LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Winter term 2023/24 14 November 2023

Topology I

Sheet 5

Exercise 1. Let $(X, \mathcal{O}_X)$ be a topological space. To X we can associate a new topological space X^+ as follows: The underlying set of X^+ is $X \cup \{\infty\}$, where ∞ is a new point not previously in X, and the topology on X^+ is defined by

$$\mathcal{O}_{X^+} = \mathcal{O}_X \cup \{(X \backslash K) \cup \{\infty\} \mid K \subseteq X \text{ compact and closed}\}.$$

If X is locally compact, non-compact, and Hausdorff, then $X \hookrightarrow X^+$ is usually called the one-point compactification of X.

(a) Show that $\mathcal{O}_{X^+}$ is a topology on X^+.

(b) Show that X^+ is compact, and that X^+ is Hausdorff if X is locally compact and Hausdorff (does weakly locally compact suffice?)

(c) Show that $(X \times Y)^+ \cong X^+ \wedge Y^+$ for all locally compact Hausdorff spaces X and Y.

(d) Show that $(\mathbb{R}^n)^+ \cong S^n$ for all $n \geq 0$. Conclude that $S^n \wedge S^m \cong S^{n+m}$ for all $n, m \geq 0$.

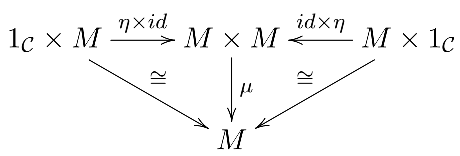
Exercise 2. Show that the space

$$S = \{(x, y) \in \mathbb{R}^2 \mid y = xm \text{ for some } m \in \mathbb{Q}\}$$

is contractible, but does not deformation retract onto $(1, 0)$. [Hint: Show that if a space X deformation retracts onto a point $x_0 \in X$, then for each neighbourhood V of x_0 there is a neighbourhood $U \subseteq V$ of x_0 such that the inclusion $U \hookrightarrow V$ is homotopic to the constant map at x_0.]

Exercise 3. Let $\mathcal{C}$ be a category with finite products. In particular, $\mathcal{C}$ has a terminal object $1_{\mathcal{C}} \in \mathcal{C}$ and there are canonical isomorphisms $M \times 1_{\mathcal{C}} \cong M \cong 1_{\mathcal{C}} \times M$. A *monoid in* $\mathcal{C}$ is a triple (M, μ, η) consisting of an object $M \in \mathcal{C}$ and morphisms $\mu\colon M \times M \to M$ and $\eta\colon 1_{\mathcal{C}} \to M$ such that the following diagrams commute:

(Unitality)



(Associativity)

$$\begin{array}{ccccc}
(M \times M) \times M & \xrightarrow{\cong} & M \times (M \times M) & \xrightarrow{id \times \mu} & M \times M \\
\big\downarrow{\scriptstyle \mu \times id} & & & & \big\downarrow{\scriptstyle \mu} \\
M \times M & & \xrightarrow{\qquad\qquad \mu \qquad\qquad} & & M
\end{array}$$

Here the isomorphisms are the canonical ones. We say that (M, μ, η) is a *group in* $\mathcal{C}$ if there is a morphism $\mathrm{inv}\colon M \to M$ such that the following diagrams commute:

$$\begin{array}{ccccc}
M & \xrightarrow{\Delta} & M \times M & \xrightarrow{\mathrm{inv} \times id} & M \times M \\
\big\downarrow & & & & \big\downarrow{\scriptstyle \mu} \\
1_{\mathcal{C}} & & \xrightarrow{\qquad \eta \qquad} & & M
\end{array}
\qquad
\begin{array}{ccccc}
M & \xrightarrow{\Delta} & M \times M & \xrightarrow{id \times \mathrm{inv}} & M \times M \\
\big\downarrow & & & & \big\downarrow{\scriptstyle \mu} \\
1_{\mathcal{C}} & & \xrightarrow{\qquad \eta \qquad} & & M
\end{array}$$

Here $\Delta = (id, id)\colon M \to M \times M$ is the diagonal.

(a) Show that a monoid or group in Set is a monoid respectively group in the usual sense.

(b) Let $\mathrm{pr}_1\colon M \times M \to M$ be the projection onto the first factor. Show that a monoid (M, μ, η) in $\mathcal{C}$ is a group if and only if the morphism $(\mathrm{pr}_1, \mu)\colon M \times M \to M \times M$ is an isomorphism.

(c) Show that a monoid (M, μ, η) in $\mathcal{C}$ is a group if and only if for all $X \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(X, M)$ together with the maps

$$\mathrm{Hom}_{\mathcal{C}}(X, M) \times \mathrm{Hom}_{\mathcal{C}}(X, M) \cong \mathrm{Hom}_{\mathcal{C}}(X, M \times M) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X, \mu)} \mathrm{Hom}_{\mathcal{C}}(X, M)$$

and $\mathrm{Hom}_{\mathcal{C}}(X, \eta)\colon \mathrm{Hom}_{\mathcal{C}}(X, 1_{\mathcal{C}}) \to \mathrm{Hom}_{\mathcal{C}}(X, M)$ is a group, natural in X.

(d) Prove the following categorical version of the Eckman-Hilton argument: Suppose that $M \in \mathcal{C}$ carries two monoid structures $(M, \star, \eta_\star)$ and $(M, \circ, \eta_\circ)$ which make the following diagram commute:

$$\begin{array}{ccccc}
M \times M \times M \times M & \xrightarrow{id \times \tau \times id} & M \times M \times M \times M & \xrightarrow{\circ \times \circ} & M \times M \\
\big\downarrow{\scriptstyle \star \times \star} & & & & \big\downarrow{\scriptstyle \star} \\
M \times M & & \xrightarrow{\qquad\qquad \circ \qquad\qquad} & & M
\end{array}$$

Here $\tau = (\mathrm{pr}_2, \mathrm{pr}_1)\colon M \times M \to M \times M$ is the morphism swapping the two factors. Show that $\star = \circ$ and both products are commutative.

This sheet will be discussed in the week of 20 November 2023.