Prof. T. Ø. Sørensen PhD
A. Groh, S. Gottwald

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Functional Analysis

Excercise Sheet 5

Problem 1 (Contractions, Compactness & Completeness).

a) Let (X, d) be a non-empty compact metric space and let $\Phi : X \to X$ be such that

$$d(\Phi(x), \Phi(y)) < d(x, y), \text{ for all } x, y \in X, x \neq y. \tag{1}$$

Prove that Φ has a unique fixed point $x_0 = \lim_{n\to\infty} \Phi^n(x)$, where $x \in X$ arbitrary.

b) Find a non-empty compact metric space (X, d) and a function $\Phi : X \to X$ such that $d(\Phi(x), \Phi(y)) \leq d(x, y)$ for all $x, y \in X$, $x \neq y$ and Φ does *not* have a fixed point. What is the difference to a)?

c) Prove that there does not exist a surjective contraction from a compact metric space with more than one elements onto itself.

d) Let (X, d) be a non-empty complete metric space and let $\Phi : X \to X$ be such that Φ^m is a contraction for some $m \in \mathbb{N}$. Prove that Φ has a unique fixed point.

e) Let (X, d) be a non-empty metric space such that any contraction $\Phi : E \to E$ on any non-empty closed subset $E \subseteq X$ has a fixed point. Prove that (X, d) is complete.

[3+2+2+2+3 Points]

Problem 2 (Topologies and convergent sequences, metrics & norms).

a) Prove or disprove: If two topological spaces $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ have the same convergent sequences then $\mathcal{T}_1 = \mathcal{T}_2$.

b) Let $(\mathbb{R}, d_1)$ and $(\mathbb{R}, d_2)$ be metric spaces, with $d_1(x, y) = |x - y|$ and $d_2(x, y) = |\phi(x) - \phi(y)|$, where $\phi(x) = x/(1 + |x|)$ for $x, y \in \mathbb{R}$. Prove that d_1 and d_2 induce the same topology on $\mathbb{R}$, but that $(\mathbb{R}, d_1)$ is complete and $(\mathbb{R}, d_2)$ not.

c) Let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be normed spaces, and let $\mathcal{T}_1, \mathcal{T}_2$ be the topologies induced by $\|\cdot\|_1$, resp. $\|\cdot\|_2$. Prove that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ (i.e. $\mathcal{T}_1 \subseteq T_2$) *iff* there exists $C > 0$ s.t. $\|x\|_1 \leq C \|x\|_2$ for all $x \in X$.

d) Prove that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on a space X *iff* there exists $c, C > 0$ s.t. $c \|x\|_2 \leq \|x\|_1 \leq C \|x\|_2$ for all $x \in X$.

[2+4+3+1 Points]

Problem 3 (Product space & Compactness). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be compact topological spaces. Prove that $(X \times Y, \mathcal{T}_{X\times Y})$ is compact, where $\mathcal{T}_{X\times Y}$ is the product topology on $X \times Y$. [10 Points]

Problem 4 (Function spaces).

a) Let $C^1([0,1])$ denote the space of differentiable functions on $[0,1]$ with continuous derivative. Find the interior of $C^1([0,1])$ in $(C([0,1]), \|\cdot\|_\infty)$, where the sup-norm is given by $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$ for $f \in C([0,1])$.

b) Let $C_{00}(\mathbb{R})$ denote the space of continuous functions with compact support[1] in $\mathbb{R}$. Consider the metric space $(C_b(\mathbb{R}), d_\infty)$ with

$$C_b(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } \sup_{x\in\mathbb{R}} |f(x)| \leq C_f \text{ for some } C_f \geq 0\}$$

and $d_\infty(f,g) := \sup_{x\in\mathbb{R}} |f(x) - g(x)|$ for all $f, g \in C_b(\mathbb{R})$.

Find the closure of $C_{00}(\mathbb{R})$ in $(C_b(\mathbb{R}), d_\infty)$.

[3+5 Points]

Deadline: May 23, 2016 14:00, *for details see http://www.math.lmu.de/~gottwald/16FA/.*

[1] The support of a continuous function $f : \mathbb{R} \to \mathbb{R}$ is the set $\operatorname{supp}(f) := \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}$.