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Functional Analysis

Tutorial 9 – Solutions of P2(*iii*)-(*v*) and P3

Problem 2 (Projections onto closed convex sets). Let $\mathcal{H}$ be a Hilbert space, and let $\Sigma \subset \mathcal{H}$ be a non-empty closed convex subset.

(*iii*) Prove that, if $x \notin \Sigma$, then $P_\Sigma(x) \in \partial\Sigma$, and $\operatorname{dist}(x,\Sigma) = \operatorname{dist}(x,\partial\Sigma)$.
[*Hint:* Use the continuity of the map $t \mapsto tx+(1-t)P_\Sigma(x)$.]

Proof. For fixed $x \in \mathcal{H}$ such that $x \notin \Sigma$, let $f : \mathbb{R} \to \mathcal{H}$, $f(t) := tx+(1-t)P_\Sigma(x)$. Then f is continuous ($\|f(t) - f(s)\| = \|x-P_\Sigma(x)\|\,|t-s|$). By contradiction, assume that $P_\Sigma(x) \in \mathring{\Sigma}$, i.e. $f(0) \in \mathring{\Sigma}$. Hence, by continuity, $f^{-1}(\mathring{\Sigma})$ is an open neighbourhood of 0, i.e. there exists $\delta \in (0,1)$ such that $|t| \leqslant \delta$ implies $f(t) \in \mathring{\Sigma}$. Hence,

$$\operatorname{dist}(x,\Sigma) \leqslant \|x-f(\delta)\| = \|f(1)-f(\delta)\| = (1-\delta)\|x-P_\Sigma(x)\| \;=\; (1-\delta)\operatorname{dist}(x,\Sigma)\,,$$

a contradiction, since $(1-\delta) < 1$. So, $P_\Sigma(x) \in \partial\Sigma$. Moreover, since Σ is closed, and thus $\partial\Sigma \subset \Sigma$, we have

$$\operatorname{dist}(x,\Sigma) = \inf_{z\in\Sigma}\|x-z\| \leqslant \inf_{z\in\partial\Sigma}\|x-z\| = \operatorname{dist}(x,\partial\Sigma)\,.$$

On the other hand, since $P_\Sigma(x) \in \partial\Sigma$, $\operatorname{dist}(x,\partial\Sigma) \leqslant \|x-P_\Sigma(x)\| = \operatorname{dist}(x,\Sigma)$. □

(*iv*) Let $f \in C^1(\mathbb{R},\mathbb{R})$ be convex and let $\Sigma \subset \mathbb{R}^2$ be given by

$$\Sigma := \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, f(x) \leqslant y\right\}.$$

Prove that, for all $(a,b) \in \mathbb{R}^2$ with $(a,b) \notin \Sigma$, we have $P_\Sigma((a,b)) = (x, f(x))$, where $x \in \mathbb{R}$ satisfies the equation $(b-f(x))f'(x) + a - x = 0$.

Proof. Since f is convex, and since Σ is the region above the graph of f, it is a convex set (by def. of a convex function). Moreover, $\Sigma = F^{-1}([0,\infty))$, where F is the continuous function $(x,y) \mapsto y-f(x)$, hence Σ is also closed. For $(a,b) \notin \Sigma$, by (*iii*), we have $P_\Sigma((a,b)) \in \partial\Sigma$, i.e. $P_\Sigma((a,b)) = (x, f(x))$ for some $x \in \mathbb{R}$. By definition of P_Σ and (*iii*), the function

$$D(t) := |(a,b) - (t, f(t))| = \sqrt{(a-t)^2 + (b-f(t))^2}$$

takes its minimum at x, i.e. x satisfies $\frac{dD}{dt}(x) = 0$, which is equivalent to the given equation. □

(v) Find the projection of the point $(1, \frac{1}{2}) \in \mathbb{R}^2$ onto $\Sigma := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 \leqslant y\}$.

Proof. By applying (iv) with $f(x) = x^2$, we get that x solves the equation

$$\left(\tfrac{1}{2} - x^2\right) 2x + 1 - x = 0 \quad \Leftrightarrow \quad x^3 = \tfrac{1}{2}\,,$$

i.e. $x = 2^{-1/3}$, and therefore $P_\Sigma((1, \frac{1}{2})) = (2^{-1/3}, 2^{-2/3})$. □

Problem 3. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Complete the proofs of Lemma 2.21, Lemma 2.22, and Remark 2.23 (2), i.e. prove the following statements:

(i) $|\langle x, y\rangle| \leqslant \|x\|\|y\|$ for all $x, y \in X$ (*Cauchy-Bunyakowsky-Schwarz inequality*).
[*Hint:* Use the fact that $\|\alpha x + y\|^2 \geqslant 0$ with $\alpha = -\overline{\langle y, x\rangle}/\langle x, x\rangle$]

Proof. We have

$$\begin{aligned} 0 \leqslant \|\alpha x + y\|^2 &= |\alpha|^2 \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\alpha \langle y, x\rangle) \\ &= \frac{|\langle y, x\rangle|^2}{\|x\|^2} + \|y\|^2 - 2\,\frac{|\langle y, x\rangle|^2}{\|x\|^2} = \|y\|^2 - \frac{|\langle y, x\rangle|^2}{\|x\|^2} \end{aligned}$$

and thus $|\langle y, x\rangle|^2 \leqslant \|y\|^2 \|x\|^2$. □

(ii) $\|x\|_X := \sqrt{\langle x, x\rangle}$ defines a norm on X (the norm *induced* by the inner product).

Proof. Homogeneity and positive definiteness for $x \neq 0$ follow directly from the properties of $\langle \cdot, \cdot \rangle$. Moreover,

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle) \leqslant \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle| \\ &\leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \;=\; (\|x\| + \|y\|)^2\,, \end{aligned}$$

where the second inequality is the Cauchy-Schwarz inequality. □

(iii) $\|x+y\|_X^2 + \|x-y\|_X^2 = 2\|x\|_X^2 + 2\|y\|_X^2$ for all $x, y \in X$ (*Parallelogram Law*).

Proof. The identity follows immediately after multiplying out the left side, since the cross-term $\mathrm{Re}(\langle x, y\rangle)$ appears in the second summand with the opposite sign. □