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Summer term 2016
May 9, 2016

Functional Analysis

Excercise Sheet 4

Problem 1 (Compactness).

a) Consider $A := (0,1)$ as a subset of the metric space $(\mathbb{R}, d_{Eucl})$. Find an open cover of A which does not admit a finite subcover.

b) Consider $B := [0,1] \cap \mathbb{Q}$ as a subset of the metric space $(\mathbb{Q}, d_{Eucl})$. Prove that B is closed and bounded in $\mathbb{Q}$ and find an open cover of B which does not admit a finite subcover.

c) Find a Hausdorff non-compact space, a finite compact non-Hausdorff space, and an infinite compact non-Hausdorff space.

[2+4+4 Points]

Problem 2 (Homeomorphisms).

a) Let X, Y be topological spaces and $f : X \to Y$ a continuous function.

Prove that $f(X)$ is compact if X is compact.

b) Let $X := [0,1)$ and $Y := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be equipped with the relative topologies induced by the Euclidian topologies on $\mathbb{R}$ resp. $\mathbb{R}^2$.

Prove that

$$\varphi : X \to Y, \theta \mapsto (\cos(2\pi\theta), \sin(2\pi\theta))$$

is a continuous bijection, but not a homeomorphism. Are X and Y homeomorphic?

c) Let X be a compact space and let Y be a Hausdorff space.

Prove or disprove: If $f : X \to Y$ is a continuous bijection then it is a homeomorphism.

d) Let X be a compact Hausdorff space and let Y be a compact space.

Prove or disprove: If $f : X \to Y$ is continuous bijection then it is a homeomorphism.

[2+2+3+3 Points]

Problem 3 (ℚ CAN BE OPEN).

a) Prove that $\mathbb{Q}$ is neither open nor closed in $(\mathbb{R}, d_{Eucl})$.

b) Prove that $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ with

$$d(x,y) := |x-y| + \sum_{n=1}^{\infty} \frac{1}{2^n} \min\left\{1, \left| \frac{1}{\min_{j\le n} |x-q_j|} - \frac{1}{\min_{j\le n} |y-q_j|} \right| \right\},$$

where $\{q_j\}_{j\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$, is a metric on $\mathbb{R}$. (*Hint*: By convention $1/0 = \infty$, $|\infty - \infty| = 0$ and $|\infty - a| = |a - \infty| = \infty$ for all $a \in \mathbb{R}$.)

c) Prove that $\{q\}$ is open in $(\mathbb{R}, d)$ for all $q \in \mathbb{Q}$.

d) Prove that $\mathbb{Q}$ is open in $(\mathbb{R}, d)$.

[3+4+2+1 Points]

Problem 4 (HOMEOMORPHISMS AND COMPLETENESS).

a) Find two metric spaces that are homeomorphic as topological spaces and such that one is complete whereas the other is not.

Hint: There exist easy examples. In particular the examples given below will not be accepted as an answer here.

Let d be the metric defined in Problem 3b) restricted to $(\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q})$.

b) Prove that $(\mathbb{R} \setminus \mathbb{Q}, d_{Eucl})$ and $(\mathbb{R} \setminus \mathbb{Q}, d)$ are homeomorphic.

c) Prove that $(\mathbb{R} \setminus \mathbb{Q}, d_{Eucl})$ is not complete whereas $(\mathbb{R} \setminus \mathbb{Q}, d)$ is complete.

[2+4+4 Points]

Deadline: May 18, 2016 10:00, *for details see http://www.math.lmu.de/˜gottwald/16FA/.*