LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis
Tutorial 9

Problem 1. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $A \subset \mathcal{H}$ be a non-trivial closed linear subspace. Complete the proof of the Projection Theorem (Thm. 2.28), by showing that the orthogonal projection onto A is linear.

Problem 2 (Projections onto closed convex sets). Let $\mathcal{H}$ be a Hilbert space, and let $\Sigma \subset \mathcal{H}$ be a non-empty closed convex subset.

(*i*) Prove that for all $x \in \mathcal{H}$ there exists a unique element $P_\Sigma(x) \in \Sigma$ such that

$$\operatorname{dist}(x, \Sigma) = \big\|x - P_\Sigma(x)\big\| .$$

[*Hint:* Pick an approximate sequence and prove that it is a Cauchy sequence by using the Parallelogram Law.]

(*ii*) Prove that, if Σ is a non-empty closed linear subspace of $\mathcal{H}$, then P_Σ is the orthogonal projection onto Σ given by the Projection Theorem.

(*iii*) Prove that, if $x \notin \Sigma$, then $P_\Sigma(x) \in \partial\Sigma$, and $\operatorname{dist}(x, \Sigma) = \operatorname{dist}(x, \partial\Sigma)$.
[*Hint:* Use the continuity of the map $t \mapsto tx+(1-t)P_\Sigma(x)$.]

(*iv*) Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be convex and let $\Sigma \subset \mathbb{R}^2$ be given by

$$\Sigma := \big\{(x, y) \in \mathbb{R}^2 \,\big|\, f(x) \leqslant y\big\} .$$

Prove that, for all $(a, b) \in \mathbb{R}^2$ with $(a, b) \notin \Sigma$, we have $P_\Sigma((a, b)) = (x, f(x))$, where $x \in \mathbb{R}$ satisfies the equation $(b-f(x))f'(x) + a - x = 0$.

(*v*) Find the projection of the point $(1, \frac{1}{2}) \in \mathbb{R}^2$ onto $\Sigma := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 \leqslant y\}$.

Problem 3. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Complete the proofs of Lemma 2.21, Lemma 2.22, and Remark 2.23 (2), i.e. prove the following statements:

(*i*) $|\langle x, y\rangle| \leqslant \|x\|\|y\|$ for all $x, y \in X$ (*Cauchy-Bunyakowsky-Schwarz inequality*).
[*Hint:* Use the fact that $\|\alpha x+y\|^2 \geqslant 0$ with $\alpha = -\overline{\langle y, x\rangle}/\langle x, x\rangle$]

(*ii*) $\|x\|_X := \sqrt{\langle x, x\rangle}$ defines a norm on X (the norm *induced* by the inner product).

(*iii*) $\|x+y\|_X^2 + \|x-y\|_X^2 = 2\|x\|_X^2 + 2\|y\|_X^2$ for all $x, y \in X$ (*Parallelogram Law*).