LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis

Excercise Sheet 1

Remark. The purpose of this excercise sheet is to get used to topological notions. Do not forget writing up solutions *carefully* and *in detail*. What did you learn from these exercises?

Problem 1. (Basic facts about closure, interior, and boundary.) Let $(X, \mathcal{T})$ be a topological space and $E \subseteq X$. Use the definitions given in class of closed sets in X, closure $\overline{E}$, interior $\mathring{E}$, and boundary ∂E of E.

a) Prove that $X \setminus \mathring{E} = \overline{X \setminus E}$.

b) Prove that $\partial E = \overline{E} \setminus \mathring{E} = \overline{E} \cap (\overline{X \setminus E})$.

c) Prove that $\overline{E} = \mathring{E} \sqcup \partial E$ ($\sqcup$ meaning disjoint union).

[3+6+1 Points]

Problem 2. (Co-finite topology.)

a) Let X be a set and $\mathcal{T}$ the family of subsets U of X such that $X \setminus U$ is finite, together with the empty set $\emptyset$.

   i) Prove that $(X, \mathcal{T})$ is a topologial space.
   ii) Let $E \subseteq X$. Find the closure $\overline{E}$ of E in the topological space $(X, \mathcal{T})$.

b) Consider $X = \mathbb{Z}$ with $\mathcal{T}$ as defined above.

   i) Prove that the sequence $1, 2, 3, \ldots$ is convergent in $(\mathbb{Z}, \mathcal{T})$ to *each* point of $\mathbb{Z}$.
   ii) Find *all* convergent sequences in $(\mathbb{Z}, \mathcal{T})$.

[2+2+2+4 Points]

Problem 3. (Relative topology) Let $(X, \mathcal{T})$ be a topological space, $S \subseteq X$, and $\mathcal{T}_S := \{O \cap S \mid O \in \mathcal{T}\}$ the *relative topology* on S induced by $\mathcal{T}$.

a) Let $E \subseteq S$. Prove that E is $\mathcal{T}_S$-closed (i.e. *relatively closed*) if and only if $E = S \cap C$ for some $\mathcal{T}$-closed set $C \subseteq X$.

b) Let $E \subseteq S$. Prove that the closure of E with respect to $\mathcal{T}_S$ (i.e. the *relative closure* of E in S) is $\overline{E} \cap S$, where $\overline{E}$ denotes the closure of E w.r.t. $\mathcal{T}$.

[5+5 Points]

Problem 4. (RELATIVE CLOSURE/CONVERGENCE.) Let $(X, \mathcal{T})$ be a topological space, $S \subseteq X$, and $(S, \mathcal{T}_S)$ be the topological space consisting of the subset S equipped with the relative topology $\mathcal{T}_S$ induced by $\mathcal{T}$.

a) Let $A \subseteq X$. Prove that the $\mathcal{T}_S$-closure of $A \cap S$ in S is contained in $\overline{A} \cap S$, where $\overline{A}$ denotes the $\mathcal{T}$-closure of A in X.

b) Let $\{x_n\}_n \subseteq S$ be a sequence and $x \in S$. Prove that $\{x_n\}_n$ is convergent to x in $(S, \mathcal{T}_S)$ *if and only if* it is convergent to x in $(X, \mathcal{T})$

[5+5 Points]

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