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Functional Analysis

Excercise Sheet 6

Problem 1 (Banach vs. absolute convergence). Let $(X, \|\cdot\|)$ be a normed space. Prove that X is a Banach space *iff* every absolutely convergent series in X is convergent in X. [8 Points]

Problem 2 (Examples of open/closed sets in ℓ^p).

a) Consider the sets

$$c_{00} := \{x = (x_n)_n \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n > N : x_n = 0\}$$

and

$$c_0 := \{x = (x_n)_n \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}.$$

Prove that $\overline{c_{00}}^{\|\cdot\|_\infty} = c_0$ and that c_0 is a closed linear subspace of ℓ^∞.

b) Let $a = (a_n)_n \subseteq (0, \infty)$ be a sequence and let

$$S^{(a)} := \{x = (x_n)_n \in \ell^2 \mid |x_n| < a_n \text{ for all } n \in \mathbb{N}\}.$$

Prove that $S^{(a)}$ is open in ℓ^2 *iff* $\inf_{n\in\mathbb{N}} a_n > 0$.

c) Let $p \in [1, \infty)$ and let

$$E := \left\{ x = (x_n)_n \in \ell^p \mid \sum_{n=1}^{\infty} x_n = 0 \right\}.$$

Prove that E is closed in ℓ^p *iff* $p = 1$.

[4+4+4 Points]

Problem 3 (Closure of ℓ^p in ℓ^q).

a) Let $1 \leq p < q < \infty$. Prove that ℓ^p is a proper dense subspace of ℓ^q.

b) Let $1 \leq p < \infty$. Find the closure of ℓ^p in ℓ^∞.

[5+5 Points]

Problem 4 (Norms and metrics)**.** Let X be a vector space (on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

a) A metric d on X is called *translation invariant iff* $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in X$, and it is called *homogeneous iff* $d(\alpha x, \alpha y) = |\alpha| d(x, y)$ for all $x, y \in X$ and all $\alpha \in \mathbb{K}$.

   Prove that there is a one-to-one correspondence between norms on X and metrics on X that are translation invariant and homogeneous.

b) Let $p : X \to [0, \infty)$ be given such that

   (i) $p(x) = 0$ iff $x = 0$,

   (ii) $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and all $\alpha \in \mathbb{K}$.

   Prove that p is a norm *iff* $K := \{x \in X \mid p(x) \leq 1\}$ is convex.

   Recall: A subset $K \subseteq X$ is called *convex iff* $tx + (1 - t)y \in K$ for all $x, y \in K$ and all $t \in [0, 1]$.

[5+5 Points]

Deadline: May 30, 2016 14:00*, for details see http://www.math.lmu.de/˜gottwald/16FA/.*