LUDWIG-
MAXIMILIANS-
UNIVERSITÄT
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MATHEMATISCHES INSTITUT

Winter term 2023/24 7 November 2023

Topology I

Sheet 4

Exercise 1. Let X be a topological space and $A \subseteq X$ a subset. For $x \in X$ show that $x \in \overline{A}$ if and only if there is a filter $\mathcal{F}$ on A converging to x in the sense that $\mathcal{U}(x) \cap A \subseteq \mathcal{F}$, where $\mathcal{U}(x)$ is the neighbourhood filter of x.

Exercise 2. Let X be a compact Hausdorff space and let $x \in X$. Show that the connected component $C(x)$ of x is precisely the intersection of all open and closed subsets of X containing x.

Exercise 3. Let $f\colon X \to Y$ be a quotient map and suppose that f is *proper*, that is, if $K \subseteq Y$ is compact, then $f^{-1}(K) \subseteq X$ is compact as well. Show that for any space Z the induced map

$$\begin{aligned} f^*\colon \operatorname{Map}(Y, Z) &\to \operatorname{Map}(X, Z) \\ g &\mapsto gf \end{aligned}$$

is an embedding.

Exercise 4. Show that for any X, Y, Z the map

$$\begin{aligned} \operatorname{Map}(X, Y) \times Z &\to \operatorname{Map}(X, Y \times Z) \\ (f, z) &\mapsto (x \mapsto (f(x), z)) \end{aligned}$$

is continuous.

Exercise 5. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and consider the projective space $\mathbb{KP}^n$.

(a) Show that $\mathbb{KP}^n$ is Hausdorff.

(b) Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere. Show that the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1} \backslash \{0\}$ descends to a homeomorphism $S^n/\sim \,\cong \mathbb{RP}^n$, where $\sim$ is the equivalence relation generated by $x \sim -x$ for all $x \in S^n$.

(c) Similarly, show that $\mathbb{KP}^n$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$ is homeomorphic to the quotient of a sphere by a suitable equivalence relation.

This sheet will be discussed in the week of 13 November 2023.