LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

PROF. T. Ø. SØRENSEN PHD
A. Groh, S. Gottwald

Summer term 2016
Apr 25, 2016

FUNCTIONAL ANALYSIS

EXCERCISE SHEET 2

Remark. The purpose of this excercise sheet is to get used to topological notions. Do not forget writing up solutions *carefully* and *in detail*. What did you learn from these exercises?

Problem 1 (CLOSURE, INTERIOR, BOUNDARY W.R.T. COMPLEMENT, INCLUSION, UNION, AND INTERSECTION)**.** Let $(X, \mathcal{T})$ be a topological space and $E, F \subseteq X$.

a) Prove that $\partial E = \partial(X \setminus E)$.

b) Prove that $\mathring{\mathring{E}} = \mathring{E}$.

c) Prove that $\overline{E \cup F} = \overline{E} \cup \overline{F}$.

d) Prove that $(E \overset{\circ}{\cup} F) \supset \mathring{E} \cup \mathring{F}$ and find an example of strict inclusion.

e) Prove that $(E \overset{\circ}{\cap} F) = \mathring{E} \cap \mathring{F}$.

f) Prove that $\partial(E \cup F) \subset \partial E \cup \partial F$ and find an example of strict inclusion.

g) Find examples for $\partial(E \cap F) \subsetneq \partial E \cap \partial F$ and for $\partial(E \cap F) \supsetneq \partial E \cap \partial F$.

[1+1+1+2+1+2+2 Points]

Problem 2 (BASE OF A TOPOLOGY)**.** Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{T}$ such that any $U \in \mathcal{T}$ is the union of sets in $\mathcal{B}$ is called a *base* for the topology $\mathcal{T}$.

a) Prove that every topology has a base.

b) Prove that $\mathcal{B} \subseteq \mathcal{T}$ is a base for $\mathcal{T}$ *iff* for all $x \in X$ the family $\mathcal{B}_x := \{B \in \mathcal{B} | x \in B\}$ is a neighbourhood basis for x.

c) Prove that $\mathcal{B} \subseteq \mathcal{P}(X)$ is the base of a topology of X *iff* $\mathcal{B}$ has the following properties:

   i) For all $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.

   ii) Let $x \in X$ and $B_1, B_2 \in \mathcal{B}$. If $x \in B_1 \cap B_2$ then there exists $B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$.

d) Prove that $\mathcal{B} := \{\, [a, b) \mid a, b \in \mathbb{R}, a \leq b\}$ is the base for a topology in $\mathbb{R}$.

[1+3+3+3 Points]

Problem 3 (Characterisation of closed sets). Let $(X, \mathcal{T})$ be a topological space and $E \subseteq X$. Prove that the following properties are equivalent:

i) E is closed.

ii) $\overline{E} = E$.

iii) $\partial E \subseteq E$.

iv) $E = \{\text{adherent points of } E\}$.

v) For all $x \in X$, if every neighbourhood of x intersects E, then $x \in E$.

vi) $\{\text{limit points of } E\} \subseteq E$.

[10 Points]

Problem 4 (Sequential continuity $\not\Longrightarrow$ continuity). Let X be a set and let $\mathcal{T}_1 := \{\emptyset\} \cup \{A \subseteq X \mid X \setminus A \text{ is at most countable}\}$.

a) Prove that $(X, \mathcal{T}_1)$ is a topological space.

Consider the topological spaces $(X, \mathcal{T}_X) := (\mathbb{R}, \mathcal{T}_1)$ and $(Y, \mathcal{T}_Y) := (\mathbb{R}, \mathcal{T}_{Eucl})$.

b) Prove that every map $f : X \to Y$ is sequentially continuous.

c) Prove that $g : X \to Y, x \mapsto x$, is not continuous.

[2+6+2 Points]

Deadline: May 2, 2016, *for details see http://www.math.lmu.de/˜gottwald/16FA/.*