mathbb{K}), \|\cdot\|_\infty)$ is a Banach space.

[2+1+2+5 Points]

[1] A $\mathbb{K}$-algebra $A = (V, \phi)$ is a $\mathbb{K}$-vector space V equipped with a $\mathbb{K}$-bilinear map $\phi : V \times V \to V$ called multiplication.

Problem 4 (Operator norm)**.** Let $X \coloneqq \{x = (x_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N} : x_n \in \mathbb{R}\}$ denote the set of all sequences in $\mathbb{R}$. Consider the operator $H : X \to X$ defined by

$$(Hx)_n \coloneqq \frac{1}{n}\sum_{j=1}^{n} x_j, \qquad n \in \mathbb{N}. \tag{1}$$

a) For $p > 1$, and $a = (a_n)_{n=1}^{\infty} \in X$ with $a_n \geq 0$ for all $n \in \mathbb{N}$, prove that

$$(Ha)_n^p - \frac{p}{p-1}(Ha)_n^{p-1}a_n \leq \frac{1}{p-1}((n-1)(Ha)_{n-1}^p - n(Ha)_n^p). \tag{2}$$

Hint: Young's inequality.

b) For $p > 1$, $N \in \mathbb{N}$, and $a = (a_n)_{n=1}^{\infty} \in X$ with $a_n \geq 0$ for all $n \in \mathbb{N}$, prove that

$$\left(\sum_{n=1}^{N}(Ha)_n^p\right)^{1/p} \leq \frac{p}{p-1}\left(\sum_{n=1}^{N} a_n^p\right)^{1/p}. \tag{3}$$

Hint: Hölder's inequality.

c) Prove that $H : \ell^p \to \ell^p$ is a well-defined, bounded linear operator.

d) Compute $\|H\|_{B(\ell^p)}$.

[2+2+3+3 Points]

Deadline: June 6, 2016 14:00*, for details see http://www.math.lmu.de/~gottwald/16FA/.*