LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis
Excercise Sheet 0

Remark. The purpose of this excercise sheet is to practice (on hopefully easy excercises) writing up solutions *carefully* and *in detail*.

Problem 1. Prove that $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is continuous using the $\varepsilon - \delta -$ definition.
[10 Points]

Problem 2. Let $\{x_n\}_n \subseteq \mathbb{R}$ be a sequence such that the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. Prove that $\sum_{n=1}^{\infty} x_n$ exists.
[10 Points]

Problem 3. Let $I \subseteq \mathbb{R}$ be an open intervall, $t_0 \in I$, and $f : I \times \mathbb{R} \to \mathbb{R}$, such that:

i) For all $t \in I$ the function $x \mapsto f(t, x)$ is integrable,

ii) For all $x \in \mathbb{R}$ the partial derivative $\frac{\partial f}{\partial t}(t_0, x)$ exists,

iii) There exists a neighbourhood U of t_0 and an integrable function $g : \mathbb{R} \to \mathbb{R}$, such that, for all $t \in U \cap I$, $t \neq t_0$ and all $x \in \mathbb{R}$, $\left|\frac{f(t,x)-f(t_0,x)}{t-t_0}\right| \leq g(x)$.

Prove that the function

$$F : I \to \mathbb{R}, t \mapsto \int_{\mathbb{R}} f(t, x)\mathrm{d}x \tag{1}$$

is differentiable at t_0, and that

$$F'(t_0) = \int_{\mathbb{R}} \frac{\partial f}{\partial t}(t_0, x)\mathrm{d}x. \tag{2}$$

[10 Points]

Problem 4.

a) Prove that $U := \{g \in C^1([0, 1]) \mid g(0) = 0\}$ is a linear subspace of $C^1([0, 1])$.

b) Prove that the map $T : C([0, 1]) \to C^1([0, 1])$, with $(Tf)(s) := \int_0^s f(t)\mathrm{d}t$ for all $s \in [0, 1]$, is well-defined and linear.

c) Prove that $\mathrm{Ran}(T) = U$.

Hint: $\mathrm{Ran}(T) := \{g \in C^1([0, 1]) \mid \exists f \in C([0, 1]) \text{ s.t. } Tf = g\}$. You may use without proof, that $C([0, 1])$ and $C^1([0, 1])$ are vector spaces.
[10 Points]

For general informations please visit http://www.math.lmu.de/~gottwald/16FA/